A GENERALIZATION OF A CONTRACTION PRINCIPLE IN PROBABILISTIC METRIC SPACES. PART II

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A fixed point theorem concerning probabilistic contractions satisfying an implicit relation, which generalizes a well-known result of Hadžić, is proved.

1. Preliminaries

In this section we recall some useful facts from the probabilistic metric spaces theory. For more details concerning this problematic we refer the reader to the books [1, 3, 9].

1.1. *t*-norms. A *triangular norm* (shortly *t-norm*) is a binary operation $T:[0,1]\times[0,1]\to[0,1]:=I$ which is commutative, associative, monotone in each place, and has 1 as the unit element.

Basic examples are $T_L : I\times I \to I$, $T_L(a,b) = \mathrm{Max}(a+b-1,0)$ (*Łukasiewicz t*-norm), $T_P(a,b) = ab$, and $T_M(a,b) = \mathrm{Min}\{a,b\}$. We also mention the following families of t-norms:

(i) *Sugeno-Weber family* $(T_\lambda^{SW})_{\lambda\in(-1,\infty)}$, defined by $T_\lambda^{SW} = \max(0,(x+y-1+\lambda xy)/(1+\lambda))$,

(ii) *Domby family* $(T_\lambda^{D})_{\lambda\in(0,\infty)}$, defined by $T_\lambda^{D} = (1+(((1-x)/x)^\lambda+((1-y)/y)^\lambda)^{1/\lambda})^{-1}$,

(iii) *Aczel-Alsina family* $(T_\lambda^{AA})_{\lambda\in(0,\infty)}$, defined by $T_\lambda^{AA} = e^{-(|\log x|^\lambda+|\log y|^\lambda)^{1/\lambda}}$.

Definition 1.1 [2, 3]. It is said that the t-norm T is *of Hadžić-type* (*H-type* for short) and $T\in\mathcal{H}$ if the family $\{T^n\}_{n\in N}$ of its iterates defined, for each x in $[0,1]$, by

$$T^0(x) = 1, \qquad T^{n+1}(x) = T\big(T^n(x),x\big), \quad \forall n\geq 0, \tag{1.1}$$

is equicontinuous at $x=1$, that is,

$$\forall \varepsilon\in(0,1)\ \exists\delta\in(0,1) \quad \text{such that } x>1-\delta \Longrightarrow T^n(x)>1-\varepsilon, \quad \forall n\geq 1. \tag{1.2}$$

There is a nice characterization of continuous t-norms T of the class $\mathcal{H}$ [8].

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(i) If there exists a strictly increasing sequence $(b_n)_{n\in N}$ in $[0,1]$ such that $\lim_{n\to\infty} b_n = 1$ and $T(b_n,b_n) = b_n$ $\forall n \in N$, then T is of Hadžić-type.

(ii) If T is continuous and $T \in \mathcal{H}$, then there exists a sequence $(b_n)_{n\in N}$ as in (i).

The t-norm T_M is an trivial example of a t-norm of H-type, but there are t-norms T of Hadžić-type with $T \neq T_M$ (see, e.g., [3]).

Definition 1.2 [3]. If T is a t-norm and $(x_1,x_2,\dots,x_n) \in [0,1]^n$ $(n \in N)$, then $T_{i=1}^n x_i$ is defined recurrently by 1, if $n = 0$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 1$. If $(x_i)_{i\in N}$ is a sequence of numbers from $[0,1]$, then $T_{i=1}^\infty x_i$ is defined as $\lim_{n\to\infty} T_{i=1}^n x_i$ (this limit always exists) and $T_{i=n}^\infty x_i$ as $T_{i=1}^\infty x_{n+i}$. In fixed point theory in probabilistic metric spaces there are of particular interest the t-norms T and sequences $(x_n) \subset [0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1$. Some examples of t-norms with the above property are given in the following proposition.

Proposition 1.3 [3]. (i) *For $T \geq T_L$ the following implication holds:*

$$\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty. \tag{1.3}$$

(ii) *(1.3) also holds for $T = T_\lambda^{SW}$.*

(iii) *If $T \in \mathcal{H}$, then for every sequence $(x_n)_{n\in N}$ in I such that* $\lim_{n\to\infty} x_n = 1$, *one has* $\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1$.

(iv) *If $T \in \{T_\lambda^D, T_\lambda^{AA}\}$, then* $\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1 - x_n)^\lambda < \infty$.

Note [4, Remark 13] that if T is a t-norm for which there exists a sequence $(x_n) \subset [0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} T_{i=1}^\infty x_{n+i} = 1$, then $\sup_{t<1} T(t,t) = 1$.

1.2. Menger spaces and generalized Menger spaces. Probabilistic contractions of Sehgal type. Let Δ_+ be the class of *distance distribution functions* [9], that is, the class of all functions $F : [0,\infty) \to [0,1]$ with the properties

(a) $F(0) = 0$;

(b) F is nondecreasing;

(c) F is left continuous on $(0,\infty)$.

D_+ is the subset of Δ_+ containing the functions F which also satisfy the condition $\lim_{x\to\infty} F(x) = 1$.

A special element of D_+ is the function ε_0, defined by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t = 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{1.4}$$

A sequence (F_n) in Δ_+ is said to be *weakly convergent to* $F \in \Delta_+$ (shortly $F_n \xrightarrow{w} F$) if $\lim_{n\to\infty} F_n(x) = F(x)$ for every continuity point x of F.

If X is a nonempty set, a mapping $F : X \times X \to \Delta_+$ is called *a probabilistic distance on X* and $F(x,y)$ is denoted by F_{xy}.

The triple (X,F,T), where X is a nonempty set, F is a probabilistic distance on X, and T is a t-norm, is called *a generalized Menger space* (or a *Menger space in the sense of*

Schweizer and Sklar) if the following conditions hold:

$$F_{xy} = \varepsilon_0 \Longleftrightarrow x = y, \tag{1.5}$$

$$F_{xy} = F_{yx}, \quad \forall x, y \in X, \tag{1.6}$$

$$F_{xy}(t+s) \geq T\big(F_{xz}(t), F_{zy}(s)\big), \quad \forall x, y, z \in X, \ \forall t, s > 0. \tag{1.7}$$

A *Menger space* is a generalized Menger space with the property Range $(F) \subset D_+$.

If (X, F, T) is a generalized Menger space with $\sup_{t<1} T(t,t) = 1$, then the family

$$\{U_{\varepsilon,\lambda}\}_{\varepsilon>0,\lambda\in(0,1)}, \quad U_{\varepsilon,\lambda} = \{(x,y) \in X \times X : F_{xy}(\varepsilon) > 1-\lambda\} \tag{1.8}$$

is a base for a metrizable uniformity on X, named the *F-uniformity* and denoted by $\mathcal{U}_F$.

$\mathcal{U}_F$ naturally determines a topology on X, called *the F-topology*:

$$O \in \mathcal{T}_F \Longleftrightarrow \forall x \in O \ \exists \varepsilon > 0, \ \exists \lambda \in (0,1) \quad \text{such that } U_{\varepsilon,\lambda}(x) \subset O. \tag{1.9}$$

$\mathcal{U}_F$ is also generated by the family $\{V_\delta\}_{\delta>0}$ where $V_\delta := U_{\delta,\delta}$. In what follows the topological notions refer to the F-topology. Thus, a sequence $(x_n)_{n\in N}$ is *F-convergent to* $x \in X$ if for all $\varepsilon > 0$, $\lambda \in (0,1)$ there exists $k \in N$ such that $F_{xx_n}(\varepsilon) > 1-\lambda$ for all $n \geq k$.

Definition 1.4. A sequence $(x_n)_{n\in N}$ in X is called *F-Cauchy* if for each $\varepsilon > 0$, $\lambda \in (0,1)$ there exists $k \in N$ such that $F_{x_r x_s}(\varepsilon) > 1-\lambda$ for all $s \geq r \geq k$.

Probabilistic contractions were first defined and studied by *V. M. Sehgal* in his doctoral dissertation at Wayne State University.

Definition 1.5 [10]. Let S be a nonempty set and let F be a probabilistic distance on S. A mapping $f : S \to S$ is called *a probabilistic contraction* (or *B-contraction*) if there exists $k \in (0,1)$ such that

$$F_{f(p)f(q)}(kt) \geq F_{pq}(t), \quad \forall p, q \in S, \ \forall t > 0. \tag{1.10}$$

In [10] it is showed that any contraction map on a complete Menger space in which the triangle inequality is formulated under the strongest triangular norm T_M has a unique fixed point. In [11] *Sherwood* showed that one can construct a complete Menger space under T_L and a fixed-point-free contraction map on that space. *Hadžić* [2] introduced the class $\mathcal{H}$ which have the property that Sehgal's result can be extended to any continuous triangular norm in that class. Completing the result of *Hadžić, Radu* solved the problem of the existence of fixed points for probabilistic contractions in complete Menger spaces (S, F, T) with T continuous. Namely, the following theorem holds.

Theorem 1.6 [7]. *Every B-contraction in a complete Menger space* (S, F, T) *with* T *continuous has a (unique) fixed point if and only if* T *is of Hadžić-type.*

However, under some additional growth conditions on the probabilistic metric F one may replace the t-norm of H-type in the above theorem, as in Tardiff's paper [13]. Corollary 2.6 in our paper gives another result in this respect.

2. Main results

The main result of this paper is Theorem 2.4 concerning contractive mappings satisfying an implicit relation similar to that in [6, 12]. This theorem generalizes the mentioned result of Hadžić (see Corollary 2.7). Note that we work in generalized Menger spaces.

We begin with an auxiliary result, which is formulated as follows.

Lemma 2.1. *Let (X,F,T) be a generalized Menger space and let $(x_n)_{n\in N}$ be a sequence in X such that, for some $k\in(0,1)$,*

$$F_{x_nx_{n+1}}(kt)\geq F_{x_{n-1}x_n}(t),\quad \forall n\geq 1,\ \forall t>0. \tag{2.1}$$

If there exists $\gamma>1$ such that

$$\lim_{n\to\infty} T_{i=n}^{\infty}F_{x_0x_1}(\gamma^i)=1, \tag{2.2}$$

then $(x_n)_{n\in N}$ is an F-Cauchy sequence.

Proof. First note [4] that if the condition $\lim_{n\to\infty} T_{i=n}^{\infty}F_{x_0x_1}(\gamma^i)=1$ holds for some $\gamma=\gamma_0>1$, then it is satisfied for all $\gamma>1$. Indeed, if $\lim_{n\to\infty} T_{i=n}^{\infty}F_{x_0x_1}(\gamma_0^i)=1$ and $\gamma\geq\gamma_0$, then $\lim_{n\to\infty} T_{i=n}^{\infty}F_{x_0x_1}(\gamma^i)\geq\lim_{n\to\infty} T_{i=n}^{\infty}F_{x_0x_1}(\gamma_0^i)=1$ and therefore $\lim_{n\to\infty} T_{i=n}^{\infty}F_{x_0x_1}(\gamma^i)=1$, while if $\gamma<\gamma_0$, then $\gamma^s>\gamma_0$, for some $s\in N$, and now $\lim_{n\to\infty} T_{i=n+s}^{\infty}F_{x_0x_1}(\gamma^i)\geq\lim_{n\to\infty} T_{i=n}^{\infty}F_{x_0x_1}(\gamma_0^i)=1$.

We will prove that

$$\forall\varepsilon>0,\ \exists n_0=n_0(\varepsilon): F_{x_nx_{n+m}}(\varepsilon)>1-\varepsilon,\quad \forall n\geq n_0,\ \forall m\in N. \tag{2.3}$$

Let $\mu\in(k,1)$ and let $\delta=k/\mu$. From the above remark it follows that

$$\lim_{n\to\infty} T_{i=n}^{\infty}F_{x_0x_1}\left(\frac{1}{\mu^i}\right)=1. \tag{2.4}$$

Let $\varepsilon>0$ be given and $y_i:=F_{x_0x_1}(1/\mu^i)$. From $\lim_{n\to\infty} T_{i=1}^{\infty}y_{n+i}=1$ it follows that there exists $n_1\in N$ such that $T_{i=1}^{m}y_{n+i-1}>1-\varepsilon$, for all $n\geq n_1$, for all $m\in N$.

Since the series $\sum_{n=1}^{\infty}\delta^n$ is convergent, there exists $n_2\in N$ such that $\sum_{n=n_2}^{\infty}\delta^n<\varepsilon$.

Let $n_0=\max\{n_1,n_2\}$. Then, for all $n\geq n_0$ and $m\in N$, we have

$$\begin{aligned} F_{x_nx_{n+m}}(\varepsilon)&\geq F_{x_nx_{n+m}}\left(\sum_{i=n}^{n+m-1}\delta^i\right)\\ &\geq T_{i=0}^{m-1}F_{x_{n+i}x_{n+i+1}}(\delta^{n+i})\geq T_{i=0}^{m-1}y_{n+i}>1-\varepsilon, \end{aligned} \tag{2.5}$$

where the last "$\geq$" inequality follows from $F_{x_sx_{s+1}}(\delta^s)=F_{x_sx_{s+1}}(k/\mu)^s\geq F_{x_0x_1}(1/\mu^s)$ for all $s\geq 1$, which immediately can be proved by induction. □

In the following we deal with the class Φ of all continuous functions $\varphi:[0,1]^4\to\mathbb{R}$ with the property:

$$\varphi(u,v,v,u)\geq 0\Longrightarrow u\geq v. \tag{2.6}$$

Next we give some examples of functions in Φ.

Example 2.2. If $a,b,c,d \in \mathbb{R}$ and $a+b+c+d=0$, then $\varphi(t_1,t_2,t_3,t_4) := at_1+bt_2+ct_3+dt_4 \in \Phi$ if and only if $a+d>0$.

Indeed, $a+d \le 0 \Rightarrow b+c \ge 0$. Choosing $u=0$, $v=1$ we have $u<v$ and $\varphi(u,v,v,u) = (a+d)u+(b+c)v = b+c \ge 0$.

Conversely, if $a+d>0$ and $\varphi(u,v,v,u) \ge 0$, then $(a+d)u \ge -(b+c)v$, that is $(a+d)u \ge (a+d)v$, which implies that $u \ge v$.

Thus, the functions φ_1, φ_2,

$$\begin{aligned} \varphi_1(t_1,t_2,t_3,t_4) &= t_1 - t_2, \\ \varphi_2(t_1,t_2,t_3,t_4) &= t_1 - t_3, \end{aligned} \tag{2.7}$$

are in Φ.

Also, the function φ defined by $\varphi(t_1,t_2,t_3,t_4) = t_1^2 - t_2t_3$ and, more generally, $\varphi(t_1,t_2,t_3,t_4) = t_1^2 - (at_2^2+bt_3^2) - t_2t_3$ with $a+b=0$ are in Φ.

In the proof of Theorem 2.4 we need the following lemma, which is the analog of uniform continuity of a metric (note that $([0,1],T)$ is rather a semigroup than a group).

Lemma 2.3. *Let (S,F,T) be a generalized Menger space with T continuous in $(a,1)$ for all $a \in (0,1)$, that is,*

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = 1 \Longrightarrow \lim_{n\to\infty} T(a_n,b_n) = a. \tag{2.8}$$

If $p,q \in S$ and (p_n) is a sequence in S such that $p_n \to p$, then $F_{p_nq} \xrightarrow{w} F_{pq}$.

Proof. Let $p,q \in S$, $p_n \to p$ and t be a continuity point of F_{pq}. By (1.7) it follows that for all $0<\varepsilon<t$,

$$\begin{aligned} F_{p_nq}(t) &\ge T(F_{p_np}(\varepsilon), F_{pq}(t-\varepsilon)), \\ F_{pq}(t+\varepsilon) &\ge T(F_{p_np}(\varepsilon), F_{p_nq}(t)). \end{aligned} \tag{2.9}$$

Therefore, $\lim_n \inf F_{p_nq}(t) \ge F_{pq}(t-\varepsilon)$ and $F_{pq}(t+\varepsilon) \ge \lim_n \sup F_{p_nq}(t)$. Letting $\varepsilon \to 0$ we obtain $\lim_n \sup F_{p_nq}(t) \le F_{pq}(t) \le \lim_n \inf F_{p_nq}(t)$, and thus $\lim_{n\to\infty} F_{p_nq}(t) = F_{pq}(t)$. □

Theorem 2.4. *Let (X,F,T) be an F-complete generalized Menger space under a t-norm T which is continuous in $(a,1)$ for all $a \in (0,1)$, $k \in (0,1)$, and $\varphi \in \Phi$. If $f : X \to X$ is a mapping such that*

$$(\varphi_f) : \varphi(F_{f(x)f(y)}(kt), F_{xy}(t), F_{xf(x)}(t), F_{yf(y)}(kt)) \ge 0, \quad \forall x,y \in X,\ \forall t>0 \tag{2.10}$$

and there exist $x_0 \in X$ and $\gamma > 1$ for which $\lim_{n\to\infty} T_{i=n}^{\infty} F_{x_0f(x_0)}(\gamma^i) = 1$, then f has a fixed point.

Proof. Let $x_0 \in X$ be such that $\lim_{n\to\infty} T_{i=n}^{\infty} F_{x_0f(x_0)}(\gamma^i) = 1$ and, for all $n \ge 1$, $x_n = f(x_{n-1})$. Note that (φ_f) implies that

$$F_{f(x)f^2(x)}(kt) \ge F_{xf(x)}(t), \quad \forall x \in X,\ \forall t>0. \tag{2.11}$$

On taking in this relation $x = x_n$ we obtain

$$\varphi(F_{x_{n+1}x_{n+2}}(kt), F_{x_n x_{n+1}}(t), F_{x_n x_{n+1}}(t), F_{x_{n+1}x_{n+2}}(kt)) \geq 0, \quad \forall n \in N,\ \forall t > 0. \tag{2.12}$$

It follows that $F_{x_{n+1}x_{n+2}}(kt) \geq F_{x_n x_{n+1}}(t)$, for all $n \in N$, for all $t > 0$ and therefore, by Lemma 2.1, (x_n) is a Cauchy sequence.

By the F-completeness of X it follows that there exists $u \in X$ such that $\lim_{n\to\infty} F_{ux_n}(t) = 1$, for all $t > 0$.

Notice that from $F_{x_{n+1}x_{n+2}}(kt) \geq F_{x_n x_{n+1}}(t)$, for all $n \in N$, for all $t > 0$ it follows that $\lim_{n\to\infty} F_{x_n x_{n+1}}(t) = 1$, for all $t > 0$, for $\lim_{n\to\infty} T_{i=n}^{\infty} F_{x_0 f(x_0)}(\gamma^i) = 1$ implies that $\lim_{n\to\infty} F_{x_0 f(x_0)}(\gamma^n) = 1$ (therefore $F_{x_0 f(x_0)} \in D_+$) and $F_{x_n x_{n+1}}(t) \geq F_{x_0 x_1}(t/k^n)$, for all $n \in N$, for all $t > 0$.

Next, on taking $x = x_n$, $y = u$ in (φ_f) one obtains

$$\varphi(F_{x_{n+1}f(u)}(kt), F_{x_n u}(t), F_{x_n x_{n+1}}(t), F_{uf(u)}(kt)) \geq 0, \quad \forall n \in N,\ \forall t > 0. \tag{2.13}$$

If kt is a continuity point of $F_{uf(u)}$, then, on taking $n \to \infty$ in the above inequality and using Lemma 2.3, we get

$$\varphi(F_{uf(u)}(kt), 1, 1, F_{uf(u)}(kt)) \geq 0. \tag{2.14}$$

Thus $F_{uf(u)}(kt) = 1$. Since $F_{uf(u)}$ is increasing, the set of its discontinuity points is at most countable. Hence $F_{uf(u)}(kt) = 1$ for all $t > 0$, from which (using (1.5)) we obtain $u = f(u)$. This completes the proof. □

Corollary 2.5 [5, Theorem 2.1]. *Let (X,F,T) be an F-complete generalized Menger space under a continuous t-norm $T \in \mathcal{H}$, $k \in (0,1)$, and $\varphi \in \Phi$. If $f : X \to X$ is a mapping such that*

$$\varphi(F_{f(x)f(y)}(kt), F_{xy}(t), F_{xf(x)}(t), F_{yf(y)}(kt)) \geq 0, \quad \forall x, y \in X,\ \forall t > 0 \tag{2.15}$$

and there exists $x_0 \in X$ for which $F_{x_0 f(x_0)} \in D_+$, then f has a fixed point.

Proof. Choose a $\mu > 1$. Since $\lim_{n\to\infty} \mu^n = \infty$ and $F_{x_0 x_1} \in D_+$, it follows that $\lim_{n\to\infty} F_{x_0 f(x_0)}(\mu^n) = 1$. Therefore, by Proposition 1.3(iii),

$$\lim_{n\to\infty} T_{i=n}^{\infty} F_{x_0 f(x_0)}(\mu^i) = 1. \tag{2.16}$$

Now apply Theorem 2.4. □

Corollary 2.6. *Let (X,F,T_L) be an F-complete generalized Menger space and $\varphi \in \Phi$. If $f : X \to X$ is a mapping such that*

$$\varphi(F_{f(x)f(y)}(kt), F_{xy}(t), F_{xf(x)}(t), F_{yf(y)}(kt)) \geq 0, \quad \forall x, y \in X,\ \forall t > 0, \tag{2.17}$$

and $\sum_{n=1}^{\infty}(1 - F_{x_0 f(x_0)}(\gamma^n)) < \infty$ for some $x_0 \in X$ and $\gamma > 1$, then f has a fixed point.

For the proof see Proposition 1.3.

COROLLARY 2.7. *Let* (X,F,T) *be an* F*-complete generalized Menger space under* $T \in \{T_\lambda^D, T_\lambda^{AA}\}$, $k \in (0,1)$, *and* $\varphi \in \Phi$. *If* $f : X \to X$ *is a mapping such that*

$$\varphi(F_{f(x)f(y)}(kt), F_{xy}(t), F_{xf(x)}(t), F_{yf(y)}(kt)) \geq 0, \quad \forall x,y \in X,\ \forall t > 0 \tag{2.18}$$

and $\sum_{n=1}^{\infty}(1 - F_{x_0 f(x_0)}(\gamma^n))^\lambda < \infty$ *for some* $x_0 \in X$ *and* $\gamma > 1$, *then* f *has a fixed point.*

COROLLARY 2.8. *Let* (X,F,T) *be an* F*-complete generalized Menger space under a continuous t-norm* $T \in \mathcal{H}$ *and* $k \in (0,1)$. *If* $f : X \to X$ *is a mapping satisfying one of the following conditions:*

$$F_{f(x)f(y)}(kt) \geq F_{xy}(t), \quad \forall x,y \in X,\ \forall t > 0, \tag{2.19}$$

$$F^2_{f(x)f(y)}(kt) \geq F_{xy}(t)F_{xf(x)}(t), \quad \forall x,y \in X,\ \forall t > 0, \tag{2.20}$$

$$F_{f(x)f(y)}(kt) \geq 2F_{xy}(t) - F_{xf(x)}(t), \quad \forall x,y \in X,\ \forall t > 0 \tag{2.21}$$

and there exists $x_0 \in X$ *for which* $F_{x_0 f(x_0)} \in D_+$, *then* f *has a fixed point.*

As a final result for this section, we consider an example to see the generality of Theorem 2.4.

Example 2.9. Let X be a set containing at least two elements and the mapping F from $X \times X$ to Δ_+, defined by

$$F_{xy}(t) = \begin{cases} 0, & \text{if } t \leq 1 \\ \dfrac{1}{2}, & \text{if } t > 1 \end{cases} \quad \text{for } x,y \in X,\ x \neq y, \qquad F_{xx} = \varepsilon_0, \quad \forall x \in X. \tag{2.22}$$

It is easy to show (see [14]) that (X,F,T_M) is a complete Menger space.

We are going to prove that the mapping $f : X \to X$, $f(x) = x$ satisfies the contractivity condition (2.21) from the above corollary with $b = 2$, $c = -1$, however it is not a B-contraction (here we took advantage of working in Δ_+ rather than in D_+).

First, we show that

$$F_{xy}(kt) + 1 \geq 2F_{xy}(t), \quad \forall x,y \in X,\ \forall t > 0. \tag{2.23}$$

Indeed, the above inequality holds with equality if $x = y$, while if $x \neq y$ then the right-hand member is at most 1.

Next, for every $t \in (1, 1/k]$, $F_{xy}(kt) = 0$, while $F_{xy}(t) = 1/2$, which means that f is not a Sehgal contraction.

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