ON INJECTIVE L-MODULES

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The concepts of free modules, projective modules, injective modules, and the like form an important area in module theory. The notion of free fuzzy modules was introduced by Muganda as an extension of free modules in the fuzzy context. Zahedi and Ameri introduced the concept of projective and injective L-modules. In this paper, we give an alternate definition for injective L-modules and prove that a direct sum of L-modules is injective if and only if each L-module in the sum is injective. Also we prove that if J is an injective module and μ is an injective L-submodule of J, and if $0 \to \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \to 0$ is a short exact sequence of L-modules, then $\nu \simeq \mu \oplus \eta$.

1. Introduction

Though the notion of a fuzzy set was introduced by L. A. Zadeh in 1965, its application to algebraic concepts started only in 1971 when A. Rosenfeld introduced fuzzy subgroups of a group. Tremendous and rapid growth of fuzzy algebraic concepts resulted in a vast literature. The book of Mordeson and Malik [7] gives an account of all these up to 1998. The notion of L-modules as an extension of classical module theory is available in this book. However, there are many concepts in abstract algebra which are to be analyzed in the fuzzy context. The notion of free fuzzy modules was introduced by Muganda [8] as an extension of free modules in the fuzzy context. The concept of a free L-module is available in [7]. Zahedi and Ameri [9] introduced the concepts of fuzzy projective and injective modules.

In our earlier paper [5], we introduced an alternate definition for a projective L-module and proved some related results. In this paper, in Section 2, we give the essential preliminaries and in Section 3, we give an alternate definition for an injective L-module and prove some results using this definition. Throughout this paper, unless otherwise stated, $L(\vee, \wedge, 1, 0)$ represents a complete Brouwerian lattice with maximal element "1" and minimal element "0;" R a ring with unity "1" and M a left module over R. "$\vee$" denotes the supremum and "$\wedge$" the infimum in L. We call L a regular lattice if $a \wedge b > 0$ for all $a, b > 0$ in L. "$\subseteq$" denotes the inclusion and "$\subset$" the proper inclusion. The set of all L-subsets of M, that is, the set of all functions from M to L, is denoted by L^M.

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For $x \in M$, $a \in L$, the L-subset which takes the value a at x and 0 elsewhere is denoted by $a_{\{x\}}$. That is,

$$a_{\{x\}}(y) = \begin{cases} a & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases} \tag{1.1}$$

2. Preliminaries

In this section, we review some definitions and results which will be used later. For details, reference may be made to Mordeson and Malik [7], for preliminaries regarding lattices Birkhoff [1], and for theory of modules, Goodearl and Warfield [2] and Hungerford [3].

Definition 2.1 (see [7]). For $\mu, \nu \in L^M$, define $\mu + \nu$ and $-\mu$ as follows.
For $x \in M$,

$$(\mu + \nu)(x) = \vee\{\mu(y) \wedge \nu(z) : y, z \in M,\ y + z = x\}, \qquad (-\mu)(x) = \mu(-x). \tag{2.1}$$

Also for an arbitrary family $\mu_i \in L^M$, $i \in I$, of L-subsets of M, define

$$\sum_{i \in I} \mu_i(x) = \vee\left\{ \underset{i \in I}{\wedge} \mu_i(x_i) : x_i \in M,\ i \in I,\ \sum_{i \in I} x_i = x \right\}, \tag{2.2}$$

where in the expression $x = \sum_{i \in I} x_i$, at most finitely many x_i's are not equal to 0.

Definition 2.2 (see [7]). For $\mu \in L^M$, define the following:
(i) $\mu * = \{x \in M : \mu(x) > 0\}$, called the support of μ,
(ii) for $a \in L$, $\mu_a = \{x \in M : \mu(x) \geq a\}$, called the a-cut or a-level subset of μ, and $\mu_a^> = \{x \in M : \mu(x) > a\}$, called the strict a-cut or strict a-level subset of μ.

Definition 2.3 (see [7]). Let f be a mapping from X into Y, and let $\mu \in L^X$ and $\nu \in L^Y$. The L-subsets $f(\mu) \in L^Y$ and $f^{-1}(\nu) \in L^X$, defined by, for all $y \in Y$,

$$f(\mu)(y) = \begin{cases} \vee\{\mu(x) : x \in X,\ f(x) = y\} & \text{if } f^{-1}(y) \neq \phi, \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

and for all $x \in X$,

$$f^{-1}(\nu)(x) = \nu(f(x)), \tag{2.4}$$

are called, respectively, the image of μ under f and the preimage of ν under f.

Definition 2.4 (see [7]). Let $\mu \in L^M$. Then μ is said to be an L-submodule of M if
(i) $\mu(0) = 1$,
(ii) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in M$,
(iii) $\mu(rx) \geq \mu(x)$ for all $r \in R$, for all $x \in M$.

Saying μ is a left L-module means that μ is an L-submodule of some left module M over a ring R. The set of all L-submodules of M is denoted by $L(M)$.

Remark 2.5. We note from [7] that if $\mu, \eta \in L(M)$, then $\mu + \eta \in L(M)$. Also if $\mu_i \in L(M)$, $i \in I$, then $\sum_{i\in I} \mu_i \in L(M)$. From [6], we see that $\mu \in L(M)$ if and only if μ_a is an R-module for all $a \in L$.

Definition 2.6 (see [7]). Let M and N be R-modules and let $\mu \in L(M)$ and $\nu \in L(N)$. An isomorphism f of M onto N is called a weak isomorphism of μ into ν if $f(\mu) \subseteq \nu$. If f is a weak isomorphism of μ into ν, then say that μ is weakly isomorphic to ν and write $\mu \simeq \nu$.

An isomorphism f of M onto N is called an isomorphism of μ onto ν if $f(\mu) = \nu$. If f is an isomorphism of μ onto ν, then say that μ is isomorphic to ν and write $\mu \cong \nu$.

Definition 2.7 (see [4]). Let A_i, $i \in \mathbb{Z}$, be R-modules and let $\mu_i \in L(A_i)$. Suppose that $\cdots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \cdots$ is an exact sequence of R-modules. Then the sequence $\cdots \xrightarrow{f_{i-1}} \mu_{i-1} \xrightarrow{f_i} \mu_i \xrightarrow{f_{i+1}} \mu_{i+1} \xrightarrow{f_{i+2}} \cdots$ of L-modules is said to be exact if, for all $i \in \mathbb{Z}$, the set of integers,

(i) $f_{i+1}(\mu_i) \subseteq \mu_{i+1}$,

(ii) $f_i(\mu_{i-1})(x) > 0$ if $x \in \operatorname{Ker} f_{i+1}$ and $f_i(\mu_{i-1})(x) = 0$ if $x \notin \operatorname{Ker} f_{i+1}$.

Definition 2.8 (see [4]). Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of R-modules. Let $\mu \in L(A)$, $\eta \in L(B)$, and $\nu \in L(C)$. Then an exact sequence of L-modules of the form $0 \to \mu \xrightarrow{f} \eta \xrightarrow{g} \nu \to 0$ is called a short exact sequence of L-modules.

Definition 2.9 (see [4]). Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\xi} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array} \tag{2.5}$$

be two isomorphic short exact sequences of R-modules with the given isomorphisms ϕ, ψ, and ξ. Let $\mu \in L(A)$, $\nu \in L(B)$, $\eta \in L(C)$, $\mu' \in L(A')$, $\nu' \in L(B')$, and $\eta' \in L(C')$ be such that

$$0 \longrightarrow \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \longrightarrow 0, \tag{2.6}$$

$$0 \longrightarrow \mu' \xrightarrow{f'} \nu' \xrightarrow{g'} \eta' \longrightarrow 0 \tag{2.7}$$

are short exact sequences of L-modules. Then the sequence (2.6) is said to be weakly isomorphic to the sequence (2.7) if $\varphi(\mu) \subseteq \mu'$, $\psi(\nu) \subseteq \nu'$, and $\xi(\eta) \subseteq \eta'$.

The sequence (2.6) is said to be isomorphic to the sequence (2.7) if $\varphi(\mu) = \mu'$, $\psi(\nu) = \nu'$, and $\xi(\eta) = \eta'$.

Definition 2.10 (see [7]). Let $\mu,\eta,\nu \in L(M)$. Then μ is said to be the direct sum of η and ν if

(i) $\mu = \eta + \nu$,

(ii) $\eta \cap \nu = 1_{\{0\}}$.

In this case, write $\mu = \eta \oplus \nu$.

Definition 2.11 (see [4]). Let A and B be two R-modules, $\mu \in L(A)$, $\eta \in L(B)$. Consider the direct sum $A \oplus B$. Extend the definition of μ and η to $A \oplus B$ to get μ' and η' in $L(A \oplus B)$ as follows:

$$\mu'(x) = \begin{cases} \mu(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases} \qquad \text{i.e., } \mu'(a,b) = \begin{cases} \mu(a) & \text{if } b = 0, \\ 0 & \text{if } b \neq 0, \end{cases} \qquad \text{for } (a,b) \in A \oplus B,$$
$$\eta'(x) = \begin{cases} \eta(x) & \text{if } x \in B, \\ 0 & \text{if } x \notin B, \end{cases} \qquad \text{i.e., } \eta'(a,b) = \begin{cases} \eta(b) & \text{if } a = 0, \\ 0 & \text{if } a \neq 0, \end{cases} \qquad \text{for } (a,b) \in A \oplus B. \tag{2.8}$$

Then $\mu',\eta' \in L(A \oplus B)$. Moreover

$$(\mu' \cap \eta')(x) = \mu'(x) \wedge \eta'(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases} \tag{2.9}$$

Therefore $\mu' + \eta'$ is in fact a direct sum and is denoted by $\mu \oplus \eta$.

Remark 2.12. Note that

$$\begin{aligned} &(\mu \oplus \eta)(a,b) \\ &\quad = (\mu' + \eta')(a,b) \\ &\quad = \vee\{\mu'(a_1,b_1) \wedge \eta'(a_2,b_2) : (a_1,b_1),\ (a_2,b_2) \in A \oplus B;\ (a_1,b_1) + (a_2,b_2) = (a,b)\} \\ &\quad = \mu'(a,0) \wedge \eta'(0,b) = \mu(a) \wedge \eta(b). \end{aligned} \tag{2.10}$$

Definition 2.13 (see [7]). Let $\mu,\mu_i \in L(M)$, for all $i \in I$, then μ is said to be the direct sum of $\{\mu_i : i \in I\}$, denoted by $\bigoplus_{i\in I} \mu_i$, if

(i) $\mu = \sum_{i\in I} \mu_i$,

(ii) $\mu_j \cap \sum_{i\in I-\{j\}} \mu_i = 1_{\{0\}}$ for all $j \in I$.

3. Injective L-modules

The concept of free fuzzy modules was introduced by Muganda [8], which is later generalized to that of free L-modules (cf. [7]). Zahedi and Ameri [9] introduced the concepts of fuzzy projective and injective modules. In this section, we give an alternate definition for injective L-modules and prove that a direct sum of L-modules is injective if and only if each summand in the sum is injective. Also we prove that if $\mu \in L(J)$ is an injective L-module, and if $0 \to \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \to 0$ is a short exact sequence of L-modules, then $\nu \simeq \mu \oplus \eta$.

Definition 3.1. Let J be an injective R-module and let $\mu \in L(J)$. Then μ is said to be an injective L-module if for R-modules A, B, and $\eta \in L(A)$, $\nu \in L(B)$, g any monomorphism from A to B such that $g(\eta) = \nu$ on $g(A)$, and $f : A \to J$ any R-module homomorphism such that $f(\eta) = \mu$ on $f(A)$, there exists an R-module homomorphism $h : B \to J$ such that $hg = f$ and $h(\nu) \subseteq \mu$.

From the crisp module theory, it is known that an R-module J is injective if and only if every short exact sequence $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits so that $B \cong J \oplus C$. An analogous result exists in the case of L-modules.

To prove this we need the following theorem.

Theorem 3.2 (see [4]). *Let $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ be a short exact sequence of R-modules and let $\mu_1 \in L(A_1)$, $\mu_2 \in L(A_2)$, $\eta \in L(B)$ be such that $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ is a short exact sequence of L-modules. If there exists an R-module homomorphism $k : B \to A_1$ with $kf = I_{A_1}$, the identity map on A_1, such that $k(\eta) \subseteq \mu_1$, then the given short exact sequence $0 \to \mu_1 \xrightarrow{f} \eta \xrightarrow{g} \mu_2 \to 0$ is weakly isomorphic to the short exact sequence $0 \to \mu_1 \xrightarrow{i} \mu_1 \oplus \mu_2 \xrightarrow{\pi} \mu_2 \to 0$. In particular $\eta \simeq \mu_1 \oplus \mu_2$.*

Theorem 3.3. *Let J be an injective module and $\mu \in L(J)$ an injective L-module. If $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of R-modules and $\nu \in L(B)$ and $\eta \in L(C)$ are such that $0 \to \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \to 0$ is a short exact sequence of L-modules where $f(\mu) = \nu$ on $f(J)$, then ν is weakly isomorphic to $\mu \oplus \eta$. That is, $\nu \simeq \mu \oplus \eta$.*

Proof. Since J is injective, it is well known that any short exact sequence $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits and $B \cong J \oplus C$, and the sequence $0 \to J \xrightarrow{f} B \xrightarrow{g} C \to 0$ is isomorphic to $0 \to J \xrightarrow{i} J \oplus C \xrightarrow{\pi} C \to 0$. Now since $\mu \in L(J)$ is an injective L-module, and since $f(\mu) = \nu$ on $f(J)$, from the definition, we get $h(\nu) \subseteq \mu$. Thus there exists a homomorphism $h : B \to J$ such that $hf = I_J$ and $h(\nu) \subseteq \mu$. Then, by the above theorem, $0 \to \mu \xrightarrow{f} \nu \xrightarrow{g} \eta \to 0$ is weakly isomorphic to $0 \to \mu \xrightarrow{i} \mu \oplus \eta \xrightarrow{\pi} \eta \to 0$, in particular $\nu \simeq \mu \oplus \eta$. □

In the crisp theory, we have the theorem that a direct sum of modules is injective if and only if each summand is injective. The same is true in the fuzzy case.

Theorem 3.4. *Let Q_α, $\alpha \in I$ be injective R-modules and $\mu_\alpha \in L(Q_\alpha)$, $\alpha \in I$. Then $\bigoplus_{\alpha \in I} \mu_\alpha \in L(\bigoplus_{\alpha \in I} Q_\alpha)$ is injective if and only if μ_α is injective for all $\alpha \in I$.*

Proof. As we have already mentioned, the module $\bigoplus_{\alpha \in I} Q_\alpha$ is injective if and only if Q_α is injective for all $\alpha \in I$. Let $i_\alpha : Q_\alpha \to \bigoplus_{\alpha \in I} Q_\alpha$ and $\pi_\alpha : \bigoplus_{\alpha \in I} Q_\alpha \to Q_\alpha$ be, respectively, the canonical injection and projection. Obviously $\bigoplus_{\alpha \in I} \mu_\alpha \in L(\bigoplus_{\alpha \in I} Q_\alpha)$ and suppose $\bigoplus_{\alpha \in I} \mu_\alpha$ is an injective L-submodule of $\bigoplus_{\alpha \in I} Q_\alpha$. We prove that μ_α is injective for all $\alpha \in I$. Let A, B be R-modules, $\eta \in L(A)$, $\nu \in L(B)$, g any monomorphism from A to B such that $g(\eta) = \nu$ on $g(A)$. For $\alpha \in I$ if $f_\alpha : A \to Q_\alpha$ is any R-module homomorphism such that

$f_\alpha(\eta) = \mu_\alpha$ on $f_\alpha(A)$, then we have to show that there exists an R-module homomorphism $h_\alpha : B \to Q_\alpha$ such that $h_\alpha g = f_\alpha$ and $h_\alpha(\nu) \subseteq \mu_\alpha$,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[dr, "f_\alpha"'] & B \arrow[d, "h_\alpha"] \\ & & Q_\alpha \arrow[d, shift right, "i_\alpha"'] \\ & & \bigoplus Q_\alpha \arrow[u, shift right, "\pi_\alpha"'] \end{tikzcd} \tag{3.1}$$

Let $\bigoplus_{\alpha\in I} Q_\alpha$ be injective. Consider $i_\alpha f_\alpha : A \to \bigoplus_{\alpha\in I} Q_\alpha$. First of all, we show that $(i_\alpha f_\alpha)(\eta) = \bigoplus_{\alpha\in I}\mu_\alpha$ on $(i_\alpha f_\alpha)(A)$.

We have $(i_\alpha f_\alpha)(\eta) \in L(\bigoplus_{\alpha\in I} Q_\alpha)$, and if $x = \bigoplus x_\alpha \in (i_\alpha f_\alpha)(A) \subseteq \bigoplus_{\alpha\in I} Q_\alpha$ where $x_\alpha \in Q_\alpha$ $(\alpha \in I)$, then $x = (i_\alpha f_\alpha)(a)$ for some $a \in A$. That is, $x = i_\alpha(f_\alpha(a))$ where $f_\alpha(a) \in Q_\alpha$. Then

$$\begin{aligned} (i_\alpha f_\alpha)(\eta)(x) &= \vee\{\eta(a') : a' \in A;\ (i_\alpha f_\alpha)(a') = x\} \\ &= \vee\{\eta(a') : a' \in A;\ (i_\alpha f_\alpha)(a') = (i_\alpha f_\alpha)(a)\} \\ &= \vee\{\eta(a') : a' \in A;\ i_\alpha(f_\alpha(a')) = i_\alpha(f_\alpha(a))\} \\ &= \vee\{\eta(a') : a' \in A;\ f_\alpha(a') = f_\alpha(a)\}. \end{aligned} \tag{3.2}$$

Also

$$\begin{aligned} \bigoplus_{\alpha\in I}\mu_\alpha(x) &= \vee\left\{\wedge\,\mu_\alpha(x_\alpha) : x_\alpha \in Q_\alpha,\ \alpha \in I;\ x = \sum_{\alpha\in I} x_\alpha\right\} \\ &= \mu_\alpha(f_\alpha(a)) \quad (\text{since the supremum is attained for the direct sum} \\ &\qquad\qquad \text{decomposition } x = 0+0+\cdots+0+f_\alpha(a)+0+\cdots+0) \\ &= f_\alpha(\eta)(f_\alpha(a)) = \vee\{\eta(a') : a' \in A;\ f_\alpha(a') = f_\alpha(a)\}. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we get $(i_\alpha f_\alpha)(\eta)(x) = \bigoplus_{\alpha\in I}\mu_\alpha(x)$ for all $x \in (i_\alpha f_\alpha)(A)$.

Now since $\bigoplus_{\alpha\in I}\mu_\alpha$ is injective, we get that $i_\alpha f_\alpha : A \to \bigoplus_{\alpha\in I} Q_\alpha$ has an extension $k : B \to \bigoplus_{\alpha\in I} Q_\alpha$ satisfying $kg = i_\alpha f_\alpha$ and $k(\nu) \subseteq \bigoplus_{\alpha\in I}\mu_\alpha$. Take $h_\alpha = \pi_\alpha k$. Then $h_\alpha : B \to Q_\alpha$ is an extension of $f_\alpha : A \to Q_\alpha$ satisfying $h_\alpha g = f_\alpha$. It remains to prove that $h_\alpha(\nu) \subseteq \mu_\alpha$.

We have $k(\nu) \subseteq \bigoplus_{\alpha\in I}\mu_\alpha$. Therefore

$$\pi_\alpha(k(\nu)) \subseteq \pi_\alpha\left(\bigoplus_{\alpha\in I}\mu_\alpha\right). \tag{3.4}$$

Now for $x_\alpha \in Q_\alpha$ $(\alpha \in I)$,

$$
\begin{aligned}
&\left(\pi_\alpha\left(\bigoplus_{\alpha\in I}\mu_\alpha\right)\right)(x_\alpha) \\
&\qquad = \vee\left\{\bigoplus_{\alpha\in I}\mu_\alpha(y) : y \in \bigoplus_{\alpha\in I} Q_\alpha;\ \pi_\alpha(y) = x_\alpha\right\} \\
&\qquad = \mu_\alpha(x_\alpha) \quad \text{(since the supremum is attained for } y = (0,\dots,0,x_\alpha,0,\dots,0)).
\end{aligned}
\tag{3.5}
$$

Thus $\pi_\alpha(\bigoplus_{\alpha\in I}\mu_\alpha) = \mu_\alpha$, and so from (3.4), we get $(\pi_\alpha k)(\nu) \subseteq \mu_\alpha$. That is, $h_\alpha(\nu) \subseteq \mu_\alpha$ as required.

Conversely, suppose that μ_α is injective for all $\alpha \in I$. We prove that $\bigoplus_{\alpha\in I}\mu_\alpha$ is injective,

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "\pi_\alpha f"] \arrow[dd, bend right, "f"'] & B \arrow[dl, "k_\alpha"] \arrow[ddl, "k"] \\
& Q_\alpha \arrow[d, shift right, "i_\alpha"'] & \\
& \bigoplus Q_\alpha \arrow[u, shift right, "\pi_\alpha"'] &
\end{tikzcd}
\tag{3.6}
$$

Since Q_α is injective for all $\alpha \in I$, we have that $\bigoplus_{\alpha\in I} Q_\alpha$ is injective. Let A, B be R-modules, $\eta \in L(A)$, $\nu \in L(B)$, g any monomorphism from A to B such that $g(\eta) = \nu$ on $g(A)$, and suppose that $f : A \to \bigoplus_{\alpha\in I} Q_\alpha$ is a module homomorphism satisfying $f(\eta) = \bigoplus_{\alpha\in I}\mu_\alpha$ on $f(A)$. Since $f(\eta) = \bigoplus_{\alpha\in I}\mu_\alpha$ on $f(A)$, we get that $(\pi_\alpha f)(\eta) = \pi_\alpha(\bigoplus_{\alpha\in I}\mu_\alpha) = \mu_\alpha$ on $(\pi_\alpha f)(A)$. This together with the fact that each μ_α is injective imply that each $\pi_\alpha f : A \to Q_\alpha$ admits an extension $k_\alpha : B \to Q_\alpha$ such that $\pi_\alpha f = k_\alpha g$. These homomorphisms k_α give $k : B \to \bigoplus_{\alpha\in I} Q_\alpha$ such that $\pi_\alpha k = k_\alpha$ and for each $x \in A$, $(\pi_\alpha k)(g(x)) = k_\alpha(g(x)) = (\pi_\alpha f)(x)$ for all $\alpha \in I$. Therefore $k(g(x)) = f(x)$ for all $x \in A$. Therefore k is an extension of f such that $kg = f$ and also $k_\alpha(\nu) \subseteq \mu_\alpha$. Now

$$
k_\alpha(\nu) \subseteq \mu_\alpha \Longrightarrow (\pi_\alpha k)(\nu) \subseteq \mu_\alpha \Longrightarrow \pi_\alpha(k(\nu)) \subseteq \mu_\alpha = \pi_\alpha\left(\bigoplus_{\alpha\in I}\mu_\alpha\right). \tag{3.7}
$$

Since this is true for every $\alpha \in I$, it follows that $k(\nu) \subseteq \bigoplus_{\alpha\in I}\mu_\alpha$. This completes the proof. $\square$

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