GENERALIZATIONS OF HOPFIAN AND CO-HOPFIAN MODULES

YONGDUO WANG

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Let R be a ring and M a left R-module. M which satisfies DCC on essential submodules is GCH, and M which satisfies ACC on small submodules is WH. If $M[X]$ is GCH $R[X]$-module, then M is GCH R-module. Examples show that a GCH module need not be co-Hopfian and a WH module need not be Hopfian.

1. Introduction and preliminaries

In this paper, all rings are associative with identity and all modules are unital left modules unless otherwise specified.

Let R be a ring and M a module. $N \leq M$ will mean N is a submodule of M. A submodule E of M is called *essential* in M (notation $E \leq_e M$) if $E \cap A \neq 0$ for any nonzero submodule A of M. Dually, a submodule S of M is called *small* in M (notation $S \ll M$) if $M \neq S + T$ for any proper submodule T of M. M is said to be Hopfian (co-Hopfian) in case any surjective (injective) homomorphism is automatically an isomorphism. M is called generalized Hopfian (GH) if any of its surjective endomorphisms has a small kernel. M is called weakly co-Hopfian if any injective endomorphism of M is essential. In this paper, we introduce the concepts of GCH modules and WH modules. It is shown that (1) a module M which satisfies DCC on essential submodules is GCH and a module M which satisfies ACC on small submodules is WH; (2) if $M[X]$ is GCH $R[X]$-module, then M is GCH R-module. Examples show that a GCH module need not be co-Hopfian and a WH module need not be Hopfian. The notions which are not explained here will be found in [4].

Lemma 1.1 (see [6, 17.3(2)]). *Let K, L, and M be modules. Then two monomorphisms $f : K \to L$, $g : L \to M$ are essential if and only if gf is essential.*

Lemma 1.2 (see [6, 19.3(1)]). *Let K, L, and M be modules. Then two epimorphisms $f : K \to L$, $g : L \to M$ are small if and only if gf is small.*

Lemma 1.3 (see [2, Proposition 5.20]). *Suppose that $K_1 \leq M_1 \leq M$, $K_2 \leq M_2 \leq M$, and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \leq_e M_1 \oplus M_2$ if and only if $K_1 \leq_e M_1$ and $K_2 \leq_e M_2$.*

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2. Modules whose essential injective endomorphisms are isomorphic

Let M be a module. M is said to be a generalized co-Hopfian module (GCH module) if any essential injective endomorphism of M is isomorphic.

PROPOSITION 2.1. *Suppose that M satisfies the condition that N is GCH for every proper essential submodule. Then M itself is GCH.*

Proof. Suppose on the contrary that M is not GCH, then there exists an essential injective homorphism $g : M \to M$ which is not an isomorphism. Let $N = \mathrm{Im}g$. Then $N \neq M$ and g induces an isomorphism $\bar{g} : M \to N$. Then $\bar{g}|_N : N \to N$ is an essential injective morphism which is not an isomorphism. This is a contradiction since N is a GCH module. □

PROPOSITION 2.2. *Let M be a GCH module and K be a direct summand of M. Then K is GCH.*

Proof. Let $M = K \oplus K'$ and let $f : K \to K$ be an essential injection of K. f induces an injective endomorphism of M, that is, $f \oplus 1_{K'} : M \to M$ with $(f \oplus 1_{K'})(k+k') = f(k) + k'$. Since $\mathrm{Im}(f \oplus 1_{K'}) = \mathrm{Im}f \oplus K' \leq_e K \oplus K' = M$ by Lemma 1.3, $f \oplus 1_{K'}$ is essential. Since M is GCH, $f \oplus 1_{K'}$ is isomorphic, and hence f is an isomorphism, as required. □

PROPOSITION 2.3. *Let $M = M_1 \oplus M_2$ and let M_1, M_2 be invariant submodules under any injection of M. Then M is GCH if and only if M_1, M_2 are GCH.*

Proof. "If" part is clear by Proposition 2.2.

"Only if" part. Let $f : M \to M$ be an essential injection. Then $f|_{M_i} : M_i \to M_i$ is essential injection. By assumption, $f|_{M_i}$ is isomorphic. Thus, $f(M) = f(M_1 + M_2) = f(M_1) + f(M_2) = M_1 + M_2 = M$, and hence f is surjective, as desired. □

It is easy to know that a module M is co-Hopfian if and only if M is both a weakly co-Hopfian module and a GCH module. In [2], Haghany and Vedadi proved that if DCC holds on nonessential submodules of M, then M is weakly co-Hopfian. We also know that Artinian modules are co-Hopfian modules. Thus it is natural that we consider a module with DCC on essential submodules.

THEOREM 2.4. *Let M be a module with DCC on essential submodules. Then M is GCH.*

Proof. Let $f : M \to M$ be an essential injection of M. Then, $\cdots \leq \mathrm{Im}f^2 \leq \mathrm{Im}f$ is a descending chain on essential submodules of M by Lemma 1.1. Since M satisfies DCC on essential submodules, there exists n such that $\mathrm{Im}f^n = \mathrm{Im}f^{n+1}$. For any $m \in M$, since $f^n(m) \in \mathrm{Im}f^{n+1}$, there exists $m_1 \in M$ such that $f^n(m) = f^{n+1}(m_1)$. Since f is injective, $m = f(m_1)$. Thus f is surjective, as required. □

LEMMA 2.5. *A quasicontinuous module M is continuous if and only if it is a GCH module.*

A continuous module is a quasicontinuous module, but a quasicontinuous module need not be a continuous module. Thus the following result gives a sufficient condition such that a quasicontinuous module is a continuous module.

COROLLARY 2.6. *Let M be a module with DCC on essential submodules. Then M is quasicontinuous if and only if M is continuous.*

Proof. It follows from Theorem 2.4 and Lemma 2.5. □

Proposition 2.7. *Let P be a property of modules preserved under isomorphism. If a module M has the property P and satisfies DCC on essential submodules with property P, then M is GCH.*

Proof. Suppose that M is not GCH. Then there exists a proper essential submodule N_1 of M with $N_1 \simeq M$. Thus N_1 is not GCH and enjoys P. We have a proper essential submodule N_2 of N_1 with $N_2 \simeq N_1$. Clearly, N_2 is not GCH and satisfies P. Repeating, we obtain a strictly descending chain $N_1 > N_2 > \cdots$ of proper essential submodules each with property P, a contradiction. □

Corollary 2.8. *If M has DCC on its non-GCH submodules, then M is GCH.*

Proof. Suppose not, and let P be the property of being non-GCH. Applying Proposition 2.7, we arrive at a contradiction. Thus M must be GCH. □

Example 2.9. A semisimple module M is weakly co-Hopfian if and only if any homogeneous component of M is finitely generated (see [2, Corollary 1.12]). Thus a semisimple module need not be a weakly co-Hopfian module, and hence it is not a co-Hopfian module. However, any semisimple module is GCH.

Let M be a module. The elements of $M[X]$ are formal sums of the form $a_0 + a_1X + \cdots + a_kX^k$ with k an integer greater than or equal to 0 and $a_i \in M$. We denote this sum by $\Sigma_{i=1}^{k} a_iX^i$ (a_0X^0 is to be understood as the element $a_0 \in M$). Addition is defined by adding the corresponding coefficients. The $R[X]$-module structure is given by

$$(\Sigma_{i=0}^{k}\lambda_iX^i)\cdot(\Sigma_{j=0}^{z}a_jX^j) = \Sigma_{\mu=0}^{k+z}c_\mu X^\mu, \tag{2.1}$$

where $c_\mu = \Sigma_{i+j=\mu}\lambda_i a_j$, for any $\lambda_i \in R$, $a_j \in M$.

Lemma 2.10 (see [5, Lemma 1.7]). *Let $N \le M$. Then N is essential in M as an R-module if and only if $N[X]$ is essential in $M[X]$ as an $R[X]$-module.*

Theorem 2.11. *Let M be a module. If $M[X]$ is GCH $R[X]$-module, then M is GCH R-module.*

Proof. Let $f : M \to M$ be an essential injective endomorphism of M. Then $f[X] : M[X] \to M[X]$ with $f[X](\Sigma m_iX^i) = \Sigma f(m_iX^i)$ is an injective endomorphism of $M[X]$. Next, we will show that $\mathrm{Im} f[X] \le_e M[X]$. It is easy to verify that $\mathrm{Im} f[X] = (\mathrm{Im} f)[X]$. Since $\mathrm{Im} f \le_e M$, $\mathrm{Im} f[X] \le_e M[X]$ by Lemma 2.10. Thus $f[X]$ is isomorphic by assumption. The surjectivity of f follows by that of $f[X]$. This completes the proof. □

3. Modules whose small surjective endomorphisms are isomorphic

Let M be a module. M is said to be a weakly Hopfian module (WH module) if any small surjection of M is isomorphic.

Theorem 3.1. *Let M be a module. If M satisfies the condition that M/N is WH for every small submodule, $0 \ne N \le M$. Then M itself is WH.*

Proof. Suppose on the contrary that M is not WH. Then there exists a small surjection f of M which is not an isomorphism. Let $N = \mathrm{Ker} f$. Then $0 \neq N$ and f induces an isomorphism $\bar{f} : M/N \to M$. If $\eta : M \to M/N$ denotes the canonical quotient morphism, then $\eta\bar{f} : M/N \to M/N$ is a small surjection which is not an isomorphism. This is a contradiction. □

Proposition 3.2. *Let M be a torsion-free module. Then M is WH.*

Proof. Let $f : M \to M$ be any small surjection. Let $0 \neq x \in M \setminus \mathrm{Ker} f$, then $-x \in M \setminus \mathrm{Ker} f$. For any $r \in R$, $f(xr) = f(x)r$. Since $x \overline{\in} \mathrm{Ker} f$, $f(x) \neq 0$. By assumption, $f(x)r \neq 0$ and hence $xr \in M \setminus \mathrm{Ker} f$. Thus, $(M \setminus \mathrm{Ker} f) \cup \{0\}$ is a submodule of M. Since $(M \setminus \mathrm{Ker} f) \cup \{0\} + \mathrm{Ker} f = M$ and f is a small surjection, $(M \setminus \mathrm{Ker} f) \cup \{0\} = M$ and so $\mathrm{Ker} f = 0$, as required. □

Proposition 3.3. *Let M be WH and K is a direct summand of M. Then K is WH.*

Proof. Since K is a direct summand of M, there exists $K' \leq M$ such that $M = K \oplus K'$. Let $f : K \to K$ be a surjection with $\mathrm{Ker} f \ll K$, then f induces a small surjection $f \oplus 1_{K'} : M \to M$ with $(f \oplus 1_{K'})(k + k') = f(k) + k'$ by Lemma 1.2. Since M is WH, $f \oplus 1_{K'}$ is isomorphic and hence f is an isomorphism, as desired. □

Proposition 3.4. *Let $M = M_1 \oplus M_2$ and let M_1, M_2 be invariant submodules under any surjection of M. Then M is WH if and only if M_1, M_2 are WH.*

Proof. "If" part is clear by Proposition 3.3.

"Only if" part. Let $f : M \to M$ be a small epimorphism, then $f|_{M_i} : M_i \to M_i$ is a small surjection. By assumption, $f|_{M_i}$ is isomorphic. Let $f(m_1 + m_2) = 0$, then $f(m_1) + f(m_2) = 0$ and so $m_1 = m_2 = 0$. Thus f is injective. This completes the proof. □

We know that Noetherian modules are Hopfian modules. It is also easy to know that a module M is Hopfian if and only if M is both a generalized Hopfian module and a WH module. In [1], Ghorbani and Haghany proved that if ACC holds on nonsmall submodules of M, then M is generalized Hopfian. Thus it is natural that we consider the following result.

Theorem 3.5. *Let M be a module with ACC on small submodules. Then M is WH.*

Proof. Let $f : M \to M$ be a small surjection of M. Then $\mathrm{Ker} f \leq \mathrm{Ker} f^2 \leq \mathrm{Ker} f^3 \leq \cdots$ is an ascending chain on small submodule of M by Lemma 1.2. Since M satisfies ACC on small submodules, there exists a positive number n such that $\mathrm{Ker} f^n = \mathrm{Ker} f^{n+1}$. Next, we will prove that $\mathrm{Ker} f = 0$. Let $x \in \mathrm{Ker} f$, then $f(x) = 0$. Since f is surjective, there exists $y_1 \in M$ such that $f(y_1) = x$. Since f is surjective, there exists $y_2 \in M$ such that $f(y_2) = y_1$. Repeating the process, we obtain $y_{n-1} \in M$ with $f(y_n) = y_{n-1}$. Thus $f(y_1) = f^2(y_2) = \cdots = f^n(y_n) = x$. Since $x \in \mathrm{Ker} f$, $f(x) = f(f^n(y_n)) = 0$, that is, $f^{n+1}(y_n) = 0$. So $y_n \in \mathrm{Ker} f^{n+1} = \mathrm{Ker} f^n$. Consequently, $f^n(y_n) = 0$ and hence $x = 0$, as required. □

Lemma 3.6. *A quasidiscrete module M is discrete if and only if M is a WH module.*

Proof. The necessity of the condition is obvious. Conversely, let $f : M \to N$ be an epimorphism, with kernel K, onto the direct summand N of M. As M is quasidiscrete, there is a decomposition $M = A \oplus B$ with $A \leq K$ and $B \cap K \ll B$. Now, $N \simeq M/K = K + B/K \simeq B/(B \cap K)$. Theorem 4.24 in [4] yields $N \simeq B$, let $g : N \to B$ be an isomorphism. Then,

$$M = A \oplus B \longrightarrow A \oplus N \longrightarrow A \oplus B = M \tag{3.1}$$

is an epimorphism with the small kernel $B \cap K$, where $1 \oplus f|_B : M \to A \oplus N$ and $1 \oplus g : A \oplus N \to A \oplus B = M$. By assumption, it is an isomorphism, that is, $B \cap K = 0$. Then $K = A$ is a direct summand of M and f splits. This completes the proof. □

A discrete module is a quasidiscrete module, but a quasidiscrete module need not be a discrete module. Thus the following result gives a sufficient condition such that a quasidiscrete module is a discrete module.

Corollary 3.7. *Let M be a module with ACC on small submodules. Then M is quasidiscrete if and only if M is discrete.*

Proof. It follows from Theorem 3.5 and Lemma 3.6. □

Proposition 3.8. *Let P be a property of modules preserved under isomorphism. If a module M has the property P and satisfies ACC on nonzero small submodules N such that M/N has the property P, then M is WH.*

Proof. Suppose that M is not WH. Then there exists a submodule N_1 with $N_1 \neq 0$ and $M/N_1 \simeq M$. Thus M/N_1 is not WH but satisfies P. Hence, there exists a submodule $N_2 \leq N_1$ with $N_1 \neq N_2$ and $M/N_2 \simeq M/N_1$. So we get $0 < N_1 < N_2$ with $M/N_i \simeq M$ for $i = 1, 2$. Repeating the process yields a chain of submodules of the type that contradicts our hypothesis. □

Corollary 3.9. *If M is GCH and satisfies ACC on small submodules N such that M/N is GCH, then M is WH.*

Proof. It follows by Proposition 3.8. □

Remark 3.10. Following Proposition 2.7, if a module M is WH and has DCC on essential WH submodules, then M is GCH.

Corollary 3.11. *If M has ACC on its nonzero small submodules N such that M/N is not WH, then M is WH.*

Proof. Let P be the property of being WH, and suppose that M is not WH. By Proposition 3.8, M must be WH. This contradiction proves that M is WH. □

Example 3.12. Let M be a torsion-free Abelian group of finite rank such that End(M) is a principal ideal domain and let $B = \oplus_n M$, where n is an integer. Then the kernel of endomorphism of B is a direct summand which, again, is a direct sum of copies of M. It is clear to see that M is WH, but it need not be Hopfian.

Example 3.13. If R is a semisimple Artinian ring, then a module M is Hopfian if and only if M has finite length (see [3]). Note that a vector space V over a field F is Hopfian if and only if it is finite dimensional. Thus an infinite-dimensional vector space is WH, but it is not Hopfian.

Let R be a ring and $\mathscr{C}$ a class of R-modules, we will say that $\mathscr{C}$ is *socle fine* whenever for any $M, N \in \mathscr{C}$, we have $\mathrm{Soc}(M) \simeq \mathrm{Soc}(N)$ if and only if $M \simeq N$.

THEOREM 3.14. *For any ring R, the following statements are equivalent.*

(1) *R is semisimple.*
(2) *The class of all GCH modules is socle fine.*
(3) *The class of all WH modules is socle fine.*

Proof. It is easy to prove that quasiprojective (quasiinjective) and semisimple modules are GCH(WH) modules. If R is semisimple, then the class of all R-modules is socle fine and hence (1) implies (2) and (3).

(2)⇒(1) Since R and $\mathrm{Soc}(R)$ are quasiprojective and $\mathrm{Soc}(R) = \mathrm{Soc}(\mathrm{Soc}(R))$, we have $R \simeq \mathrm{Soc}(R)$ by (2). Thus R is semisimple.

(3)⇒(1) Since $\mathrm{Soc}(E(R)) = \mathrm{Soc}(R) = \mathrm{Soc}(\mathrm{Soc}(R))$ and both of $E(R)$ and $\mathrm{Soc}(R)$ are quasiinjective, we have $E(R) \simeq \mathrm{Soc}(R)$ is semisimple, and so is R. □

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Yongduo Wang: Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou 730050, China
E-mail address: ydwang333@sohu.com