GROUPS WITH THE SAME ORDERS OF SYLOW NORMALIZERS AS THE MATHIEU GROUPS

BEHROOZ KHOSRAVI AND BEHNAM KHOSRAVI

Received 17 October 2004

There exist many characterizations for the sporadic simple groups. In this paper we give two new characterizations for the Mathieu sporadic groups. Let M be a Mathieu group and let p be the greatest prime divisor of $|M|$. In this paper, we prove that M is uniquely determined by $|M|$ and $|N_M(P)|$, where $P \in \mathrm{Syl}_p(M)$. Also we prove that if G is a finite group, then $G \cong M$ if and only if for every prime q, $|N_M(Q)| = |N_G(Q')|$, where $Q \in \mathrm{Syl}_q(M)$ and $Q' \in \mathrm{Syl}_q(G)$.

1. Introduction

Let G be a finite group. We denote by $\pi(G)$ the set of all prime divisors of $|G|$. It was proved that if G is an alternating group, a finite projective special linear group, a Janko sporadic simple group, or a finite projective special symplectic group, then G is characterizable by the orders of normalizers of its Sylow subgroups [1, 2, 3, 4, 10].

Mazurov and Shi [11, 12, 13, 14] and Deng [7] proved that some of the almost sporadic simple groups are characterizable by the set of element orders. Chen [5] and A. Khosravi [9] proved that some of the almost sporadic simple groups are characterizable by the set of order components.

Notation 1. For a prime number q, we define

$$n_q(G) = |N_G(Q)| \quad \text{where } Q \in \mathrm{Syl}_q(G). \tag{1.1}$$

In this paper, as the main result, the following theorems are proved.

Theorem 1.1. *Let M be a Mathieu group and let G be a finite group. If for every prime number q, $n_q(M) = n_q(G)$, then $G \cong M$.*

Theorem 1.2. *Let M be a Mathieu group and let G be a finite group. Let p be the greatest prime divisor of $|M|$. If $|G| = |M|$ and $n_p(M) = n_p(G)$, then $G \cong M$.*

In this paper, all groups are finite. All further unexplained notations are standard and refer to [6], for example. Let m be a positive integer and let q be a prime number. Then m_q denotes the q-part of m. In other words, $m_q = q^k$ if $q^k \| m$ (i.e., $q^k \mid m$ but $q^{k+1} \nmid m$).

International Journal of Mathematics and Mathematical Sciences 2005:9 (2005) 1449–1453
DOI: 10.1155/IJMMS.2005.1449

We would like to note that $|\mathbb{Z}_6| = |S_3| = 6$, $\mathbb{Z}_3 \lhd \mathbb{Z}_6$, and $\mathbb{Z}_3 \lhd S_3$. Therefore $|N_{\mathbb{Z}_6}(P)| = |N_{S_3}(P)| = 6$, where $P \cong \mathbb{Z}_3$ is the 3-Sylow subgroup of $\mathbb{Z}_6$ and S_3. But $\mathbb{Z}_6 \ncong S_3$ and hence they are not characterizable by these conditions.

2. Preliminary results

The following lemma is an immediate consequence of [15, Theorem 2.1.17].

Lemma 2.1. *Let G be a q-group and let $|G| = q^k$, for some $k > 0$. Then p divides $\prod_{i=1}^{k}(q^i - 1)$, for any prime divisor p of $|\mathrm{Aut}(G)|$ with $p \neq q$.*

Lemma 2.2 [8]. *The following results hold:*

(i) $|M_{11}| = 7920$, $n_2(M_{11}) = 16$, $n_3(M_{11}) = 144$, $n_5(M_{11}) = 20$, *and* $n_{11}(M_{11}) = 55$;

(ii) $|M_{12}| = 95\,040$, $n_2(M_{12}) = 64$, $n_3(M_{12}) = 108$, $n_5(M_{12}) = 40$, *and* $n_{11}(M_{12}) = 55$;

(iii) $|M_{22}| = 443\,520$, $n_2(M_{22}) = 128$, $n_3(M_{22}) = 72$, $n_5(M_{22}) = 20$, $n_7(M_{22}) = 21$, *and* $n_{11}(M_{22}) = 55$;

(iv) $|M_{23}| = 10\,200\,960$, $n_2(M_{23}) = 128$, $n_3(M_{23}) = 144$, $n_5(M_{23}) = 60$, $n_7(M_{23}) = 42$, $n_{11}(M_{23}) = 55$, *and* $n_{23}(M_{23}) = 253$;

(v) $|M_{24}| = 244\,823\,040$, $n_2(M_{24}) = 1024$, $n_3(M_{24}) = 216$, $n_5(M_{24}) = 240$, $n_7(M_{24}) = 126$, $n_{11}(M_{24}) = 110$, *and* $n_{23}(M_{24}) = 253$.

Remark 2.3. Note that we can calculate the crucial normalizer orders $n_{11}(M_{11})$, $n_{11}(M_{12})$, $n_{11}(M_{22})$, $n_{23}(M_{23})$, and $n_{23}(M_{24})$ without using GAP. For example, from the character table in atlas [6], we can see that there are two conjugacy classes of elements of order 11 in M_{11}, each of order 720, for a total of 1440 elements of order 11. Since a Sylow 11-subgroup of M_{11} has order 11, that means there are 144 Sylow 11-subgroups. Therefore, $n_{11}(M_{11}) = |M_{11}|/144 = 55$. Similar calculations work in the other cases.

Lemma 2.4. *Let M be a Mathieu group and let G be a finite group. If for every prime number q, $n_q(M) = n_q(G)$, then $|G| = |M|$.*

Proof. Obviously if Q is a q-Sylow subgroup of G, then $Q \leq N_G(Q)$ and hence $|Q|$ divides $|N_G(Q)|$. Also $|G|_q = |Q|$. Since $n_q(G) = n_q(M)$, we conclude that $(n_q(G))_q = (n_q(M))_q$. Therefore for every prime q, we have $|G|_q = |M|_q$. Hence $|G| = |M|$. □

Remark 2.5. Lemma 2.4 shows that if Theorem 1.2 is proved, then Theorem 1.1 is an immediate consequence of Theorem 1.2. Therefore in the next section we only prove Theorem 1.2.

3. Characterizations of the Mathieu groups

We note that in the proof of Theorem 1.2, we use the classification of finite simple groups. We prove this theorem by using the following lemmas.

Lemma 3.1. *Let G be a finite group and let M be M_{11}, M_{12}, or M_{22}. If $|G| = |M|$ and $n_{11}(M) = n_{11}(G)$, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple group, 11 is a divisor of $|K/H|$, and $|G/K| \mid 5$.*

Proof. Let $1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_k = G$ be a chief series of G. Since $11 \| \, |M|$, there exists some i such that 11 is a divisor of $|G_{i+1}/G_i|$. So let $G_i = H$ and $G_{i+1} = K$. Since K/H

is a chief factor of G, it follows that K/H is a minimal normal subgroup of G/H. Also K/H is characteristically simple which implies that K/H be a simple group or a product of isomorphic simple subgroups. We know that $11 \mid |K/H|$, but $11^2 \nmid |G/H|$ and hence K/H is a simple group.

Therefore $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ is a normal series of G, such that K/H is a simple group and 11 is a divisor of $|K/H|$. Let P be a 11-Sylow subgroup of G. Therefore $P \leq K$, which implies that $G = KN_G(P)$, by the Frattini argument. Then

$$\frac{G}{K} \cong \frac{KN_G(P)}{K} \cong \frac{N_G(P)}{N_G(P) \cap K} \cong \frac{N_G(P)}{N_K(P)}. \tag{3.1}$$

Therefore $|G/K|$ is a divisor of $|N_G(P)| = 55$. Since 11 divides $|K/H|$, we conclude that $|G/K|$ divides 5 and the proof is complete. □

Lemma 3.2. *Let G be a finite group and let M be M_{11}, M_{12}, or M_{22}. If $|G| = |M|$ and $n_{11}(M) = n_{11}(G)$, then $G \cong M$.*

Proof. By Lemma 3.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple group, 11 is a divisor of $|K/H|$, and $|G/K|$ divides 5. We claim that $|H|$ is a divisor of 5. Since 11 divides $|K/H|$, we have $11 \nmid |H|$. If $q \in \pi(H)$, then let Q be a q-Sylow subgroup of H. Now $H \lhd K$ and $Q \in \mathrm{Syl}_q(H)$. Hence by using the Frattini argument, we conclude that $K = HN_K(Q)$. We know that $C_K(Q) \trianglelefteq N_K(Q)$ and $N_H(Q) \trianglelefteq N_K(Q)$. Therefore $N_H(Q) \trianglelefteq C_K(Q)N_H(Q) \trianglelefteq N_K(Q)$. Also

$$\frac{K}{H} \cong \frac{HN_K(Q)}{H} \cong \frac{N_K(Q)}{H \cap N_K(Q)} \cong \frac{N_K(Q)}{N_H(Q)}, \tag{3.2}$$

and K/H is a simple group. So $C_K(Q)N_H(Q) = N_H(Q)$ or $C_K(Q)N_H(Q) = N_K(Q)$. Now we consider these cases separately.

Case 1. If $C_K(Q)N_H(Q) = N_H(Q)$ and $|Q| = q^k$, then $C_K(Q)$ is a subgroup of $N_H(Q)$. Therefore $|C_K(Q)|$ divides $|N_H(Q)|$, and hence $|N_K(Q)/N_H(Q)|$ is a divisor of $|N_K(Q)/C_K(Q)|$. We know that $N_K(Q)/C_K(Q)$ is isomorphic to a subgroup of $\mathrm{Aut}(Q)$. Also $11 \mid |K/H|$, $11 \nmid |H|$, $q \in \pi(H)$, and $|Q| = q^k$ (where $Q \in \mathrm{Syl}_q(H)$) which implies that $11 \mid \prod_{i=1}^{k}(q^i - 1)$, by Lemma 2.1.

But easy calculations show that $11 \nmid \prod_{i=1}^{7}(2^i - 1)$, $11 \nmid \prod_{i=1}^{3}(3^i - 1)$, $11 \nmid (5 - 1)$, and $11 \nmid (7 - 1)$. Therefore this case is impossible.

Case 2. If $C_K(Q)N_H(Q) = N_K(Q)$, then

$$\frac{K}{H} \cong \frac{N_K(Q)}{N_H(Q)} \cong \frac{C_K(Q)N_H(Q)}{N_H(Q)} \cong \frac{C_K(Q)}{C_K(Q) \bigcap N_H(Q)}. \tag{3.3}$$

Since 11 divides $|K/H|$, we conclude that 11 divides $|C_K(Q)|$, and hence $P \leq C_K(Q)$, where $P \in \mathrm{Syl}_{11}(G)$. Therefore $Q \leq C_K(P) \leq N_G(P)$. Hence $|Q|$ is a divisor of $|N_G(P)| = 55$. Since $11 \nmid |Q|$, we conclude that $|Q|$ is a divisor of 5, and hence $|H| \mid 5$.

Therefore $|G/K| \cdot |H|$ is a divisor of 5. If $|G/K| \cdot |H| = 5$, then K/H is a simple group of order $|M|/5$, which is a contradiction by the classification of finite simple groups (see [6, pages 239–241]). (There exists no simple group of order 1584, 19008, or 88704.) So $|G/K| = 1$, $|H| = 1$, which implies that G be a simple group of order M. Since there exists only one simple group of order $|M|$, it follows that $G \cong M$. □

Lemma 3.3. *Let G be a finite group and let M be M_{23} or M_{24}. If $|G| = |M|$ and $n_{23}(M) = n_{23}(G)$, then $G \cong M$.*

Proof. Similar to Lemma 3.1 we conclude that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple group, 23 is a divisor of $|K/H|$, and $|G/K|$ is a divisor of 11. Since $23 \mid |K/H|$ and $23^2 \nmid |G|$, it follows that $23 \nmid |H|$. Also $23 \nmid \prod_{i=1}^{10}(2^i - 1)$, $23 \nmid \prod_{i=1}^{3}(3^i - 1)$, $23 \nmid (5-1)$, $23 \nmid (7-1)$, and $23 \nmid (11-1)$. Therefore if Q is a q-Sylow subgroup of H, then $|Q|$ divides $n_{23}(G) = 253$. Hence $|H|$ divides 11. Therefore $|G/K| \cdot |H|$ is a divisor of 11. If $|G/K| \cdot |H| = 11$, then K/H is a simple group of order $|M|/11$, which is a contradiction by the classification of finite simple groups (see [6, pages 239–241]). (There exists no simple group of order 927360 or 22256640.) So $|G/K| = 1$, $|H| = 1$, which implies that G be a simple group of order M. Since there exists only one simple group of order $|M|$, it follows that $G \cong M$. □

Acknowledgments

The authors express their gratitude to Professor Bell and the referee for useful suggestions which improved the manuscript. The first author would like to thank the Institute for Studies in Theoretical Physics and Mathematics (IPM) for the financial support. Also we want to thank Professor Thomas Breuer for his help in computing the orders of normalizers of Sylow subgroups of the Mathieu sporadic simple groups. We dedicate this paper to our parents: Professor Amir Khosravi and Mrs. Soraya Khosravi for their unending love. The first author was supported in part by a Grant from IPM (no. 82200031). We also have to thank the authors of the atlas [6] for the rich mine of information they have provided.

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Behrooz Khosravi: Department of Pure Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), 424 Hafez Avenue, Tehran 15914, Iran; Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O. Box 19395-5746, Tehran, Iran

E-mail address: bkhosravi@aut.ac.ir

Behnam Khosravi: Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, Evin, Tehran 19838, Iran

E-mail address: khosravi@saba.tmu.ac.ir