ON THE L^p-CONVERGENCE FOR MULTIDIMENSIONAL ARRAYS OF RANDOM VARIABLES

LE VAN THANH

Received 14 August 2004 and in revised form 4 March 2005

For a d-dimensional array of random variables $\{X_n,\ n \in \mathbb{Z}_+^d\}$ such that $\{|X_n|^p,\ n \in \mathbb{Z}_+^d\}$ is uniformly integrable for some $0 < p < 2$, the L^p-convergence is established for the sums $(1/|n|^{1/p})(\sum_{j \prec n}(X_j - a_j))$, where $a_j = 0$ if $0 < p < 1$, and $a_j = EX_j$ if $1 \le p < 2$.

1. Introduction

Let $\mathbb{Z}_+^d$, where d is an integer, denote the positive integer d-dimensional lattice points. The notation $m \prec n$, where $m = (m_1, m_2, \ldots, m_d)$ and $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}_+^d$, means that $m_i \le n_i$, $1 \le i \le d$, $|n|$ is used for $\prod_{i=1}^d n_i$.

Gut [2] proved that if $\{X, X_n,\ n \in \mathbb{Z}_+^d\}$ is a d-dimensional array of i.i.d. random variables with $E|X|^p < \infty$ $(0 < p < 2)$ and $EX = 0$ if $1 \le p < 2$, then

$$\frac{\sum_{j \prec n} X_j}{|n|^{1/p}} \longrightarrow 0 \text{ in } L^p \quad \text{as } \min_{1 \le i \le d} n_i \longrightarrow \infty, \tag{1.1}$$

where $(n_1, n_2, \ldots, n_d) = n \in \mathbb{Z}_+^d$.

In 1999, Hong and Hwang [3] proved that if $\{X_{mn},\ m \ge 1,\ n \ge 1\}$ is a double array of pairwise independent random variables such that

$$P\{|X_{mn}| > t\} \le P\{|X| > t\}, \quad t \ge 0,\ m \ge 1,\ n \ge 1, \tag{1.2}$$

where X is a random variable, then the condition $E(|X|^p \log^+ |X|) < \infty$ $(1 < p < 2)$ implies that

$$\frac{\sum_{k=1}^m \sum_{l=1}^n (X_{kl} - EX_{kl})}{(mn)^{1/p}} \longrightarrow 0 \text{ in } L^1 \quad \text{as } \max\{m, n\} \longrightarrow \infty. \tag{1.3}$$

In this note, we provide conditions for $(1/|n|^{1/p})(\sum_{j \prec n}(X_j - a_j)) \to 0$ in L^p as $|n| \to \infty$, where $n \in \mathbb{Z}_+^d$, $j \in \mathbb{Z}_+^d$, $a_j = 0$ if $0 < p < 1$, and $a_j = EX_j$ if $1 \le p < 2$.

International Journal of Mathematics and Mathematical Sciences 2005:8 (2005) 1317–1320
DOI: 10.1155/IJMMS.2005.1317

2. Result

Theorem 2.1. *Let* $\{X_n, n \in \mathbb{Z}_+^d\}$ *be a d-dimensional array of random variables such that* $\{|X_n|^p, n \in \mathbb{Z}_+^d\}$ *is uniformly integrable for some* $0 < p < 2$. *Assume that* $\{X_n, n \in \mathbb{Z}_+^d\}$ *is pairwise independent if* $p = 1$ *and* $\{X_n, n \in \mathbb{Z}_+^d\}$ *is independent if* $1 < p < 2$. *Then,*

$$\frac{\sum_{j \prec n}(X_j - a_j)}{|n|^{1/p}} \longrightarrow 0 \quad \text{in } L^p \text{ as } |n| \longrightarrow \infty, \tag{2.1}$$

where $a_j = 0$ *if* $0 < p < 1$, *and* $a_j = EX_j$ *if* $1 \le p < 2$.

Proof. For arbitrary $\epsilon > 0$, there exists $M > 0$ such that

$$E(|X_n|^p I(|X_n| > M)) < \epsilon \quad \forall n \in \mathbb{Z}_+^d. \tag{2.2}$$

Set

$$\begin{aligned} X_n' &= X_n I(|X_n| \le M), \quad n \in \mathbb{Z}_+^d, \\ X_n'' &= X_n I(|X_n| > M), \quad n \in \mathbb{Z}_+^d. \end{aligned} \tag{2.3}$$

For all $n \in \mathbb{Z}_+^d$,

$$E|X_n'' - EX_n''|^p \le 4E|X_n''|^p < 4\epsilon. \tag{2.4}$$

If $0 < p < 1$, then

$$\begin{aligned} E\left|\sum_{j \prec n} X_j\right|^p &\le E\left|\sum_{j \prec n} X_j'\right|^p + E\left|\sum_{j \prec n} X_j''\right|^p \le E\left|\sum_{j \prec n} X_j'\right|^p + \sum_{j \prec n} E|X_j''|^p \\ &\le (|n|M)^p + |n|\epsilon \quad (\text{by } (2.2)). \end{aligned} \tag{2.5}$$

The conclusion (2.1) follows from (2.5).

If $p = 1$ and $\{X_n, n \in \mathbb{Z}_+^d\}$ is pairwise independent, then

$$\begin{aligned} E\left|\sum_{j \prec n}(X_j - EX_j)\right| &\le E\left|\sum_{j \prec n}(X_j' - EX_j')\right| + \sum_{j \prec n} E|X_j'' - EX_j''| \\ &\le \left[E\left|\sum_{j \prec n}(X_j' - EX_j')\right|^2\right]^{1/2} + \sum_{j \prec n} E|X_j'' - EX_j''| \\ &\qquad (\text{by the Jensen inequality (see [1, page 103])}) \\ &\le \left[\sum_{j \prec n} E(X_j' - EX_j')^2\right]^{1/2} + 4|n|\epsilon \quad (\text{by } (2.4)) \\ &\le (|n|M^2)^{1/2} + 4|n|\epsilon \\ &\qquad \left(\text{since } E(X_j' - EX_j')^2 = E(X_j')^2 - (EX_j')^2 \le M^2,\ j \in \mathbb{Z}_+^d\right) \\ &= o(|n|) \quad \text{as } |n| \longrightarrow \infty. \end{aligned} \tag{2.6}$$

If $1 < p < 2$ and $\{X_n,\ n \in \mathbb{Z}_+^d\}$ is independent, then

$$\begin{aligned}
E\left|\sum_{j\prec n}(X_j - EX_j)\right|^p &\le 2^{p-1}\left[E\left|\sum_{j\prec n}(X_j' - EX_j')\right|^p + E\left|\sum_{j\prec n}(X_j'' - EX_j'')\right|^p\right] \\
&\le 2^{p-1}\left[\left(E\left|\sum_{j\prec n}(X_j' - EX_j')\right|^2\right)^{p/2} + 2\sum_{j\prec n}E\left|X_j'' - EX_j''\right|^p\right] \\
&\quad \text{(by the Jensen inequality [1] and the von Bahr-Esseen} \\
&\quad \text{inequality [4])} \\
&\le 2^{p-1}\left(\sum_{j\prec n}E(X_j' - EX_j')^2\right)^{p/2} + 2^{p+2}|n|\epsilon \quad \text{(by (2.4))} \\
&\le 2^{p-1}\left(|n|M^2\right)^{p/2} + 2^{p+2}|n|\epsilon \\
&\quad \left(\text{since } E(X_j' - EX_j')^2 = E(X_j')^2 - (EX_j')^2 \le M^2,\ j \in \mathbb{Z}_+^d\right) \\
&= o(|n|) \quad \text{as } |n| \longrightarrow \infty,
\end{aligned} \tag{2.7}$$

again establishing (2.1). □

Note that if $\{X, X_n,\ n \in \mathbb{Z}_+^d\}$ are random variables such that $E|X|^p < \infty$ $(p > 0)$ and $\sup_{n\in\mathbb{Z}_+^d} P\{|X_n| > t\} \le P\{|X| > t\}$ for all $t \ge 0$, then $\{|X_n|^p,\ n \in \mathbb{Z}_+^d\}$ is uniformly integrable. The following corollary follows immediately from Theorem 2.1.

Corollary 2.2. *Let $\{X, X_n,\ n \in \mathbb{Z}_+^d\}$ be random variables such that $E|X|^p < \infty$ for some $0 < p < 2$, and $\sup_{n\in\mathbb{Z}_+^d} P\{|X_n| > t\} \le P\{|X| > t\}$ for all $t \ge 0$. Assume that $\{X_n,\ n \in \mathbb{Z}_+^d\}$ is pairwise independent if $p = 1$ and $\{X_n,\ n \in \mathbb{Z}_+^d\}$ is independent if $1 < p < 2$. Then,*

$$\frac{\sum_{j\prec n}(X_j - a_j)}{|n|^{1/p}} \longrightarrow 0 \text{ in } L^p \quad \text{as } |n| \longrightarrow \infty, \tag{2.8}$$

where $a_j = 0$ if $0 < p < 1$, and $a_j = EX_j$ if $1 \le p < 2$.

Acknowledgment

The author is grateful to Professor Andrew Rosalsky (University of Florida, USA) for the reference [2].

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Le Van Thanh: Department of Mathematics, Vinh University, Nghe An 42118, Vietnam
E-mail address: lvthanhvinh@yahoo.com