FUZZY TL-UNIFORM SPACES

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The main purpose of this paper is to introduce a new structure that is a fuzzy TL-uniform space. We show that our structure generates a fuzzy topological space, precisely, a fuzzy T-locality space. Also, we deduce the concept of level uniformities of a fuzzy TL-uniformity. We connect the category of fuzzy TL-uniform spaces with the category of uniform spaces. We establish a necessary and sufficient condition, under which a fuzzy TL-uniformity is probabilistic pseudometrizable. Finally, we define a functor from the category of fuzzy TL-uniform spaces into the category of fuzzy T-locality spaces and we show that it preserves optimal lifts.

1. Introduction

In this paper, we introduce, for each continuous triangular norm T, a new structure that is a fuzzy TL-uniform space, which generates a fuzzy topological space. We proceed as follows.

In Section 2, we present some basic definitions and ideas, also, we supply three lemmas on the α-cuts of fuzzy subsets on a universe set X for all α in the interval $[0,1]$. We need those lemmas for proofs scattered in subsequent sections.

In Section 3, we introduce our definition of fuzzy TL-uniform spaces, fuzzy uniform map, and the fuzzy topology associated with a fuzzy TL-uniform space.

In Section 4, we study the relationship between a fuzzy TL-uniformity and its α-level uniformities. Also, we generate fuzzy TL-uniformities from classical uniformities.

In Section 5, we prove that each category of fuzzy TL-uniform spaces and uniform maps between them is a topological category, and we describe the fuzzy TL-uniform space of the optimal lift of a source in this category. Also, we define a functor from the category of fuzzy TL-uniform spaces into the category of fuzzy T-locality spaces, and we show that it preserves optimal lifts.

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2. Prerequisites

A triangular norm (cf. [1, 2]) is a binary operation on the unit interval $I = [0,1]$ that is associative, symmetric, isotone in each argument and has neutral element 1. A continuous triangular norm T is uniformly continuous, because its domain $I \times I$ is compact. This means that for all $\varepsilon > 0$, there is $\theta = \theta_{T,\varepsilon} > 0$ in such a way that for every $(\alpha,\beta) \in I \times I$, we have

$$(\alpha T\beta) - \varepsilon \leq (\alpha - \theta)T(\beta - \theta) \leq \alpha T\beta \leq (\alpha + \theta)T(\beta + \theta) \leq (\alpha T\beta) + \varepsilon. \tag{2.1}$$

Given a lower semicontinuous triangular norm T, the following binary operation on I:

$$j_T(\alpha,\gamma) = \sup\{\theta \in I : \alpha T\theta \leq \gamma\}, \quad (\alpha,\gamma) \in I \times I, \tag{2.2}$$

is called the residual implication of T [3], we often simplify j_T to j. The best well-known triangular norms are the simplest three, namely Min (also denoted by $\wedge$), T_m (the Lukasiewicz conjunction), and π (product), where for all $\alpha,\beta \in I$,

$$\alpha T_m\beta = (\alpha + \beta)\underline{\wedge}1, \qquad \alpha\pi\beta = \alpha\beta, \tag{2.3}$$

where the binary operation $\underline{\wedge}$, above, is the truncated subtraction, defined on nonnegative real numbers by $r\underline{\wedge}\, s = \max\{r - s, 0\}$, $r,s \geq 0$.

Proposition 2.1 [4]. *The T-residual implication j, of a lower semicontinuous triangular norm T, enjoys the following properties for all $\alpha,\beta,\gamma \in I$:*
(IT1) $\alpha T\beta \leq \gamma$ *if and only if* $\alpha \leq j(\beta,\gamma)$,
(IT2) $\alpha T\beta > \gamma$ *if and only if* $\alpha > j(\beta,\gamma)$,
(IT3) $j(\alpha,\gamma) = 1$ *if and only if* $\alpha \leq \gamma$,
(IT4) $j(1,\gamma) = \gamma$,
(IT5) $j(\alpha T\beta,\gamma) = j(\alpha, j(\beta,\gamma))$,
(IT6) $\alpha T j(\alpha,\gamma) \leq \gamma$,
(IT7) $j(\alpha,\beta)T j(\theta,\gamma) \leq j(\alpha T\theta, \beta T\gamma)$,
(IT8) j *is antimonotone in the left argument and monotone in the right argument.*

A fuzzy set λ in a universe X, introduced by Zadeh in [5], is a function $\lambda : X \to I$. The collection of all fuzzy subsets of X is denoted by I^X. Ordinary subsets of a universe X will frequently occur in what follows. We will often need to consider a subset $M \subseteq X$ as a fuzzy subset of X, said to be a crisp subset of X, which we will denote by the symbol $\mathbf{1}_M$. We do this by identifying $\mathbf{1}_M$ with its characteristic function. The collection of all crisp subsets of X is denoted by 2^X. The symbol $\underline{\underline{\alpha}}$ denotes the constant fuzzy set in a universe X with value $\alpha \in I$.

For the fuzzy sets $\mu,\lambda \in I^X$, the degree of containment of μ in λ according to $j = j_T$ is the real number in I [6]:

$$j\langle\mu,\lambda\rangle = \inf_{x\in X} j(\mu(x),\lambda(x)). \tag{2.4}$$

Last, we denote by $j|\mu,\lambda|$ the following fuzzy subset of X:

$$j|\mu,\lambda|(x) = j\big(\mu(x),\lambda(x)\big), \quad x \in X. \tag{2.5}$$

Given a fuzzy set $\lambda \in I^X$ and a real number $\alpha \in I_1 = [0,1[$, the strong α-cut of λ is the following subset of X:

$$\lambda^\alpha = \{x \in X : \lambda(x) > \alpha\}, \tag{2.6}$$

and for a real number $\alpha \in I$, the weak α-cut of λ is the subset of X:

$$\lambda_{\alpha^*} = \{x \in X : \lambda(x) \geq \alpha\}, \tag{2.7}$$

for both strong α-cuts and weak α-cuts of λ, we identify crisp sets with their characteristic functions.

It is direct to verify that every $\lambda \in I^X$ has the following resolutions:

$$\lambda = \bigvee_{\alpha\in I_1}\left[\underline{\underline{\alpha}} \wedge \lambda^\alpha\right] = \bigvee_{\alpha\in I_0}\left[\underline{\underline{\alpha}} \wedge \lambda_{\alpha^*}\right]. \tag{2.8}$$

We follow Lowen's definition of a fuzzy interior operator on a set X [7].

This is an operator ${}^o : I^X \to I^X$ that satisfies $\mu^o \leq \mu$, $(\mu \wedge \lambda)^o = \mu^o \wedge \lambda^o$ for all $\mu,\lambda \in I^X$, and $\underline{\underline{\alpha}}^o = \underline{\underline{\alpha}}$ for all $\alpha \in I$.

The pair $(X,{}^o)$ is called a fuzzy topological space (FTS).

The category of all fuzzy topological spaces and continuous functions between them (cf. [7]) is denoted by FTS.

Given a fuzzy topological space $(X,{}^o) = (X,\boldsymbol{\tau})$, and $\alpha \in I_1$, the set

$$\boldsymbol{\iota}_\alpha(\boldsymbol{\tau}) = \{\lambda^\alpha \subseteq X : \lambda \in \boldsymbol{\tau}\} \tag{2.9}$$

is a topology on X, said to be the α-level topology of $\boldsymbol{\tau}$ [8]. It is direct to see that its interior operator is given by

$$\operatorname{int}_{\iota_\alpha(\boldsymbol{\tau})}(M) = \left[\left(\underline{\underline{\alpha}} \vee \mathbf{1}_M\right)^o\right]^\alpha, \quad M \subseteq X, \tag{2.10}$$

and, $\boldsymbol{\iota}(\boldsymbol{\tau}) = \sup_{\alpha\in I_1} \boldsymbol{\iota}_\alpha(\boldsymbol{\tau})$ is called the topology modification of $\boldsymbol{\tau}$ [8].

Prefilters and prefilterbases were introduced by Lowen in [9]. A prefilter in a universe X is a nonempty collection $\mathfrak{I} \subset I^X$ which satisfies $\underline{\underline{0}} \notin \mathfrak{I}$, $\mathfrak{I}$ is closed under finite meets and contains all the fuzzy supersets of its individual members. A prefilterbase in X is a nonempty collection $\mathcal{B} \subset I^X$ which satisfies $\underline{\underline{0}} \notin \mathcal{B}$ and the meets of two members of $\mathcal{B}$ contain a member of $\mathcal{B}$. A prefilterbase $\mathcal{B}$ is said to be prefilterbasis for a prefilter $\mathfrak{I}$ if $\mathfrak{I} = [\mathcal{B}]$, where $[\mathcal{B}] = \{\lambda \in I^X : \text{there exists } \nu \in \mathcal{B} \text{ with } \nu \leq \lambda\}$.

Definition 2.2 [10]. The T-saturation operator is the operator $\sim T$ which sends a prefilterbase $\mathcal{B}$ in X to the following subset of I^X:

$$\mathcal{B}^{\sim T} = \left[\left[\left\{ \bigvee_{\gamma \in [0,1[} \underline{\gamma} T \mu_\gamma : \mu_\gamma \in \mathcal{B} \ \forall \gamma \in [0,1[\right\}\right]\right], \tag{2.11}$$

said to be the T-saturation of $\mathcal{B}$.

Theorem 2.3 [10]. *The T-saturation operator $\sim T$ is isotone. Also, given a prefilterbase $\mathcal{B}$ in X, $\mathcal{B}^{\sim T}$ satisfies*

$$\mathcal{B}^{\sim T} \text{ is a prefilter}, \ \supseteq [\mathcal{B}] \supseteq \mathcal{B}, \quad \sup_{\nu \in \mathcal{B}} j\langle \nu, \lambda \rangle = \sup_{\mu \in \mathcal{B}^{\sim T}} j\langle \mu, \lambda \rangle, \tag{2.12}$$

moreover, $\mathcal{B}^{\sim T}$ equals the following subset of I^X:

$$\left\{ \lambda \in I^X : j \left| \underline{\beta}, \lambda \right| \in [\mathcal{B}] \ \forall \beta < 1 \right\}. \tag{2.13}$$

Definition 2.4 [10]. A fuzzy T-locality space is a fuzzy topological space $(X,{}^o)$ whose fuzzy interior operator is induced by some indexed family $\mathcal{B} = (\mathcal{B}(x))_{x \in X}$, of prefilterbases in I^X, in the following manner:

$$\mu^o(x) = \sup_{\nu \in \mathcal{B}(x)} j\langle \nu, \mu \rangle, \quad \forall (\mu, x) \in I^X \times X. \tag{2.14}$$

The family $\mathcal{B}$ is said to be a T-locality basis for $(X,{}^o)$, the fuzzy topology of $(X,{}^o)$ will be denoted by $\boldsymbol{\tau}(\mathcal{B})$. The full subcategory, of FTS, of fuzzy T-locality spaces is denoted by T-FLS.

Theorem 2.5 [10]. *A family of prefilterbases in X, $\mathcal{B} = (\mathcal{B}(x))_{x \in X}$, will be a T-locality base in X if and only if it satisfies the following two conditions for all $x \in X$.*

(TLB1) $\nu(x) = 1$ *for all* $\nu \in \mathcal{B}(x)$.

(TLB2) *Every $\nu \in \mathcal{B}(x)$ has a T-kernel. This consists of two families $(\nu_\gamma \in \mathcal{B}(x))_{\gamma \in I_1}$ and $(\nu_{y\gamma\theta} \in \mathcal{B}(y))_{(y,\gamma,\theta) \in X \times I_1 \times I_0}$ such that for all*

$$(y, \gamma, \theta) \in X \times I_1 \times I_0, \qquad [(\gamma T \nu_\gamma(y)) \underline{\wedge} \theta] T \nu_{y\gamma\theta} \leq \nu, \quad \text{where } I_0 =]0,1]. \tag{2.15}$$

Theorem 2.6 [10]. (i) *A family $\mathcal{B} = (\mathcal{B}(x))_{x \in X}$ will be a T-locality basis for a T-locality space $(X,{}^o)$ if and only if $\mathcal{B}^{\sim T}$ is so. In this case, $\mathcal{B}^{\sim T}$ is the greatest T-locality basis for $(X,{}^o)$ (and $\mathcal{B}^{\sim T}$ is called the T-locality system of $(X,{}^o)$). It is the unique T-saturated T-locality basis for $(X,{}^o)$. In this case, $\mathcal{B}$ is a basis for the T-locality system $\mathcal{B}^{\sim T}$.*

(ii) *If $\mathcal{B} = (\mathcal{B}(x))_{x \in X}$ is a T-locality basis for a T-locality space $(X,{}^o)$, then $\mathcal{B}^{\sim T} = ((\mathcal{B}(x))^{\sim T})_{x \in X}$ is obtained also by*

$$(\mathcal{B}(x))^{\sim T} = \{ \nu \in I^X : \nu^o(x) = 1 \}, \quad x \in X. \tag{2.16}$$

A distance distribution function (DDF) [2] is a function from the set $\mathbb{R}^+$ of positive real numbers to the unit interval $I = [0,1]$, which is monotone, left continuous, and has supremum 1. The set of all DDFs is denoted by D^+. Every continuous triangular norm T induces on D^+ a binary operation $\oplus_T$ by $(\eta \oplus_T \zeta)(s) = \sup\{\eta(b)T\zeta(s-b) : 0 < b < s\}$, $s > 0$. The identity element of a semigroup $(D^+, \oplus_T)$ is ε_0 defined by

$$\varepsilon_0(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases} \tag{2.17}$$

A probabilistic pseudometric on a set X (cf. [11]) is a function $\Gamma : X \times X \to D^+$ that satisfies for all x, y, z in X the following properties:

(PM1) $\Gamma(x,x) = \varepsilon_0$,

(PM2) $\Gamma(x,y) = \Gamma(y,x)$,

(PM3) $\Gamma(x,y) \oplus_T \Gamma(y,z) \geq \Gamma(x,z)$.

In [12], Höhle defines for every $\psi, \varphi \in I^{X\times X}$ and $\lambda \in I^X$, the T-section of ψ over λ by $\psi\langle\lambda\rangle_T(x) = \sup_{y\in X} \lambda(y)T\psi(y,x)$, $x \in X$, the T-composition of ψ, φ by $\psi o_T \varphi(x,y) = \sup_{z\in X} \varphi(x,z)T\psi(z,y)$, $x, y \in X$, the symmetric of ψ by ${}_s\psi(x,y) = \psi(y,x)$, $x, y \in X$.

Now, we will use the following lemmas in the sequel.

Lemma 2.7. *For all* φ, $\psi \in I^{X\times X}$ *and* $\alpha \in I_1$,

(i) $(\underline{\underline{\gamma}}T\varphi)^\alpha = \varphi^{j(\gamma,\alpha)}$ *for all* $\gamma \in I$,

(ii) $(\varphi o_T \psi)^\alpha = \bigcup_{\theta T\beta>\alpha}(\varphi_{\theta^*} o \psi_{\beta^*}) = \bigcup_{\theta T\beta \leq \alpha}(\varphi^\theta o \psi^\beta)$.

Proof. (i) For every $(x,y) \in X \times X$, we get the equivalences

$$\begin{aligned} &(x,y) \in \left(\underline{\underline{\gamma}}T\varphi\right)^\alpha \\ &\quad \text{iff} \quad \gamma T\varphi(x,y) > \alpha \\ &\quad \text{iff} \quad \varphi(x,y) > j(\gamma,\alpha), \quad \text{by (IT2)}, \\ &\quad \text{iff} \quad (x,y) \in \varphi^{j(\gamma,\alpha)}. \end{aligned} \tag{2.18}$$

This proves (i).

(ii) For the first equality, let $(x,y) \in X \times X$, we get the equivalences

$$\begin{aligned} &(x,y) \in (\varphi o_T \psi)^\alpha \\ &\quad \text{iff} \quad (\varphi o_T \psi)(x,y) > \alpha \\ &\quad \text{iff} \quad \sup_{z\in X} \psi(x,z)T\varphi(z,y) > \alpha \\ &\quad \text{iff} \quad \exists z \in X, \quad \beta,\theta \in I_o \text{ with } \beta T\theta > \alpha \text{ such that} \\ &\psi(x,z) \geq \beta,\ \varphi(z,y) \geq \theta \\ &\quad \text{iff} \quad \exists z \in X, \quad \beta,\theta \in I_o \text{ with } \beta T\theta > \alpha \text{ such that} \\ &(x,z) \in \psi_{\beta^*},\ (z,y) \in \varphi_{\theta^*} \\ &\quad \text{iff} \quad (x,y) \in \bigcup_{\theta T\beta>\alpha} (\varphi_{\theta^*} o \psi_{\beta^*}). \end{aligned} \tag{2.19}$$

which proves our assertion.

For the second equality of (ii), we get also

$$\begin{aligned}
&(x,y) \notin (\varphi o_T \psi)^{\alpha} \\
&\qquad \text{iff} \quad (\varphi o_T \psi)(x,y) \le \alpha \\
&\qquad \text{iff} \quad \sup_{z\in X} \psi(x,z) T \varphi(z,y) \le \alpha \\
&\qquad \text{iff} \quad \forall z \in X,\ \exists \beta,\theta \in I_0 \text{ with } \beta T \theta \le \alpha \text{ such that} \\
&\psi(x,z) \le \beta,\ \varphi(z,y) \le \theta \\
&\qquad \text{iff} \quad \forall z \in X,\ \exists \beta,\theta \in I_0 \text{ with } \beta T \theta \le \alpha \text{ such that} \\
&(x,z) \notin \psi^{\beta},\ (z,y) \notin \varphi^{\theta} \\
&\qquad \text{iff} \quad (x,y) \notin \bigcup_{\theta T \beta \le \alpha} (\varphi^{\theta} o \psi^{\beta}).
\end{aligned} \tag{2.20}$$

This completes the proof. □

Lemma 2.8 [13]. *For all* $\psi \in I^{X\times X}$ *and* $\alpha \in I_1$,

$$({}_s\psi)^{\alpha} = {}_s(\psi^{\alpha}). \tag{2.21}$$

Lemma 2.9 [14]. *Let* $(\psi_n)_{n\in N}$ *be a sequence of fuzzy sets on* $X \times X$ *for which* $\psi_n(x,x) = 1$ *for all* $x \in X$; $\psi_n = {}_s(\psi_n)$; $\psi_{n+1} o_T \psi_{n+1} o_T \psi_{n+1} \le \psi_n$. *Then, there is a probabilistic pseudometric* Γ *on* X *satisfying*

$$\psi_{n+1}(x,y) \le \Gamma(x,y)(2^{-n}) \le \psi_n(x,y), \quad \forall (x,y) \in X \times X. \tag{2.22}$$

3. Fuzzy TL-uniform spaces

The fuzzy TL-uniform spaces are introduced in this section, and some of their properties are given. The uniform maps are defined. Also, we generate a fuzzy topology from a fuzzy TL-uniform space.

Definition 3.1. (i) A fuzzy TL-uniform base on a set X is a subset $\boldsymbol{v} \subset I^{X\times X}$ which fulfills the following properties.

(FLUB1) $\boldsymbol{v}$ is a prefilterbase.

(FLUB2) For all $\varphi \in \boldsymbol{v}$ and $x \in X$, $\varphi(x,x) = 1$.

(FLUB3) For all $\varphi \in \boldsymbol{v}$ and $\gamma \in I_1$, there is $\varphi_\gamma \in \boldsymbol{v}$ with $\underline{\underline{\gamma}} T \varphi_\gamma \le {}_s\varphi$.

(FLUB4) For all $\varphi \in \boldsymbol{v}$ and $\gamma \in I_1$, there is $\varphi_\gamma \in \boldsymbol{v}$ with $\underline{\underline{\gamma}} T (\varphi_\gamma o_T \varphi_\gamma) \le \varphi$.

Obviously, (FLUB3) and (FLUB4) can be replaced by the single condition

(FLUB3$^{\backslash}$). For all $\varphi \in \boldsymbol{v}$ and $\gamma \in I_1$, there is $\varphi_\gamma \in \boldsymbol{v}$ with $\underline{\underline{\gamma}} T (\varphi_\gamma o_T \varphi_\gamma) \le {}_s\varphi$.

(ii) A fuzzy TL-uniformity on X is a T-saturated fuzzy TL-uniform base on X.

(iii) If $\boldsymbol{\mu}$ is a fuzzy TL-uniformity on X, then $\boldsymbol{v}$ is a basis for $\boldsymbol{\mu}$ if $\boldsymbol{v}$ is a prefilterbase and $\boldsymbol{v}^{\sim T} = \boldsymbol{\mu}$.

It follows that for a fuzzy TL-uniformity $\boldsymbol{\mu}$ on a set X, and all $\varphi \in \boldsymbol{\mu}$, ${}_s\varphi \in \boldsymbol{\mu}$. The pair $(X,\boldsymbol{\mu})$ consisting of a set X and a fuzzy TL-uniformity $\boldsymbol{\mu}$ on X is called fuzzy TL-uniform space. The elements of $\boldsymbol{\mu}$ are called fuzzy vicinities.

Proposition 3.2. *Let $\boldsymbol{v}$ be a fuzzy TL-uniform base on a set X, then the prefilter $\boldsymbol{v}^{\sim T}$ determined by $\boldsymbol{v}$ is a fuzzy TL-uniformity on X.*

Conversely, if $\boldsymbol{\mu}$ is a fuzzy TL-uniformity on X, then conditions (FLUB2)–(FLUB4) are satisfied by every prefilterbase $\boldsymbol{v}$ which determined $\boldsymbol{\mu}$.

Proof. Suppose $\boldsymbol{v}$ is a fuzzy TL-uniform base on X. We show that $\boldsymbol{v}^{\sim T}$ is a fuzzy TL-uniformity.

Obviously, $\boldsymbol{v}^{\sim T}$ is T-saturated prefilter.

Now, to prove (FLUB2), let $\varphi \in \boldsymbol{v}^{\sim T}$, then there exists $(\psi_\alpha \in \boldsymbol{v})_{\alpha\in I_1}$ such that $\varphi \geq \bigvee_{\alpha\in I_1}(\underline{\underline{\alpha}}T\psi_\alpha)$. Hence for every $x \in X$, we have

$$\begin{aligned}\varphi(x,x) &\geq \bigvee_{\alpha\in I_1}\left(\underline{\underline{\alpha}}T\psi_\alpha\right)(x,x)\\ &= \bigvee_{\alpha\in I_1}(\alpha T1) \quad \text{for } \psi_\alpha \in v\\ &= \bigvee_{\alpha\in I_1}(\alpha) = 1.\end{aligned} \tag{3.1}$$

Also, to prove (FLUB3$^\backslash$), let $\varphi \in \boldsymbol{v}^{\sim T}$ and $\gamma \in I_1$, then for all $\alpha \in I_1$, there exists $\psi_\alpha \in \boldsymbol{v}$ such that

$$\varphi \geq \underline{\underline{\alpha}}T\psi_\alpha. \tag{3.2}$$

But for each $\beta \in I_1$, there is ${}^\beta\psi_\alpha \in \boldsymbol{v}$ with

$$\underline{\underline{\beta}}T\left({}^\beta\psi_\alpha o_T {}^\beta\psi_\alpha\right) \leq {}_s\psi_\alpha. \tag{3.3}$$

By continuity of T, we can choose $\beta \in I_1$ such that $\beta T\beta = \gamma$, and by taking $\varphi_\gamma = {}^\beta\psi_\beta \in \boldsymbol{v} \subseteq \boldsymbol{v}^{\sim T}$, we get

$$\begin{aligned}\underline{\underline{\gamma}}T\left(\varphi_\gamma o_T \varphi_\gamma\right) &= \underline{\underline{\beta}}T\underline{\underline{\beta}}T\left({}^\beta\psi_\beta o_T {}^\beta\psi_\beta\right)\\ &\leq \underline{\underline{\beta}}T{}_s\psi_\beta, \quad \text{by (3.3)},\\ &\leq {}_s\varphi, \quad \text{directly from (3.2)}.\end{aligned} \tag{3.4}$$

This proves that $\boldsymbol{v}^{\sim T}$ is a fuzzy TL-uniformity.

Conversely, suppose that $\boldsymbol{\mu}$ is a fuzzy TL-uniformity, and let $\boldsymbol{v}$ be a prefilterbase which determines $\boldsymbol{\mu}$.

Since $\boldsymbol{v} \subseteq \boldsymbol{\mu}$, then obviously $\boldsymbol{v}$ satisfies (FLUB2), while (FLUB3) follows immediately from the fact that for all $\psi \in \boldsymbol{\mu}$, we get ${}_s\psi \in \boldsymbol{\mu}$.

To prove (FLUB4), let $\varphi \in \boldsymbol{v}$ and $\gamma \in I_1$, then $\varphi \in \boldsymbol{\mu}$ and hence, there is a family $(\psi_\gamma \in \boldsymbol{\mu})_{\gamma \in I_1}$ such that

$$\underline{\underline{\gamma}} T(\psi_\gamma o_T \psi_\gamma) \leq \varphi. \tag{3.5}$$

By continuity of T, we can find $\alpha \in I_1$ such that $\gamma \leq \alpha T \alpha T \alpha$, and by Theorem 2.6, there is φ_γ in $\boldsymbol{v}$ with $\varphi_\gamma \leq j|\underline{\underline{\alpha}}, \psi_\alpha|$, consequently,

$$\begin{aligned}\underline{\underline{\gamma}} T(\varphi_\gamma o_T \varphi_\gamma) &\leq \underline{\underline{\alpha}} T \underline{\underline{\alpha}} T \underline{\underline{\alpha}} T\Big(j\Big|\underline{\underline{\alpha}}, \psi_\alpha\Big| o_T j\Big|\underline{\underline{\alpha}}, \psi_\alpha\Big|\Big)\\ &= \underline{\underline{\alpha}} T\Big[\Big(\underline{\underline{\alpha}} T j\Big|\underline{\underline{\alpha}}, \psi_\alpha\Big|\Big) o_T \Big(\underline{\underline{\alpha}} T j\Big|\underline{\underline{\alpha}}, \psi_\alpha\Big|\Big)\Big], \quad \text{clear},\\ &\leq \underline{\underline{\alpha}} T(\psi_\alpha o_T \psi_\alpha), \quad \text{by (IT6)},\\ &\leq \varphi, \quad \text{by (3.5)},\end{aligned} \tag{3.6}$$

which proves that $\boldsymbol{v}$ satisfies (FLUB4), and completes the proof. □

Lemma 3.3. *If* $\{\psi_\alpha : \alpha \in I_1\} \subseteq I^{X \times X}$ *and* $\psi = \bigvee_{\alpha \in I_1} (\underline{\underline{\alpha}} T \psi_\alpha) \in I^{X \times X}$, *then for every* $M \in 2^X$, $\psi\langle \mathbf{1}_M \rangle_T = \bigvee_{\alpha \in I_1} [\underline{\underline{\alpha}} T \psi_\alpha \langle \mathbf{1}_M \rangle_T] \in I^X$.

Proof. For every $x \in X$,

$$\begin{aligned}\psi\langle \mathbf{1}_M \rangle_T(x) &= \sup_{y \in X} \mathbf{1}_M(y) T \psi(y,x) = \sup_{y \in M} \psi(y,x)\\ &= \sup_{y \in M} \Big\{ \bigvee_{\alpha \in I_1} [\alpha T \psi_\alpha(y,x)] \Big\}\\ &= \bigvee_{\alpha \in I_1} \Big\{ \alpha T \Big[\sup_{y \in M} \psi_\alpha(y,x) \Big] \Big\}, \quad \text{by isotonicity of } T,\\ &= \bigvee_{\alpha \in I_1} \Big[\underline{\underline{\alpha}} T \psi_\alpha \langle \mathbf{1}_M \rangle_T \Big](x).\end{aligned} \tag{3.7}$$

Hence $\psi\langle \mathbf{1}_M \rangle_T = \bigvee_{\alpha \in I_1} [\underline{\underline{\alpha}} T \psi_\alpha \langle \mathbf{1}_M \rangle_T]$. □

Lemma 3.4. *For a prefilterbase* $\coprod$ *in* $X \times X$ *and* $M \in 2^X$,

$$\Big\{ \psi\langle \mathbf{1}_M \rangle_T : \psi \in \overset{\sim T}{\coprod} \Big\} = \Big\{ \psi\langle \mathbf{1}_M \rangle_T : \psi \in \coprod \Big\}^{\sim T}. \tag{3.8}$$

Proof. First, we show that $\{\psi\langle \mathbf{1}_M \rangle_T : \psi \in \coprod^{\sim T}\}$ is T-saturated prefilter. Let $\psi, \varphi \in \coprod^{\sim T}$ and let $\mu \in I^X$ be such that $\mu \geq \psi\langle \mathbf{1}_M \rangle_T \wedge \varphi\langle \mathbf{1}_M \rangle_T$. Then we define $\phi \in I^{X \times X}$ by

$$\phi(x,y) = \begin{cases} \mu(x) & \text{if } x \in M,\\ 1 & \text{if } x \notin M. \end{cases} \tag{3.9}$$

It is easy to see that $\mu = \phi\langle \mathbf{1}_M\rangle_T$ and $\phi \geq \psi \wedge \varphi$ and hence $\phi \in \coprod^{\sim T}$. This proves that $\mu \in \{\psi\langle \mathbf{1}_M\rangle_T : \psi \in \coprod^{\sim T}\}$, demonstrates that $\{\psi\langle \mathbf{1}_M\rangle_T : \psi \in \coprod^{\sim T}\}$ is a prefilter.

Now, given a subfamily $\{\psi_\gamma : \gamma \in I_1\}$ of $\coprod^{\sim T}$. Since $\coprod^{\sim T}$ is T-saturated, $\bigvee_{\alpha \in I_1}(\underline{\underline{\alpha}} T \psi_\alpha) \in \coprod^{\sim T}$, so,

$$\begin{aligned}\bigvee_{\alpha \in I_1}\left(\underline{\underline{\alpha}} T \psi_\alpha \langle \mathbf{1}_M\rangle_T\right) &= \bigvee_{\alpha \in I_1}\left(\underline{\underline{\alpha}} T \psi_\alpha\right)\langle \mathbf{1}_M\rangle_T, \quad \text{by Lemma 3.3,}\\ &\in \left\{\psi\langle \mathbf{1}_M\rangle_T : \psi \in \overset{\sim T}{\coprod}\right\}.\end{aligned} \tag{3.10}$$

This proves our assertion. Finally, we have

$$\begin{aligned}\left\{\psi\langle \mathbf{1}_M\rangle_T : \psi \in \coprod\right\} &\subseteq \left\{\psi\langle \mathbf{1}_M\rangle_T : \psi \in \overset{\sim T}{\coprod}\right\}\\ &= \left\{\psi\langle \mathbf{1}_M\rangle_T : \forall \gamma \in I_1\ \exists \psi_\gamma \in \coprod \text{ with } \underline{\underline{\gamma}} T \psi_\gamma \leq \psi\right\}\\ &\subseteq \left\{\psi\langle \mathbf{1}_M\rangle_T : \forall \gamma \in I_1\ \exists \psi_\gamma \in \coprod \text{ with } \underline{\underline{\gamma}} T \psi_\gamma \langle 1_M\rangle_T \leq \psi\langle 1_M\rangle_T\right\}\\ &= \left\{\psi\langle \mathbf{1}_M\rangle_T : \psi \in \coprod\right\}^{\sim T},\end{aligned} \tag{3.11}$$

since the prefilter $\{\psi\langle \mathbf{1}_M\rangle_T : \psi \in \coprod^{\sim T}\}$ is T-saturated, then

$$\begin{aligned}\left\{\psi\langle \mathbf{1}_M\rangle_T : \psi \in \coprod\right\}^{\sim T} &\subseteq \left\{\psi\langle \mathbf{1}_M\rangle_T : \psi \in \overset{\sim T}{\coprod}\right\}\\ &\subseteq \left\{\psi\langle \mathbf{1}_M\rangle_T : \psi \in \coprod\right\}^{\sim T}.\end{aligned} \tag{3.12}$$

So, equality holds, and winds up the proof. □

Theorem 3.5. *If $\boldsymbol{v}$ is a fuzzy TL-uniform base on a set X, then the indexed family $\mathcal{B} = (\mathcal{B}(x))_{x \in X}$, given by*

$$\mathcal{B}(x) = \left\{\varphi\langle \mathbf{1}_x\rangle_T : \varphi \in \boldsymbol{v}\right\}, \quad \textit{is a } T\textit{-locality base on } X. \tag{3.13}$$

Proof. Given $x \in X$ and $v \in \mathcal{B}(x)$, then there is $\varphi \in \boldsymbol{v}$ such that $v = \varphi\langle \mathbf{1}_x\rangle_T$, hence

$$v(x) = \varphi\langle \mathbf{1}_x\rangle_T(x) = \sup_{y \in X}\{[(\mathbf{1}_x)(y)]T\varphi(y,x)\} = (\mathbf{1}_x)(x)T\varphi(x,x) = 1. \tag{3.14}$$

It follows that $\mathcal{B}$ satisfies (TLB1).

Also, for every $v \in \mathcal{B}(x)$, $\gamma \in I_1$, and $z \in X$, we have

$$\begin{aligned} v(z) &= \varphi\langle \mathbf{1}_x \rangle_T(z) \quad \text{for some } \varphi \in \boldsymbol{v} \\ &= \varphi(x,z) \geq \underline{\underline{\gamma}} T(\varphi_\gamma o_T \varphi_\gamma)(x,z), \quad \text{by (FLUB4)} \\ &= \sup_{y \in X} \{\gamma T \varphi_\gamma(x,y) T \varphi_\gamma(y,z)\} \\ &\geq \sup_{y \in X} \{\gamma T \varphi_\gamma(x,y) T [\delta T(\varphi_{\gamma\delta} o_T \varphi_{\gamma\delta})(y,z)]\}, \quad \text{by (FLUB4) again} \\ &= \sup_{y \in X} \{\gamma T \varphi_\gamma(x,y) T \delta T \sup_{s \in X} [\varphi_{\gamma\delta}(y,s) T \varphi_{\gamma\delta}(s,z)]\} \\ &\geq \{\gamma T \varphi_\gamma(x,y) T \delta T \varphi_{\gamma\delta}(y,z) T \varphi_{\gamma\delta}(z,z)\} \\ &= \Big[\gamma T \varphi_\gamma \langle \mathbf{1}_x \rangle_T(y) T \delta\Big] T \Big[\varphi_{\gamma\delta} \langle \mathbf{1}_y \rangle_T(z)\Big]. \end{aligned} \tag{3.15}$$

Now, for every $\theta \in I_0$, we can get (by continuity of T) $\delta = \delta_\theta \in I_1$ for which $[\gamma T \varphi_\gamma \langle \mathbf{1}_x \rangle_T (y) T \delta] \geq [(\gamma T \varphi_\gamma \langle \mathbf{1}_x \rangle_T(y)) \underline{\triangle} \theta]$, hence take $v_{xy} = \varphi_\gamma \langle \mathbf{1}_x \rangle_T \in \mathcal{B}(x)$ and $v_{yy\theta} = \varphi_{\gamma\delta} \langle \mathbf{1}_y \rangle_T \in \mathcal{B}(y)$ which satisfy $v(z) \geq [(\gamma T v_{xy}(y)) \underline{\triangle} \theta] T v_{yy\theta}(z)$.

This shows that $\mathcal{B}$ satisfies (TLB2), which completes the proof. □

Proposition 3.6. *If $\boldsymbol{\mu}$ is a fuzzy TL-uniformity on a set X, then the indexed family $\mathcal{B} = (\mathcal{B}(x))_{x \in X}$, given by*

$$\mathcal{B}(x) = \Big\{\varphi\langle \mathbf{1}_x \rangle_T : \varphi \in \boldsymbol{\mu}\Big\}, \quad \text{is a } T\text{-locality system on } X. \tag{3.16}$$

The proof follows at once from Theorem 3.5 and Lemma 3.4.

We will see how a fuzzy TL-uniform base can generate a fuzzy topology.

Definition 3.7. The fuzzy T-locality space $(X, \boldsymbol{\tau}^T(\boldsymbol{v}))$ induced by a fuzzy TL-uniform base $\boldsymbol{v}$ on X is the T-FLS defined by means of the fuzzy T-locality base of the preceding theorem, for which its fuzzy interior operator $\boldsymbol{o}$ is given by

$$\lambda^o(x) = \sup_{\varphi \in \boldsymbol{v}} j \Big\langle \varphi\langle \mathbf{1}_x \rangle_T, \lambda \Big\rangle, \quad (\lambda, x) \in I^X \times X. \tag{3.17}$$

The fuzzy topology $\boldsymbol{\tau}^T(\boldsymbol{v})$ of that space is also denoted simply by $\boldsymbol{\tau}(\boldsymbol{v})$. Notice that due to Theorem 1.4, a fuzzy TL-uniform base $\boldsymbol{v}$ and its fuzzy TL-uniformity $\boldsymbol{v}^{\sim T}$ induce the same fuzzy T-locality space, that is, $\boldsymbol{\tau}(\boldsymbol{v}) = \boldsymbol{\tau}(\boldsymbol{v}^{\sim T})$.

Uniform-type continuity:

Definition 3.8. Let $(X, \boldsymbol{\mu})$ and $(X^\backslash, \boldsymbol{\mu}^\backslash)$ be fuzzy TL-uniform spaces and $f : X \to X^\backslash$. f is a fuzzy uniform map (or fuzzy uniformly continuous) if any of the following equivalent conditions hold.

(i) For each $\psi' \in \boldsymbol{\mu}^\backslash$, $(f \times f)^{-1}(\psi') \in \boldsymbol{\mu}$.

(ii) For each $\psi' \in \boldsymbol{\mu}^\backslash$, there is $\psi \in \boldsymbol{\mu}$ such that $\psi \leq (f \times f)^{-1}(\psi')$.

(iii) For each $\psi' \in \boldsymbol{\mu}^\backslash$, there is $\psi \in \boldsymbol{\mu}$ such that $(f \times f)(\psi) \leq \psi'$,

where $(f \times f)^{-1}(\psi')(x,y) = \psi'(f(x), f(y))$, $x, y \in X$.

The composite of two uniformly continuous functions $f:(X,\boldsymbol{\mu}) \to (Y,U)$ and $g:(Y,U) \to (Z,\wp)$ is again uniformly continuous, since $(gf \times gf)^{-1}(\psi) = (f \times f)^{-1}\{(g \times g)^{-1}(\psi)\} \in \boldsymbol{\mu}$, for all $\psi \in \boldsymbol{\wp}$.

Proposition 3.9. *If $(X,\boldsymbol{\mu})$ and $(X^\backslash,\boldsymbol{\mu}^\backslash)$ are fuzzy TL-uniform spaces, with bases $\boldsymbol{v}$ and $\boldsymbol{v}^\backslash$, respectively, and $f: X \to X^\backslash$, then f is uniformly continuous if and only if for all $\varphi' \in \boldsymbol{v}^\backslash$ and all $\gamma \in I_1$, there is $\varphi \in \boldsymbol{v}$ such that*

$$\underline{\underline{\gamma}}T\varphi \leq (f \times f)^{-1}(\varphi'). \tag{3.18}$$

Proof. Let $f:(X,\boldsymbol{\mu}) \to (X^\backslash,\boldsymbol{\mu}^\backslash)$ be uniformly continuous and $\varphi' \in v^\backslash$, $\gamma \in I_1$. Then $\varphi' \in \boldsymbol{v}^{\backslash\sim T} = \boldsymbol{\mu}^\backslash$, since f is uniformly continuous, we get $(f \times f)^{-1}(\varphi') \in \boldsymbol{\mu} = \boldsymbol{v}^{\sim T}$, hence there is $\varphi \in v$ such that $\underline{\underline{\gamma}}T\varphi \leq (f \times f)^{-1}(\varphi')$.

Conversely, suppose that the stated condition holds.

Let $\psi' \in \boldsymbol{\mu}^\backslash$, since $\boldsymbol{\mu}^\backslash = \boldsymbol{v}^{\backslash\sim T}$, then for all $\gamma \in I_1$, there exists $\varphi'_\gamma \in \boldsymbol{v}^\backslash$ such that $\psi' \geq \underline{\underline{\gamma}}T\varphi'_\gamma$, hence

$$\begin{aligned}(f \times f)^{-1}(\psi') &\geq (f \times f)^{-1}\left(\underline{\underline{\gamma}}T\varphi'_\gamma\right) = \underline{\underline{\gamma}}T\left[(f \times f)^{-1}(\varphi'_\gamma)\right] \\ &\geq \underline{\underline{\gamma}}T\underline{\underline{\theta}}T\varphi_{\gamma\theta} \quad \text{for some } \theta \in I_1,\ \varphi_{\gamma\theta} \in v, \text{ by hypothesis.}\end{aligned} \tag{3.19}$$

Since T is continuous, then for all $\alpha \in I_1$, there exists $\gamma = \gamma_\alpha \in I_1$ such that $\alpha = \gamma T \gamma$, by taking $\phi_\alpha = \varphi_{\gamma\gamma} \in \boldsymbol{v}$, we get

$$(f \times f)^{-1}(\psi) \geq \underline{\underline{\gamma}}T\underline{\underline{\gamma}}T\varphi_{\gamma\gamma} = \underline{\underline{\alpha}}T\phi_\alpha, \tag{3.20}$$

which implies that $(f \times f)^{-1}(\psi) \in \boldsymbol{\mu}$. Hence f is uniformly continuous, and winds up the proof. □

A function $f:(X,^o) = (X,\boldsymbol{\tau}) \to (Y,^o) = (Y,\boldsymbol{\tau}^\backslash)$, between two fuzzy topological spaces, is said to be continuous [7]; if $f^{-1}(\mu) \in \boldsymbol{\tau}$ for all $\mu \in \boldsymbol{\tau}^\backslash$, $\mu \in I^Y$. Equivalently, if $f^{-1}(\lambda^o) \leq [f^{-1}(\lambda)]^o$ for all $\lambda \in I^Y$.

Theorem 3.10. *Let $(X,\boldsymbol{\mu})$ and $(Y,\boldsymbol{\wp})$ be fuzzy TL-uniform spaces. If $f: X \to Y$ is uniformly continuous, then it is continuous with respect to the fuzzy topologies generated by μ and $\wp$, respectively.*

Proof. Let $\lambda \in I^Y$ and $x \in X$, we denote by o_1 and o_2 the fuzzy interior operators of $\boldsymbol{\tau}(\boldsymbol{\mu})$ and $\boldsymbol{\tau}(\boldsymbol{\wp})$, respectively. Then, we have

$$\begin{aligned}f^{-1}(\lambda^{o_2})(x) = \lambda^{o_2}(f(x)) &= \sup_{\psi \in \boldsymbol{\wp}} j\left\langle \psi\langle \mathbf{1}_{f(x)}\rangle_T, \lambda\right\rangle \\ &= \sup_{\psi \in \boldsymbol{\wp}} \inf_{y \in Y} j\left(\psi\langle \mathbf{1}_{f(x)}\rangle_T(y), \lambda(y)\right) \\ &\leq \sup_{\psi \in \boldsymbol{\wp}} \inf_{z \in X} j(\psi(f(x), f(z)), \lambda(f(z))) \quad \text{for range } f \subseteq Y\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\psi\in\wp}\inf_{z\in X} j\big((f\times f)^{-1}(\psi)(x,z), f^{-1}(\lambda)(z)\big)\\
&\le \sup_{\varphi\in\boldsymbol{\mu}}\inf_{z\in X} j\big(\varphi(x,z), f^{-1}(\lambda)(z)\big), \quad \text{by uniformly continuous of } f \text{ and (IT8)},\\
&= \sup_{\varphi\in\boldsymbol{\mu}}\inf_{z\in X} j\Big(\varphi\langle \mathbf{1}_x\rangle_T(z), [f^{-1}(\lambda)](z)\Big)\\
&= [f^{-1}(\lambda)]^{o_1}(x),
\end{aligned}
\tag{3.21}
$$

which proves the continuity of $f:(X,\boldsymbol{\tau}(\boldsymbol{\mu}))\to(Y,\boldsymbol{\tau}(\wp))$ at each point x in X. □

4. The operators $\boldsymbol{\iota}_{u,\alpha}$ and ω_u

Now, we introduce the concept of α-level uniformities for a given fuzzy *TL*-uniformity and we study the relationships between them.

For a fuzzy *TL*-uniformity $\boldsymbol{\mu}$ on a set X and $\alpha\in I_1$, we define

$$
\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu}) = \Big\{\psi^{j(\beta,\alpha)} \subseteq X\times X : \psi\in\boldsymbol{\mu},\ \beta\in]\alpha,1]\Big\},
\tag{4.1}
$$

we next show that $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$ is a uniformity on X whenever $\boldsymbol{\mu}$ is a fuzzy *TL*-uniformity, called the α-level uniformity of $\boldsymbol{\mu}$.

In the following two propositions, let $\boldsymbol{v}$ be a basis for a fuzzy *TL*-uniformity $\boldsymbol{\mu}$ on a nonempty set X.

Proposition 4.1. *$\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ is a filterbasis for the filter $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$.*

Proof. We first show that $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ is indeed a filterbase.

(a) $\varnothing\notin\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$, because every member in $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ contains the diagonal of $X(D(X))$.

(b) The intersection of two members of $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ contains a member: given $U,V\in\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$, then there are $\psi,\varphi\in\boldsymbol{v}$ and $\beta_1,\beta_2\in]\alpha,1]$ such that $U=\varphi^{j(\beta_1,\alpha)}$, $V=\psi^{j(\beta_2,\alpha)}$, without loss of generality, $\beta_1\le\beta_2$. So

$$
\begin{aligned}
U\cap V &= \varphi^{j(\beta_1,\alpha)}\cap\psi^{j(\beta_2,\alpha)}\\
&\supseteq \varphi^{j(\beta_1,\alpha)}\cap\psi^{j(\beta_1,\alpha)}, \quad \text{because } j(\beta_2,\alpha)\le j(\beta_1,\alpha)\\
&= (\varphi\wedge\psi)^{j(\beta_1,\alpha)}.
\end{aligned}
\tag{4.2}
$$

But, there is $\zeta\in\boldsymbol{v}$ with $\zeta\le\varphi\wedge\psi$ (because $\boldsymbol{v}$ is a prefilterbase). Thus $U\cap V$ contains a member $\zeta^{j(\beta_1,\alpha)}$ of $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$.

To prove that $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$ is a filter, it now suffices to show that $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$ is closed under supersets: given $U\in\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$, let $V\supseteq U$. Then

$$
U=\psi^{j(\beta,\alpha)} \quad \text{for some } \psi\in\boldsymbol{\mu},\ \beta\in]\alpha,1].
\tag{4.3}
$$

If we take $\varphi = V \bigvee \underline{\underline{j(\beta,\alpha)}} \in I^{X\times X}$, we get $\varphi \geq \psi^{j(\beta,\alpha)} \bigvee \underline{\underline{j(\beta,\alpha)}} \geq \psi$ therefore, φ is also in $\boldsymbol{\mu}$ and

$$V = \left(V \vee \underline{\underline{j(\beta,\alpha)}}\right)^{j(\beta,\alpha)} = \varphi^{j(\beta,\alpha)} \in \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu}), \tag{4.4}$$

which proves our assertion.

Finally, we show that $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$ is generated by $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$: let $U \in \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$, then $U = \psi^{j(\beta,\alpha)}$ for some $\psi \in \boldsymbol{\mu}$ and $\beta \in]\alpha,1]$. Since $\boldsymbol{v} \subseteq \boldsymbol{v}^{\sim T} \subset \boldsymbol{\mu}$, then for all $\gamma \in I_1$, there exists $\varphi_\gamma \in \boldsymbol{v}$ with

$$\underline{\underline{\gamma}} T \varphi_\gamma \leq \psi. \tag{4.5}$$

By continuity of T, we can choose $\gamma_o \in I_1$ such that $\gamma_o T \beta \in]\alpha,1]$. Then

$$\begin{aligned}
\left(\varphi_{\gamma_o}\right)^{j(\gamma_o T\beta,\alpha)} &= \left(\varphi_{\gamma_o}\right)^{j(\gamma_o,j(\beta,\alpha))}, \quad \text{by (IT5)},\\
&= \left(\gamma_o T \varphi_{\gamma_o}\right)^{j(\beta,\alpha)}, \quad \text{by Lemma 2.7(i)},\\
&\subseteq \psi^{j(\beta,\alpha)}, \quad \text{by (4.5)},\\
&= U.
\end{aligned} \tag{4.6}$$

Thus there is a member $(\varphi_{\gamma_o})^{j(\gamma_o T\beta,\alpha)}$ of $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ contained in U. Since also $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ is included in $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$.

This completes the proof that it is a filterbasis for $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$. □

Proposition 4.2. *$\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ is a uniform basis for the uniformity $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$.*

Proof. $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ is a uniform base.

(UB1) $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ is a filterbase (from Proposition 4.1).

(UB2) Each member of $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ evidently contains the diagonal $D(X)$: since $\varphi(x,x) = 1 > j(\beta,\alpha)$ for all $(\varphi,x,\beta) \in \boldsymbol{v}\times X\times]\alpha,1]$, then $(x,x) \in \varphi^{j(\beta,\alpha)}$ for all $(\varphi^{j(\beta,\alpha)},x) \in (\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v}))\times X$.

(UB3) For every $U \in \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$, its symmetric ${}_sU$ contains a member of $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$: given $U \in \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$, then there exist $\varphi \in \boldsymbol{v}$ and $\beta \in]\alpha,1]$, such that $U = \varphi^{j(\beta,\alpha)}$, hence for all $\gamma \in I_1$, there exists $\varphi_\gamma \in \boldsymbol{v}$ with

$$\underline{\underline{\gamma}} T \varphi_\gamma \leq {}_s\varphi. \tag{4.7}$$

By continuity of T, we can get $\gamma_o \in I_1$ such that $\gamma_o T\beta \in]\alpha,1]$. Then

$$\begin{aligned}
\left(\varphi_{\gamma_o}\right)^{j(\gamma_o T\beta,\alpha)} &= \left(\varphi_{\gamma_o}\right)^{j(\gamma_o,j(\beta,\alpha))}, \quad \text{by (IT5)},\\
&= \left(\gamma_o T \varphi_{\gamma_o}\right)^{j(\beta,\alpha)}, \quad \text{by Lemma 2.7(i)},\\
&\subseteq ({}_s\varphi)^{j(\beta,\alpha)}, \quad \text{by (4.7)},\\
&= {}_s\left(\varphi^{j(\beta,\alpha)}\right), \quad \text{by Lemma 2.8},\\
&= {}_sU.
\end{aligned} \tag{4.8}$$

Therefore, ${}_sU$ contains the member $(\varphi_{\gamma_o})^{j(\gamma_o T\beta,\alpha)}$ of $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$.

(UB4) For every $U \in \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$, there exists $V \in \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ such that $VoV \subset U$: given $U \in \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$, there are $\varphi \in \boldsymbol{v}$ and $\beta \in]\alpha,1]$, such that $U = \varphi^{j(\beta,\alpha)}$. Also, for all $\gamma \in I_1$, there exists $\varphi_\gamma \in \boldsymbol{v}$ with

$$\underline{\underline{\gamma}}T(\varphi_\gamma o_T \varphi_\gamma) \leq \varphi. \tag{4.9}$$

By continuity of T, we can choose $\gamma_o \in I_1$ such that $\gamma_o \geq \beta$ and $\gamma_o T\beta \in]\alpha,1]$. Then

$$\begin{aligned} U = \varphi^{j(\beta,\alpha)} &\supseteq \Big[\underline{\underline{\gamma}}_o T(\varphi_{\gamma_o} o_T \varphi_{\gamma_o})\Big]^{j(\beta,\alpha)} \\ &= (\varphi_{\gamma_o} o_T \varphi_{\gamma_o})^{j(\gamma_o, j(\beta,\alpha))}, \quad \text{by Lemma 2.7(i)}, \\ &= (\varphi_{\gamma_o} o_T \varphi_{\gamma_o})^{j(\gamma_o T\beta,\alpha)}, \quad \text{by (IT5)}, \\ &= \bigcup_{\theta T\delta \leq j(\gamma_o T\beta,\alpha)} \Big[(\varphi_{\gamma_o})^\theta o (\varphi_{\gamma_o})^\delta\Big], \quad \text{by Lemma 2.7(ii)}, \\ &\supseteq \bigcup_{\theta T\delta \leq j(\gamma_o T\beta,\alpha T\alpha)} \Big[(\varphi_{\gamma_o})^\theta o (\varphi_{\gamma_o})^\delta\Big], \quad \text{by (IT8)}, \\ &\supseteq (\varphi_{\gamma_o})^{j(\gamma_o,\alpha)} o (\varphi_{\gamma_o})^{j(\beta,\alpha)}, \quad \text{by (IT7)}, \\ &\supseteq (\varphi_{\gamma_o})^{j(\beta,\alpha)} o (\varphi_{\gamma_o})^{j(\beta,\alpha)}, \quad \text{by (IT8) for } \gamma_o \geq \beta. \end{aligned} \tag{4.10}$$

Taking $V = (\varphi_{\gamma_o})^{j(\beta,\alpha)}$, which is in $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$, proves our assertion. This completes the proof that $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ is a uniform base.

Consequently, by Proposition 4.1, $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$ is uniformity with $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{v})$ a basis. □

Corollary 4.3. *Let $(X,\boldsymbol{\mu})$ be a fuzzy TL-uniform space, in the cases $T = T_m$,* min, *the level uniformities of $(X,\boldsymbol{\mu})$ form an ascending chain, while in the case $T = \pi$, the level uniformities form an antichain.*

Proof. Let $0 \leq \alpha_1 < \alpha_2 < 1$ and $V \in \boldsymbol{\iota}_{u,\alpha_1}(\boldsymbol{\mu})$.

Then there are $\varphi \in \boldsymbol{\mu}$ and $\beta \in]\alpha_1,1]$ such that $V = \varphi^{j(\beta,\alpha_1)}$. First, whenever $T = T_m$, we have

$$\begin{aligned} j(\beta,\alpha_1) &= \min\{1-\beta+\alpha_1, 1\}, \quad \text{by (2.2)}, \\ &= \min\{1-\beta+\alpha_1-\alpha_2+\alpha_2, 1\} \\ &= j(\alpha_2+\beta-\alpha_1, \alpha_2), \end{aligned} \tag{4.11}$$

hence, $V = \varphi^{j(\beta,\alpha_1)} = \varphi^{j(\alpha_2+\beta-\alpha_1,\alpha_2)} \in \boldsymbol{\iota}_{u,\alpha_2}(\boldsymbol{\mu})$, because it is easy to see that $\alpha_2+\beta-\alpha_1 \in]\alpha_2,1]$, since $\beta > \alpha_1$, which proves that $\boldsymbol{\iota}_{u,\alpha_1}(\boldsymbol{\mu}) \subseteq \boldsymbol{\iota}_{u,\alpha_2}(\boldsymbol{\mu})$.

Second, whenever $T = \text{Min}$, we have

$$j(\beta,\alpha) = \alpha \quad \forall \beta > \alpha, \text{ by (2.2)}. \tag{4.12}$$

Hence

$$\begin{aligned} V &= \varphi^{j(\beta,\alpha_1)} = \varphi^{\alpha_1} \\ &\supseteq \varphi^{\alpha_2}, \quad \text{because } \alpha_1 < \alpha_2, \\ &= \varphi^{j(\gamma,\alpha_2)} \quad \text{for any } \gamma \in \,]\alpha_2, 1]. \end{aligned} \tag{4.13}$$

But for every $\gamma \in]\alpha_2, 1]$, $\varphi^{j(\gamma,\alpha_2)} \in \boldsymbol{\iota}_{u,\alpha_2}(\boldsymbol{\mu})$, $V \in \boldsymbol{\iota}_{u,\alpha_2}(\boldsymbol{\mu})$, because $\boldsymbol{\iota}_{u,\alpha_2}(\boldsymbol{\mu})$ is a filter. Hence $\boldsymbol{\iota}_{u,\alpha_1}(\boldsymbol{\mu}) \subseteq \boldsymbol{\iota}_{u,\alpha_2}(\boldsymbol{\mu})$.

Finally, whenever $T = \pi$, we have

$$j(\beta, \alpha) = \frac{\alpha}{\beta} \quad \forall \beta > \alpha, \text{ by (2.2)}. \tag{4.14}$$

Let $W \in \boldsymbol{\iota}_{u,\alpha_2}(\boldsymbol{\mu})$. Then $W = \psi^{j(\beta_o,\alpha_2)}$ for some $\psi \in \boldsymbol{\mu}$ and $\beta_o \in]\alpha_2, 1]$. For this number β_o, take $\gamma_0 = \alpha_1(\beta_o/\alpha_2)$, we get

$$W = \psi^{j(\beta_o,\alpha_2)} = \psi^{(\alpha_2/\beta_o)} = \psi^{(\alpha_1/\gamma_0)} = \psi^{j(\gamma_o,\alpha_1)} \in \boldsymbol{\iota}_{u,\alpha_1}(\boldsymbol{\mu}) \tag{4.15}$$

because, obviously, we can see that $\gamma_o \in]\alpha_1, 1]$, which proves that $\boldsymbol{\iota}_{u,\alpha_2}(\boldsymbol{\mu}) \subseteq \boldsymbol{\iota}_{u,\alpha_1}(\boldsymbol{\mu})$. □

Proposition 4.4. *If $(X, \boldsymbol{\mu})$ and $(Y, \wp)$ are fuzzy TL-uniform spaces, and $f : X \to Y$ is a uniform map, then it is also a uniform map when considered as a function between the uniform spaces $(X, \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu}))$ and $(Y, \boldsymbol{\iota}_{u,\alpha}(\wp))$ for all $\alpha \in I_1$.*

The proof immediately follows from the definition of α-level uniformities and the fact that $(f \times f)^{-1}(\psi^\beta) = ((f \times f)^{-1}(\psi))^\beta$ for all $\psi \in \wp$, $\beta \in I_1$.

For every fixed $\alpha \in I_1$, define a function $\boldsymbol{\iota}^{\sim}_{u,\alpha}$ from the category of fuzzy TL-uniform spaces and fuzzy uniform maps to the category of uniform spaces and uniform maps by

$$\begin{aligned} \text{on objects}: \boldsymbol{\iota}^{\sim}_{u,\alpha} \quad &\text{is } \boldsymbol{\iota}_{u,\alpha}, \\ \text{on morphisms}: \boldsymbol{\iota}^{\sim}_{u,\alpha} \quad &\text{is the identity function.} \end{aligned} \tag{4.16}$$

Then an obvious conclusion from the above proposition is that these $\boldsymbol{\iota}^{\sim}_{u,\alpha}$ are well-defined functors.

Now, for a uniformity $\mathbf{u}$ on a set X, we define

$$\omega_u(\mathbf{u}) = \left\{\psi \in I^{X \times X} : \forall \gamma \in I_1,\ \psi^\gamma \in \mathbf{u}\right\}, \tag{4.17}$$

we supply the proof that $\omega_u(\mathbf{u})$ is a fuzzy TL-uniformity on X.

Proposition 4.5. *If u is a uniformity on a set X, then $\omega_u(\mathbf{u})$ is a fuzzy TL-uniformity with* $\mathbf{u}$ *as a basis.*

Proof. First, it is easy to see that $\omega_u(\mathbf{u})$ is a fuzzy TL-uniform base. To show that $\omega_u(\mathbf{u})$ is T-saturated, let $\psi \in (\omega_u(\mathbf{u}))^{\sim T}$. Then

$$\forall \gamma \in I_1 \quad \exists \varphi_\gamma \in \omega_u(\mathbf{u}) \quad \text{with } \psi \geq \underline{\underline{\gamma}} T \varphi_\gamma. \tag{4.18}$$

For every $\alpha \in I_1$, choose $\gamma = \gamma_\alpha > \alpha$ in I_1 for which

$$\psi^\alpha \supseteq \left(\underline{\underline{\gamma}} T \varphi_\gamma\right)^\alpha = (\varphi_\gamma)^{j(\gamma,\alpha)}, \tag{4.19}$$

since $(\varphi_\gamma)^{j(\gamma,\alpha)} \in \mathbf{u}$, because $j(\gamma,\alpha) \in I_1$, then $\psi^\alpha \in \mathbf{u}$ because $\mathbf{u}$ is a filter, hence $\psi \in \omega_u(\mathbf{u})$. This shows that $(\omega_u(\mathbf{u}))^{\sim T} = \omega_u(\mathbf{u})$.

Second, to show that $\mathbf{u}$ is a basis for $\omega_u(\mathbf{u})$, we prove that $(\mathbf{u})^{\sim T} = \omega_u(\mathbf{u})$: let $\psi \in (\mathbf{u})^{\sim T}$. Then there is a family $(W_\beta \in \mathbf{u})_{\beta \in I_1}$ such that

$$\psi \geq \underline{\underline{\beta}} T \mathbf{1}_{W_\beta}. \tag{4.20}$$

Now, for every $\alpha \in I_1$, we choose $\gamma > \alpha$ in I_1 for which we get

$$\begin{aligned}\psi^\alpha &\supseteq \left(\underline{\underline{\gamma}} T \mathbf{1}_{W_\gamma}\right)^\alpha \\ &= \left(\underline{\underline{\gamma}} \wedge \mathbf{1}_{W_\gamma}\right)^\alpha \quad \text{for } \mathbf{1}_{W_\gamma} \text{ is crisp} \\ &= \mathbf{1}_{W_\gamma} = W_\gamma \in \mathbf{u},\end{aligned} \tag{4.21}$$

thus $\psi^\alpha \in \mathbf{u}$, because $\mathbf{u}$ is filter, consequently, $\psi \in \omega_u(\mathbf{u})$ renders $(\mathbf{u})^{\sim T} \subseteq \omega_u(\mathbf{u})$.

On the other hand, if $\varphi \in \omega_u(\mathbf{u})$, then

$$\varphi = \bigvee_{\alpha \in I_1} \left(\underline{\underline{\alpha}} \wedge \varphi^\alpha\right) \geq \bigvee_{\alpha \in I_1} \left(\underline{\underline{\alpha}} T \varphi^\alpha\right). \tag{4.22}$$

Since $\varphi^\alpha \in \mathbf{u}$, for all $\alpha \in I_1$, we get $\varphi \in (u)^{\sim T}$, therefore, $\omega_u(\mathbf{u}) \subseteq (\mathbf{u})^{\sim T}$, and hence equality holds. □

Proposition 4.6. *If $(X,\mathbf{u})$ and $(Y,\mathcal{W})$ are uniform spaces, and $f : X \to Y$ is uniformly continuous in the usual sense, then it is uniformly continuous when considered as a function between the fuzzy TL-uniform spaces $(X,\omega_u(\mathbf{u}))$ and $(Y,\omega_u(\mathcal{W}))$.*

Now, if we denote by TL-FUS (US) the category of fuzzy TL-uniform spaces (the category of uniform spaces) together with the uniformly continuous functions between these spaces, and define the map $\omega^\sim : \text{US} \to TL\text{-FUS}$ by setting $\omega^\sim(X,u) = (X,\omega_u(u))$ and $\omega^\sim(f) = f$, we get that $\omega^\sim$ is a well-defined functor.

In the following, we denote the fuzzy topology associated with a fuzzy TL-uniformity $\boldsymbol{\mu}$ by $\boldsymbol{\tau}(\boldsymbol{\mu})$ and the topology associated with a uniformity u by $\mathbf{T}(u)$.

Now, we introduce some compatibility between the above notions.

Theorem 4.7. (i) *The α-level topology $\boldsymbol{\iota}_\alpha(\boldsymbol{\tau}(\boldsymbol{\mu}))$ of the fuzzy T-locality space $(X,\boldsymbol{\tau}(\boldsymbol{\mu}))$, coincides with the topology $\mathbf{T}(\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu}))$ on X, induced by the α-level uniformity $\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})$ of a fuzzy TL-uniform space $(X,\boldsymbol{\mu})$.*

(ii) *For a uniform space (X,u), the fuzzy topology $\boldsymbol{\tau}(\omega_u(\mathbf{u}))$ induced by the above fuzzy TL-uniformity $\omega_u(\mathbf{u})$ is the same fuzzy topology $\omega(\mathbf{T}(\mathbf{u}))$*

(iii) $\boldsymbol{\iota}^\sim_{u,\alpha} o \omega^\sim = Id_{US}$ *for all $\alpha \in I_1$.*

Proof. (i) Let o be the fuzzy interior operator of the T-FLS $(X,\boldsymbol{\tau}(\boldsymbol{\mu}))$. Then for every nonempty $M \in 2^X$,

$$\begin{aligned}
\operatorname{int}_{\boldsymbol{\iota}\alpha(\boldsymbol{\tau}(\boldsymbol{\mu}))} M &= \left(\left(\underline{\underline{\alpha}} \vee M\right)^{o}\right)^{\alpha}, \quad \text{by (2.10)}, \\
&= \left\{x : \left(\underline{\underline{\alpha}} \vee M\right)^{o}(x) > \alpha\right\} \\
&= \left\{x : \sup_{\psi \in \boldsymbol{\mu}} \inf_{y \in X} j\left(\psi\langle \mathbf{1}_x\rangle_T(y), \left(\underline{\underline{\alpha}} \vee M\right)(y)\right) > \alpha\right\}, \quad \text{by Definition 2.4}, \\
&= \left\{x : \exists \psi \in \boldsymbol{\mu},\ \beta \in]\alpha,1[\text{ s.t. } \forall y \notin M,\ j(\psi(x,y),\alpha) \geq \beta\right\} \\
&= \left\{x : \exists \psi \in \boldsymbol{\mu},\ \beta \in]\alpha,1[\text{ s.t. } \forall y \notin M,\ \beta T \psi(x,y) \leq \alpha\right\}, \quad \text{by (IT1)}, \\
&= \left\{x : \exists \psi \in \boldsymbol{\mu},\ \beta \in]\alpha,1[\text{ s.t. } \forall y \notin M,\ (x,y) \notin \left(\underline{\underline{\beta}} T \psi\right)^{\alpha}\right\} \\
&= \left\{x : \exists \psi \in \boldsymbol{\mu},\ \beta \in]\alpha,1[\text{ s.t. } \forall y \notin M,\ (x,y) \notin \psi^{j(\beta,\alpha)}\right\} \\
&= \left\{x : \exists V \in \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu}) \text{ s.t. } \forall y \notin M,\ (x,y) \notin V\right\} \\
&= \left\{x : \exists V \in \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu}) \text{ s.t. } \forall y \notin M,\ y \notin V\langle x\rangle\right\} \\
&= \left\{x : V\langle x\rangle \subseteq M \text{ for some } V \in \boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})\right\} \\
&= \operatorname{int}_{\mathbf{T}(\boldsymbol{\iota}_{u,\alpha}(\mu))} M.
\end{aligned} \tag{4.23}$$

This demonstrates that $\boldsymbol{\iota}_\alpha(\boldsymbol{\tau}(\boldsymbol{\mu}))) = \mathbf{T}(\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu}))$, which renders (i).

(ii) Let the fuzzy set λ be $\boldsymbol{\tau}(\omega(\mathbf{u}))$-open.

If $x \notin (\operatorname{int}_{\mathbf{T}(u)} \lambda^{\varepsilon})$ for some $x \in X$ and $\varepsilon \in I_1$, we get $V\langle x\rangle \not\subset \lambda^{\varepsilon}$, for all $V \in \mathbf{u}$, thus, there exists $z \in X$ such that $z \in V\langle x\rangle$ for all $V \in \mathbf{u}$ with $z \notin \lambda^{\varepsilon}$, that is, there exists $z \in X$ such that $z \in V\langle x\rangle$ for all $V \in u$ with $\lambda(z) \leq \varepsilon$, hence

$$\begin{aligned}
\lambda(x) &= \left(\operatorname{int}_{\boldsymbol{\tau}(\omega(\mathbf{u}))} \lambda\right)(x), \quad \text{by hypothesis} \\
&= \sup_{\psi \in \omega(\mathbf{u})} \inf_{y \in X} j\left(\psi\langle \mathbf{1}_x\rangle_T(y), \lambda(y)\right) \\
&= \sup_{V \in \mathbf{u}} \inf_{y \in X} j\left(\mathbf{1}_V\langle \mathbf{1}_x\rangle_T(y), \lambda(y)\right), \quad \text{by Proposition 4.4}, \\
&\leq \sup_{V \in \mathbf{u}} j\left(\mathbf{1}_V\langle \mathbf{1}_x\rangle_T(z), \lambda(z)\right) \\
&= j(1,\lambda(z)) \\
&= \lambda(z), \quad \text{by (IT4)}, \\
&\leq \varepsilon.
\end{aligned} \tag{4.24}$$

That is, $x \notin \lambda^{\varepsilon}$, which shows that $\lambda^{\varepsilon} \subseteq \operatorname{int}_{\mathbf{T}(u)} \lambda^{\varepsilon}$, $\varepsilon \in I_1$. Consequently, $\lambda^{\varepsilon} = \operatorname{int}_{\mathbf{T}(u)} \lambda^{\varepsilon}$, that is, $\lambda^{\varepsilon} \in \mathbf{T}(u)$ for all $\varepsilon \in I_1$.

On the other hand, let λ be $\omega(\mathbf{T}(u))$-open.

Given $x \in X$, then for all $\varepsilon \in I_1$ with $\varepsilon < \lambda(x)$, we have $x \in \lambda^{\varepsilon}$, and hence, by hypothesis $x \in \operatorname{int}_{\mathbf{T}(u)} \lambda^{\varepsilon}$, thus there exists $W \in u$ such that $W\langle x\rangle \subset \lambda^{\varepsilon}$, that is, there exists $W \in u$ such that for all $z \in W\langle x\rangle$, we get $z \in \lambda^{\varepsilon}$, that is, there exists $W \in u$ such that for all $z \in W\langle x\rangle$, we get $\lambda(z) > \varepsilon$. Hence,

$$\begin{aligned}\left(\operatorname{int}_{\boldsymbol{\tau}(\omega(\mathbf{u}))}\lambda\right)(x) &= \sup_{\psi\in\omega(\mathbf{u})}\inf_{y\in X} j\Big(\psi\langle \mathbf{1}_x\rangle_T(y),\lambda(y)\Big)\\ &= \sup_{V\in\mathbf{u}}\inf_{y\in X} j\Big(\mathbf{1}_V\langle \mathbf{1}_x\rangle_T(y),\lambda(y)\Big)\\ &\geq \inf_{y\in X} j\Big(\mathbf{1}_W\langle \mathbf{1}_x\rangle_T(y),\lambda(y)\Big) \geq \varepsilon.\end{aligned} \tag{4.25}$$

This shows that $\lambda \leq \operatorname{int}_{\boldsymbol{\tau}(\omega(\mathbf{u}))}\lambda$, which proves that λ is $\boldsymbol{\tau}(\omega_u(\mathbf{u}))$-open, and winds up the proof of (ii).

(iii) Follows immediately from the definitions. □

Example 4.8. Let $(X,<)$ be an ordered set. For each point $x_o \in X$, let $V_{x_o} = \{(x,y) : x,y > x_o\}$ and define $U_{x_o} = D(X) \cup V_{x_o}$. Then the reader can easily check that the collection $\mathbf{u}^* = \{U_x : x \in X\}$ is a basis for a uniformity $\mathbf{u}$ on X. Hence by the above notions, we have

$$\omega_u(\mathbf{u}) = \Big\{\psi \in I^{X\times X} : \forall \gamma \in I_1,\ \psi^{\gamma} \in \mathbf{u}\Big\} \tag{4.26}$$

is a fuzzy TL-uniformity on X with $\mathbf{u}$ as a basis. Moreover,

$$\boldsymbol{\iota}_{u,\alpha}(\omega_u(\mathbf{u})) = \Big\{\psi^{j(\beta,\alpha)} \subseteq X\times X : \omega \in \omega_u(\mathbf{u}),\ \beta \in]\alpha,1]\Big\} = \mathbf{u}. \tag{4.27}$$

Definition 4.9. A fuzzy topological space $(X,\boldsymbol{\tau})$ is called a fuzzy TL-uniformizable if there is a fuzzy TL-uniformity $\boldsymbol{\mu}$ on X such that $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{\mu})$.

Corollary 4.10. *A topological space $(X,\mathbf{T})$ is uniformizable if and only if $(X,\omega(\mathbf{T}))$ is fuzzy TL-uniformizable.*

Proof. If $\mathbf{T} = \mathbf{T}(u)$, then

$$\omega(\mathbf{T}) = \omega(\mathbf{T}(u)) = \boldsymbol{\tau}(\omega(\mathbf{u})). \tag{4.28}$$

Conversely, let $\omega(\mathbf{T}) = \boldsymbol{\tau}(\boldsymbol{\mu})$, then

$$\begin{aligned}\mathbf{T} = \boldsymbol{\iota}(\omega(\mathbf{T})) &= \sup_{\alpha\in I_1}\boldsymbol{\iota}_{\alpha}(\omega(\mathbf{T})), \quad \text{by definition of modification topology,}\\ &= \sup_{\alpha\in I_1}\boldsymbol{\iota}_{\alpha}(\boldsymbol{\tau}(\boldsymbol{\mu})), \quad \text{by hypothesis,}\\ &= \sup_{\alpha\in I_1}\mathbf{T}(\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})), \quad \text{by Theorem 4.7(i),}\\ &= \mathbf{T}\Big(\sup_{\alpha\in I_1}\boldsymbol{\iota}_{u,\alpha}(\boldsymbol{\mu})\Big).\end{aligned} \tag{4.29}$$

□

Definition 4.11. A T-fls $(X,\boldsymbol{\tau}(\mathscr{B}))$ is said to be the following.

(i) L-T_0, if for every $x \neq y$ in X, there is $v \in \mathscr{B}(x) \cup \mathscr{B}(y)$ such that

$$v(x) \wedge v(y) < 1. \tag{4.30}$$

(ii) L-regular, if for every $(M,x,\varepsilon) \in 2^X \times X \times I_0$, there is $v \in \mathscr{B}(x)$ with $\sup_{y\in X}(v \wedge \mathbf{1}_M)(y) < \varepsilon$, then there are an open set μ and $\rho \in \mathscr{B}(x)$ such that $\mathbf{1}_M \leq \mu$ and $\sup_{y\in X}(\mu T \rho)(y) < \varepsilon$.

Theorem 4.12. *TL-uniformizability $\Rightarrow$ L-regularity.*

Proof. Suppose $(X,\boldsymbol{\mu})$ is a fuzzy TL-uniform space.

Let $M \in 2^X$, $x \in X$, and $\varepsilon \in I_0$ are such that there is $v \in \mathscr{B}(x)$ with

$$\sup_{y\in X}(v \wedge \mathbf{1}_M)(y) < \varepsilon, \tag{4.31}$$

consequently, we can find ε_0 very small such that

$$\sup_{y\in X}(v \wedge \mathbf{1}_M)(y) + \varepsilon_0 < \varepsilon. \tag{4.32}$$

Since, $v \in \mathscr{B}(x)$, then for all $\gamma \in I_1$ there are $\psi, \psi_\gamma \in \boldsymbol{\mu}$ such that $v = \psi\langle \mathbf{1}_x\rangle_T$ and $\psi \geq \underline{\underline{\gamma}}T(\psi_\gamma o_T \psi_\gamma)$, (by using (FLUB4)). Hence

$$\begin{aligned}
\varepsilon &> \sup_{z\in X}\Big(\psi\langle \mathbf{1}_x\rangle_T \wedge \mathbf{1}_M\Big)(z) + \varepsilon_0 \\
&\geq \sup_{z\in M}\Big[\underline{\underline{\gamma}}T(\omega_\gamma o_T \psi_\gamma)\langle \mathbf{1}_x\rangle_T\Big](z) + \varepsilon_0 \\
&= \sup_{z\in M}\big[\gamma T(\psi_\gamma o_T \psi_\gamma)(x,y)\big] + \varepsilon_0 \\
&= \Big\{\gamma T \sup_{z\in M}\sup_{y\in X}\big[\psi_\gamma(x,y)T\psi_\gamma(y,z)\big]\Big\} + \varepsilon_0 \\
&= \Big\{\gamma T \sup_{y\in X}\sup_{z\in M}\big[\psi_\gamma(x,y)T_s\psi_\gamma(z,y)\big]\Big\} + \varepsilon_0 \\
&\geq (\gamma+\theta)T\sup_{y\in X}\left[\psi_\gamma\langle \mathbf{1}_x\rangle_T(y)T\bigvee_{z\in M}{}_s\psi_\gamma\langle \mathbf{1}_z\rangle_T(y)\right], \quad \theta = \theta_{T,\varepsilon_0} > 0 \text{ as in (2.1)}.
\end{aligned} \tag{4.33}$$

Choosing $\gamma_0 \in I_1$ for which $(\gamma_0 + \theta) = 1$, and taking $\rho = {}_s\psi_{\gamma_0}\langle \mathbf{1}_x \rangle_T$, $\mu = (\bigvee_{z\in M} {}_s\psi_{\gamma_0}\langle \mathbf{1}_z \rangle_T)^o$, we get $\rho \in \mathcal{B}(x)$, and μ is open, with $\varepsilon > \sup_{y\in X}[\rho(y)T\mu(y)]$, also $\mu \geq \mathbf{1}_M$, because for every $x \in M$, we have

$$\begin{aligned}
\mu(x) &= \left(\bigvee_{z\in M} {}_s\psi_{\gamma_0}\langle \mathbf{1}_z \rangle_T\right)^o (x) \\
&\geq \left({}_s\psi_{\gamma_0}\langle \mathbf{1}_x \rangle_T\right)^o (x) \\
&= (v_x)^o(x) \quad \text{for } {}_s\psi_{\gamma_0} \in \mu \\
&= 1, \quad \text{by Theorem 2.3.}
\end{aligned} \tag{4.34}$$

This proves the L-regularity of $(X, \boldsymbol{\tau}(\boldsymbol{\mu}))$. □

Theorem 4.13. *If $(X,\boldsymbol{\mu})$ is a fuzzy TL-uniform space, then $(X,\boldsymbol{\tau}(\boldsymbol{\mu}))$ is L-T_0 if and only if $(\inf_{\psi\in\boldsymbol{\mu}} \psi)_{1^*} = D(X)$.*

Proof. A T-FLS $(X,\boldsymbol{\tau}(\boldsymbol{\mu}))$ is L-T_0

$$\begin{aligned}
&\text{iff} \quad \forall x \neq y \text{ in } X,\ \exists v \in \mathcal{B}(x) \cup \mathcal{B}(y) \text{ such that } v(x) \wedge v(y) < 1 \\
&\text{iff} \quad \forall x \neq y \text{ in } X,\ \exists \psi \in \boldsymbol{\mu} \text{ such that } \psi\langle \mathbf{1}_x \rangle_T(x) \wedge \psi\langle \mathbf{1}_x \rangle_T(y) < 1 \\
&\text{iff} \quad \forall x \neq y \text{ in } X,\ \exists \psi \in \boldsymbol{\mu} \text{ such that } \psi(x,y) = \psi\langle \mathbf{1}_x \rangle_T(y) < 1 \\
&\text{iff} \quad \forall x \neq y \text{ in } X,\ \exists \psi \in \boldsymbol{\mu} \text{ such that } (x,y) \notin \psi_{1^*} \\
&\text{iff} \quad \left(\inf_{\psi\in\boldsymbol{\mu}}\right)_{1^*} = \bigcap_{\psi\in\boldsymbol{\mu}} (\psi_{1^*}) = D(X).
\end{aligned} \tag{4.35}$$

□

Theorem 4.14. *Let Γ be a probabilistic pseudometric on a set X. Then there is a fuzzy TL-uniformity $\boldsymbol{\mu} = \boldsymbol{\mu}(\Gamma)$ given by*

$$\boldsymbol{\mu}(\Gamma) = \left\{\psi \in I^{X\times X} : \forall \gamma \in I_1\ \exists n \in N \text{ with } \gamma T\Gamma(x,y)(2^{-n}) \leq \psi(x,y)\right\}. \tag{4.36}$$

Proof. Obviously, $\boldsymbol{\mu}(\Gamma)$ is T-saturated prefilter.

Now, if we put $\psi_n(x,y) = \Gamma(x,y)(2^{-n})$, we get a sequence $(\psi_n)_{n\in N}$ satisfies $\psi_n(x,x) = 1$, $\psi_n = {}_s(\psi_n)$, and $\psi_{n+1} o_T \psi_{n+1} \leq \psi_n$, which shows that $\boldsymbol{\mu}(\Gamma)$ is a fuzzy TL-uniformity. □

Remark 4.15. Let $\boldsymbol{\mu}$ be a fuzzy TL-uniformity on X, a nonempty subset $\mathcal{B}$ of $\boldsymbol{\mu}$ is said to be a base of $\boldsymbol{\mu}$ if for each $\psi \in \boldsymbol{\mu}$ and each $n \in N$, there is $\varphi_n \in \mathcal{B}$ such that $\underline{\underline{(1-1/n)}}T\varphi_n \leq \psi$.

Theorem 4.16. *A fuzzy TL-uniformity $\boldsymbol{\mu}$ on a set X is probabilistic pseudo-metrizable if and only if it has a countable base.*

Proof. Let $\boldsymbol{\mu}$ be probabilistic pseudo-metrizable. Then by using (PM1) and Theorem 4.14, we get the necessary condition. For the sufficiency of the condition, suppose that $\boldsymbol{\mu}$ has a countable base $\mathcal{B} = \{\psi_1, \psi_2, \ldots, \psi_n, \ldots\}$.

We set $\varphi_1 = \psi_1 \wedge {}_s\psi_1$, then φ_1 is a symmetric member of $\boldsymbol{\mu}$.

For a fixed γ in I_1, apply (FLUB4) twice for the member $\varphi_1 \wedge \psi_2$ which is in $\boldsymbol{\mu}$, then there is a symmetric member say φ_2 in $\boldsymbol{\mu}$ such that

$$\underline{\underline{\gamma}} T(\varphi_2 o_T \varphi_2 o_T \varphi_2) \leq \varphi_1 \wedge \psi_2. \tag{4.37}$$

Next, consider $\varphi_2 \wedge \psi_3$ and get a symmetric φ_3 in $\boldsymbol{\mu}$ such that

$$\underline{\underline{\gamma}} T(\varphi_3 o_T \varphi_3 o_T \varphi_3) \leq \varphi_2 \wedge \psi_3. \tag{4.38}$$

In this manner, we proceed by induction a sequence $(\varphi_n)_{n\in N}$ of fuzzy vicinities for which

$$\varphi_n = {}_s(\varphi_n), \quad \underline{\underline{\gamma}} T(\varphi_{n+1} o_T \varphi_{n+1} o_T \varphi_{n+1}) \leq \varphi_n \wedge \psi_{n+1} \leq \varphi_n, \tag{4.39}$$

by using Lemma 2.9, there is a probabilistic pseudometric Γ on X, with

$$\underline{\underline{\gamma}} T \varphi_{n+1}(x,y) \leq \Gamma(x,y)(2^{-n}) \leq \varphi_n(x,y); \tag{4.40}$$

that is, the fuzzy TL-uniformity $\boldsymbol{\mu}$ is generated by Γ, which proves our assertion. □

5. Optimal lifts in fuzzy TL-uniform spaces

We prove that the category of fuzzy TL-uniform spaces and uniform maps between them is a topological category, by constructing optimal lifts of sources in it. For expansion on the categorical notions mentioned below, see [15].

The category FTS of fuzzy topological spaces and their continuous functions is a topological category. This means that its objects are sets with structures (here, fuzzy topologies), its morphisms are admissible functions between those sets (here, the continuous functions), morphism composition and the identity morphisms are the usual ones for functions, and the following three conditions are satisfied for all nonempty sets X.

(1) Every source $(f_j : X \to (X_j, \sigma_j)_{j\in J})$ of functions in the category FTS has a unique optimal lift (also called initial lift), namely, the coarsest structure on X making each function f_j a morphism. Specifically, the optimal lift of that source is the fuzzy topology $\boldsymbol{\tau}$ on X with subbasis [16]:

$$\boldsymbol{\tau}^* = \left\{ f_j^{-1}(\lambda) \in I^X : j \in J,\ \lambda \in \boldsymbol{\sigma}_j \right\}, \tag{5.1}$$

that is, $\boldsymbol{\tau}$ is the smallest fuzzy topology on X that contains $\boldsymbol{\tau}^*$.

(2) The class FTS(X), of all FTS-structures (i.e., fuzzy topologies) on X, is a set (smallness condition).

(3) If X is a singleton, then FTS(X) is a singleton.

An important construction is that of initial fuzzy TL-uniformities. Given a family $(f_j : X \to (X_j, \boldsymbol{\mu}_j))_{j\in J}$, where for each $j \in J$, $(X_j, \boldsymbol{\mu}_j)$ is a fuzzy TL-uniform space and f_j is a function from some set X to X_j, we want to construct on X (in accordance with the general definition of initial lifts in categories of sets with structures) a coarsest fuzzy TL-uniformity making each function f_j uniformly continuous. This fuzzy TL-uniformity is called the initial fuzzy TL-uniformity on X for the family $(f_j : X \to (X_j, \boldsymbol{\mu}_j))_{j\in J}$.

If $(Y,\wp)$ is a fuzzy TL-uniform space and $h: Y \to X$, then if X carries the initial fuzzy TL-uniformity, h is uniformly continuous if and only if each $f_j o h$ is uniformly continuous.

Lemma 5.1. *For a fuzzy TL-uniform base $\boldsymbol{v}$ on a set Y and a function $f: X \to Y$,*

$$(f\times f)^{-1}(\boldsymbol{v}^{\sim T}) \subseteq [(f\times f)^{-1}(\boldsymbol{v})]^{\sim T} = \left\{(f\times f)^{-1}(\varphi) : \varphi \in \boldsymbol{v}\right\}^{\sim T}. \tag{5.2}$$

Proof. Let $\psi \in (f\times f)^{-1}(\boldsymbol{v}^{\sim T})$ and $\gamma \in I_1$. Then there are $\varphi \in \boldsymbol{v}^{\sim T}$ and $\varphi_\gamma \in \boldsymbol{v}$, with $\psi = (f\times f)^{-1}(\varphi)$ and $\underline{\underline{\gamma}} T \varphi_\gamma \leq \varphi$.

By taking $\psi_\gamma = (f\times f)^{-1}(\varphi_\gamma) \in (f\times f)^{-1}(\boldsymbol{v})$, we have

$$\underline{\underline{\gamma}} T \psi_\gamma = \underline{\underline{\gamma}} T (f\times f)^{-1}(\varphi_\gamma) = (f\times f)^{-1}\left(\underline{\underline{\gamma}} T \varphi_\gamma\right) \leq (f\times f)^{-1}(\varphi) = \psi. \tag{5.3}$$

Therefore, $\psi \in [(f\times f)^{-1}(\boldsymbol{v})]^{\sim T}$, which completes the proof of the lemma. □

Proposition 5.2. *Let $f: X \to Y$ be a function. If $\boldsymbol{v}$ is a fuzzy TL-uniform base on Y, then $(f\times f)^{-1}(\boldsymbol{v})$ is a fuzzy TL-uniform base on X.*

Proof. (FLUB1) $(f\times f)^{-1}(\boldsymbol{v})$ is indeed a prefilterbase.

(FLUB2) For all $\psi \in (f\times f)^{-1}(\boldsymbol{v})$ and $x \in X$, we have

$$\begin{aligned}\psi(x,x) &= [(f\times f)^{-1}(\varphi)](x,x) \quad \text{for some } \varphi \in \boldsymbol{v}\\ &= \varphi(f(x), f(x))\\ &= 1.\end{aligned} \tag{5.4}$$

(FLUB3$^\backslash$) Let $\psi \in (f\times f)^{-1}(\boldsymbol{v})$ and $\gamma \in I_1$, we get φ and φ_γ in $\boldsymbol{v}$ such that

$$\psi = (f\times f)^{-1}(\varphi), \qquad \underline{\underline{\gamma}} T(\varphi_\gamma o_T \varphi_\gamma) \leq {}_s\varphi. \tag{5.5}$$

By taking $\psi_\gamma = (f\times f)^{-1}(\varphi_\gamma) \in (f\times f)^{-1}(\boldsymbol{v})$, we have

$$\begin{aligned}\underline{\underline{\gamma}} T(\psi_\gamma o_T \psi_\gamma) &= \underline{\underline{\gamma}} T[((f\times f)^{-1}(\varphi_\gamma)) o_T ((f\times f)^{-1}(\varphi_\gamma))]\\ &= \underline{\underline{\gamma}} T[(f\times f)^{-1}(\varphi_\gamma o_T \varphi_\gamma)]\\ &\leq (f\times f)^{-1}({}_s\varphi)\\ &= {}_s\psi,\end{aligned} \tag{5.6}$$

which completes the proof that $(f\times f)^{-1}(\boldsymbol{v})$ is a fuzzy TL-uniform base on X. □

By a similar proof, we also have the following proposition.

Proposition 5.3. *Let $f_j : X \to Y_j$ be a function, $j \in J$. If $\boldsymbol{v}_j$ is a fuzzy TL-uniform base on Y_j for every $j \in J$, then $\bigvee_{j\in J}(f \times f)^{-1}(\boldsymbol{v}_j)$ is a fuzzy TL-uniform base on X, where*

$$\bigvee_{j\in J}(f \times f)^{-1}(\boldsymbol{v}_j) = \left\{ \bigwedge_{j\in J_1}(f \times f)^{-1}(\varphi_j) : J_1 \text{ is a finite subset of } J,\ \varphi_j \in \boldsymbol{v}_j\ \forall j \right\}. \tag{5.7}$$

Theorem 5.4. *A source in TL-FUS $(f_j : X \to (Y_j,\boldsymbol{\mu}_j))_{j\in J}$ has an optimal lift in TL-FUS, with fuzzy TL-uniform base*

$$\boldsymbol{v} = \bigvee_{j\in J}(f_j \times f_j)^{-1}(\boldsymbol{\mu}_j). \tag{5.8}$$

Proof. Since $(f_j \times f_j)^{-1}(\boldsymbol{\mu}_j) \subseteq \bigvee_{j\in J}(f_j \times f_j)^{-1}(\boldsymbol{\mu}_j) = \boldsymbol{v} \subseteq \boldsymbol{v}^{\sim T}$, then by Proposition 3.6, all f_j are uniformly continuous: $(X,\boldsymbol{v}^{\sim T}) \to (Y_j,\boldsymbol{\mu}_j)$.

Let $(Z,\wp)$ be in TL-FUS, and let a function $h : Z \to X$ be such that $f_j oh : (Z,\wp) \to (Y_j,\boldsymbol{\mu}_j)$ are uniformly continuous for all $j \in J$. Then by Proposition 3.6 again,

$$\begin{aligned} \wp &\supseteq \left[(f_j oh) \times (f_j oh)\right]^{-1}(\boldsymbol{\mu}_j) \\ &= \left[(f_j \times f_j)o(h \times h)\right]^{-1}(\boldsymbol{\mu}_j) \quad \forall j \in J, \text{ that is,} \\ \wp &\supseteq \bigvee_{j\in J}(h \times h)^{-1}\left[(f_j \times f_j)^{-1}(\boldsymbol{\mu}_j)\right] \\ &= (h \times h)^{-1}\left[\bigvee_{j\in J}(f_j \times f_j)^{-1}(\boldsymbol{\mu}_j)\right] \\ &= (h \times h)^{-1}(\boldsymbol{v}). \end{aligned} \tag{5.9}$$

Referring to Proposition 3.6, once more, we find that h is uniformly continuous: $(Z,\wp) \to (X,\boldsymbol{v}^{\sim T})$. This establishes that $(X,\boldsymbol{v}^{\sim T})$ is the optimal lift in TL-FUS of the given source. □

Proposition 5.5. *TL-FUS is a topological category.*

Definition 5.6. The functor $\mathfrak{t}^{\sim} : TL$-FUS $\to T$-FLS leaves the functions unaltered, and is defined on objects $(X,\boldsymbol{\mu})$ in TL-FUS by $\mathfrak{t}^{\sim}(X,\boldsymbol{\mu}) = (X,\boldsymbol{\tau}(\boldsymbol{\mu}))$.

Theorem 5.7. *The above $\mathfrak{t}^{\sim}$ is a well-defined functor. The proof follows immediately from Theorem 3.10.*

Theorem 5.8. *The functor $\mathfrak{t}^{\sim}$ preserves optimal lifts. The proof can be analogously as [17, Theorem 7.3].*

Theorem 5.9. *Fuzzy TL-uniformizability is an initial property.*

Proof. This follows from Proposition 5.5 and Theorem 5.8. □

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