ON SANDWICH THEOREMS FOR SOME CLASSES OF ANALYTIC FUNCTIONS

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The purpose of this present paper is to derive some subordination and superordination results for certain normalized analytic functions in the open unit disk. Relevant connections of the results, which are presented in the paper, with various known results are also considered.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in $\Delta := \{z : |z| < 1\}$, and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_2 z^2 + \cdots$. With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Then we say that the function f is subordinate to g if there exists a Schwarz function $\omega(z)$, analytic in Δ with

$$\omega(0) = 0, \qquad |\omega(z)| < 1 \quad (z \in \Delta), \tag{1.1}$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \Delta). \tag{1.2}$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta). \tag{1.3}$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0), \qquad f(\Delta) \subset g(\Delta). \tag{1.4}$$

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Let $p,h \in \mathcal{H}$ and let $\phi(r,s,t;z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z),zp'(z),z^2p''(z);z)$ are univalent and if p satisfies the second-order superordination

$$h(z) \prec \phi(p(z),zp'(z),z^2p''(z);z), \tag{1.5}$$

then p is a solution of the differential superordination (1.5). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.5). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.5) is said to be the best subordinant. Recently Miller and Mocanu [5] obtained conditions on h, q, and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z),zp'(z),z^2p''(z);z) \Longrightarrow q(z) \prec p(z). \tag{1.6}$$

Using the results of Miller and Mocanu [5], Bulboacă [3] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [2]. Ali et al. [1] have used the results of Bulboacă [3] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \tag{1.7}$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. [8] obtained sufficient conditions for normalized analytic functions $f(z)$ to satisfy

$$\begin{gathered} q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z), \\ q_1(z) \prec \frac{z^2f'(z)}{\{f(z)\}^2} \prec q_2(z), \end{gathered} \tag{1.8}$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$, while Obradović and Owa [7] obtained subordination results with the quantity $(f(z)/z)^{\mu}$ (see also [10]).

Obradović [6] introduced a class of functions $f \in \mathcal{A}$ such that for $0 < \alpha < 1$,

$$\Re\left\{f'(z)\left(\frac{z}{f(z)}\right)^{\alpha}\right\} > 0, \quad z \in \Delta. \tag{1.9}$$

He called this class of function "non-Bazilevič" type. Tuneski and Darus [11] obtained Fekete-Szegö inequality for the non-Bazilevič class of functions. Using this non-Bazilevič class, Wang et al. [12] studied many subordination results for the class $N(\alpha,\lambda,A,B)$ defined as

$$N(\alpha,\lambda,A,B) := \left\{f \in \mathcal{A} : (1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha} \prec \frac{1+Az}{1+Bz}\right\}, \tag{1.10}$$

where $\lambda \in \mathbb{C}$, $-1 \leq B \leq 1$, $A \neq B$, $0 < \alpha < 1$.

The main object of the present sequel to the aforementioned works is to apply a method based on the differential subordination in order to derive several subordination results. Furthermore, we obtain the previous results of Srivastava and Lashin [10], Singh [9] and Obradović and Owa [7] as special cases of some of the results presented here.

2. Preliminaries

In our present investigation, we will need the following definition and results.

Definition 2.1 (see [5, Definition 2, page 817]). Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \left\{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\right\}, \tag{2.1}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Theorem 2.2 (see [4, Theorem 3.4h, page 132]). *Let $q(z)$ be univalent in the unit disk Δ and let θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*
  (1) *$Q(z)$ is starlike univalent in Δ;*
  (2) *$\Re(zh'(z))/Q(z) > 0$ for $z \in \Delta$.*
If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.2}$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.3 (see [8]). *Let q be a convex univalent function in Δ and let $\psi, \gamma \in \mathbb{C}$ with $\Re(1 + (zq''(z)/q'(z))) > \max\{0, -\Re(\psi/\gamma)\}$. If $p(z)$ is analytic in Δ and*

$$\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z), \tag{2.3}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2.4 (see [4, Corollary 3.4h.1, page 135]). *Let $q(z)$ be univalent in Δ and let $\varphi(z)$ be analytic in a domain containing $q(\Delta)$. If $zq'(z)/\varphi(q(z))$ is starlike, and*

$$zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z)), \tag{2.4}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Theorem 2.5 (see [3]). *Let $q(z)$ be convex univalent in the unit disk Δ and let ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*
  (1) *$\Re[\vartheta'(q(z))/\varphi(q(z))] > 0$ for $z \in \Delta$;*
  (2) *$zq'(z)\varphi(q(z))$ is starlike univalent in Δ.*

If $p(z) \in \mathcal{H}[q(0),1] \cap Q$, with $p(\Delta) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.5}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Lemma 2.6 (see [5, Theorem 8, page 822]). *Let q be convex univalent in Δ and $\gamma \in \mathbb{C}$. Further assume that $\Re[\overline{\gamma}] > 0$. If $p(z) \in \mathcal{H}[q(0),1] \cap Q$, and $p(z) + \gamma zp'(z)$ is univalent in Δ, then*

$$q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z) \tag{2.6}$$

implies $q(z) \prec p(z)$ and q is the best subordinant.

3. Subordination for analytic functions

By using Lemma 2.3, we first prove the following.

Theorem 3.1. *Let q be univalent in Δ, $\lambda \in \mathbb{C}$, and $0 < \alpha < 1$. Suppose q satisfies*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left\{\frac{\lambda}{\alpha}\right\}\right\}. \tag{3.1}$$

If $f \in \mathcal{A}$ satisfies the subordination

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha} \prec q(z) + \frac{\lambda zq'(z)}{\alpha}, \tag{3.2}$$

then

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec q(z) \tag{3.3}$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{z}{f(z)}\right)^{\alpha}. \tag{3.4}$$

Then

$$\frac{zp'(z)}{p(z)} = \alpha\left[1 - \frac{zf'(z)}{f(z)}\right], \tag{3.5}$$

which, in light of hypothesis (3.2) of Theorem 3.1, yields the following subordination:

$$p(z) + \frac{\lambda zp'(z)}{\alpha} \prec q(z) + \frac{\lambda zq'(z)}{\alpha}. \tag{3.6}$$

The assertion of Theorem 3.1 now follows by an application of Lemma 2.3 with $\gamma = \lambda/\alpha$ and $\psi = 1$. □

Taking $q(z) = (1+Az)/(1+Bz)$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let $-1 \le B < A \le 1$ and (3.1) hold. If $f \in \mathcal{A}$, and*

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha} \prec \frac{\lambda(A-B)z}{\alpha(1+Bz)^2} + \frac{1+Az}{1+Bz}, \tag{3.7}$$

then

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz} \tag{3.8}$$

and $(1+Az)/(1+Bz)$ is the best dominant.

Theorem 3.1 for the choice of $q(z) = (1+z)/(1-z)$ reduces to the following.

Corollary 3.3. *Let (3.1) hold. If $f \in \mathcal{A}$, and*

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha} \prec \frac{2\lambda z}{\alpha(1-z)^2} + \frac{1+z}{1-z}, \tag{3.9}$$

then

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec \frac{1+z}{1-z} \tag{3.10}$$

and $(1+z)/(1-z)$ is the best dominant.

Theorem 3.4. *Let q be univalent in Δ, $\gamma, \mu \neq 0 \in \mathbb{C}$, and $0 \le \beta \le 1$. Let $f \in \mathcal{A}$. Suppose q satisfies*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0. \tag{3.11}$$

If

$$1 + \gamma\mu\left\{\frac{zf'(z) + \beta z^2 f''(z)}{(1-\beta)f(z) + \beta z f'(z)} - 1\right\} \prec 1 + \gamma\frac{zq'(z)}{q(z)}, \tag{3.12}$$

then

$$\left[\frac{(1-\beta)f(z) + \beta z f'(z)}{z}\right]^{\mu} \prec q(z) \tag{3.13}$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \left[\frac{(1-\beta)f(z) + \beta z f'(z)}{z}\right]^{\mu}, \quad z \neq 0. \tag{3.14}$$

Then a computation shows that

$$\mu\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta zf'(z)}-1\right\}=\frac{zp'(z)}{p(z)}. \tag{3.15}$$

By setting

$$\theta(\omega):=1, \qquad \phi(\omega):=\frac{\gamma}{\omega}, \tag{3.16}$$

it can be easily observed that $\theta(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C}\setminus\{0\}$, and that

$$\phi(\omega)\neq 0 \quad (\omega\in\mathbb{C}\setminus\{0\}). \tag{3.17}$$

Also, we let

$$\begin{aligned} Q(z) &= zq'(z)\phi(q(z)) = \gamma\frac{zq'(z)}{q(z)}, \\ h(z) &= \theta\{q(z)\}+Q(z) = 1+\gamma\frac{zq'(z)}{q(z)}. \end{aligned} \tag{3.18}$$

From (3.11), we find that $Q(z)$ is starlike univalent in Δ and that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right)=\Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0 \tag{3.19}$$

by the hypothesis (3.11) of Theorem 3.4. Thus, by applying Theorem 2.2, our proof of Theorem 3.4 is completed. □

For a special case when $q(z)=1/(1-z)^{2b}$ $(b\in\mathbb{C}\setminus\{0\})$, $\beta=0$, $\gamma=1/b$, and $\mu=1$, Theorem 3.4 reduces at once to the following known result obtained by Srivastava and Lashin [10].

Corollary 3.5. *Let b be a nonzero complex number. If $f\in\mathcal{A}$, and*

$$1+\frac{1}{b}\left[\frac{zf'(z)}{f(z)}-1\right]\prec\frac{1+z}{1-z}, \tag{3.20}$$

then

$$\frac{f(z)}{z}\prec\frac{1}{(1-z)^{2b}} \tag{3.21}$$

and $1/(1-z)^{2b}$ is the best dominant.

For a special case when $q(z)=1/(1-z)^{2b}$ $(b\in\mathbb{C}\setminus\{0\})$, $\beta=1$, $\gamma=1/b$, and $\mu=1$ Theorem 3.4 reduces at once to another known result obtained by Srivastava and Lashin [10].

Corollary 3.6. *Let b be a nonzero complex number. If $f \in \mathcal{A}$, and*

$$1+\frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}, \tag{3.22}$$

then

$$f'(z) \prec \frac{1}{(1-z)^{2b}} \tag{3.23}$$

and $1/(1-z)^{2b}$ is the best dominant.

For $q(z) = (1+Bz)^{\mu(A-B)/B}$, $\gamma = 1/\mu$, and $\beta = 0$ in Theorem 3.4, we get the following known result obtained by Obradović and Owa [7].

Corollary 3.7. *Let $-1 \le B < A \le 1$. If $f \in \mathcal{A}$, and*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \tag{3.24}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec (1+Bz)^{\mu(A-B)/B} \quad (z \in \Delta;\ z \neq 0;\ \mu \in \mathbb{C};\ \mu \neq 0) \tag{3.25}$$

and $(1+Bz)^{\mu(A-B)/B}$ is the best dominant.

We remark here that $q(z) = (1+Bz)^{\mu(A-B)/B}$ is univalent if and only if $|(\mu(A-B)/B)-1| \le 1$ or $|(\mu(A-B)/B)+1| \le 1$.

For $q(z) = e^{\mu Az}$, $\gamma = 1/\mu$, and $\beta = 0$ in Theorem 3.4, we get the following known result obtained by Obradović and Owa [7].

Corollary 3.8. *If $f \in \mathcal{A}$, and*

$$\frac{zf'(z)}{f(z)} \prec 1+Az, \tag{3.26}$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec e^{\mu Az} \quad (z \in \Delta;\ z \neq 0;\ \mu \in \mathbb{C};\ \mu \neq 0) \tag{3.27}$$

and $e^{\mu Az}$ is the best dominant.

Similar to the previous corollary, the function $q(z) = e^{\mu Az}$ is univalent if and only if $|\mu A| < \pi$.

Theorem 3.9. *Let q be univalent in $\Delta, \gamma \neq 0$, $\delta, \alpha \in \mathbb{C}$, and let $0 \le \beta \le 1$. Let $f \in \mathcal{A}$. Suppose q satisfies*

$$\Re\left\{\frac{\alpha}{\gamma}+1+\frac{zq''(z)}{q'(z)}\right\} > 0, \tag{3.28}$$

and also $\Re(\alpha/\gamma) > 0$. *Let*

$$\Psi(z) := \left[\frac{(1-\beta)f(z)+\beta zf'(z)}{z}\right]^{\mu}\left\{\alpha+\gamma\mu\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta zf'(z)}-1\right\}\right\}+\delta. \tag{3.29}$$

If

$$\Psi(z) \prec \alpha q(z)+\delta+\gamma zq'(z), \tag{3.30}$$

then

$$\left[\frac{(1-\beta)f(z)+\beta zf'(z)}{z}\right]^{\mu} \prec q(z) \tag{3.31}$$

and q *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) := \left[\frac{(1-\beta)f(z)+\beta zf'(z)}{z}\right]^{\mu}. \tag{3.32}$$

Then a computation shows that

$$\mu\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta zf'(z)}-1\right\} = \frac{zp'(z)}{p(z)}, \tag{3.33}$$

and hence

$$\mu p(z)\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta zf'(z)}-1\right\} = zp'(z). \tag{3.34}$$

By setting

$$\theta(\omega) := \alpha\omega+\delta, \qquad \phi(\omega) := \gamma, \tag{3.35}$$

it can be easily observed that $\theta(\omega)$ and $\phi(\omega)$ are analytic in $\mathbb{C}$. Also, we let

$$\begin{gathered} Q(z) = zq'(z)\phi(q(z)) = \gamma zq'(z), \\ h(z) = \theta\{q(z)\}+Q(z) = \alpha q(z)+\delta+\gamma zq'(z). \end{gathered} \tag{3.36}$$

From (3.28), we find that $Q(z)$ is starlike univalent in Δ, and that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left\{\frac{\alpha}{\gamma}+1+\frac{zq''(z)}{q'(z)}\right\} > 0 \tag{3.37}$$

by the hypothesis (3.28) of Theorem 3.9. Thus, by applying Theorem 2.2, our proof of Theorem 3.9 is completed. □

For $\beta = 1$, $\delta = -\alpha$, $\gamma = 1$, we get the following corollary.

Corollary 3.10. *Let q be univalent in Δ. Let $f \in \mathcal{A}$ and $1+\alpha > 0$. Suppose f satisfies*

$$\Re\left\{\alpha + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{3.38}$$

If

$$\alpha\left\{(f'(z))^{\mu} - 1\right\} + \mu\left\{\frac{zf''(z)}{f'(z)}(f'(z))^{\mu}\right\} \prec \alpha q(z) - \alpha + zq'(z), \tag{3.39}$$

then

$$[f'(z)]^{\mu} \prec q(z) \tag{3.40}$$

and q is the best dominant.

Taking $q(z) = 1 + \lambda/(1+\alpha)z$, we obtain a recent result of Singh [9, Theorem 1(ii), page 571].

4. Superordination for analytic functions

Theorem 4.1. *Let q be convex univalent in Δ, $\lambda \in \mathbb{C}$, and $0 < \alpha < 1$. Suppose q satisfies*

$$\Re\{\lambda\} > 0 \tag{4.1}$$

and $(z/f(z))^{\alpha} \in \mathcal{H}[q(0),1] \cap Q$. Let

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha} \tag{4.2}$$

be univalent in Δ. If

$$q(z) + \frac{\lambda z q'(z)}{\alpha} \prec (1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha}, \tag{4.3}$$

then

$$q(z) \prec \left(\frac{z}{f(z)}\right)^{\alpha} \tag{4.4}$$

and q is the best subordinant.

Proof. Define the function $p(z)$ by

$$p(z) := \left(\frac{z}{f(z)}\right)^{\alpha}. \tag{4.5}$$

Then a computation shows that

$$p(z)+\frac{\lambda}{\alpha}zp'(z)=(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha}-\lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha}. \tag{4.6}$$

Theorem 4.1 follows as an application of Lemma 2.6. □

Taking $q(z)=(1+Az)/(1+Bz)$ in Theorem 4.1, we get the following corollary.

Corollary 4.2. *Let $-1\leq B<A\leq 1$. Let q be convex univalent in Δ. Suppose q satisfies $\Re(\lambda)>0$ and $(z/f(z))^{\alpha}\in\mathcal{H}[q(0),1]\cap Q$. Let*

$$(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha}-\lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha} \tag{4.7}$$

be univalent in Δ. If

$$\frac{\lambda(A-B)z}{\alpha(1+Bz)^2}+\frac{1+Az}{1+Bz}\prec(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha}-\lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha}, \tag{4.8}$$

then

$$\frac{1+Az}{1+Bz}\prec\left(\frac{z}{f(z)}\right)^{\alpha} \tag{4.9}$$

and $(1+Az)/(1+Bz)$ is the best subordinant.

Since the proof of Theorem 5.2 is similar to the proof of Theorem 4.1, we state the theorem without proof.

Theorem 4.3. *Let q be convex univalent in Δ, $\gamma\in\mathbb{C}$, $0\leq\beta\leq 1$, and $f\in\mathcal{A}$. Suppose $[((1-\beta)f(z)+\beta zf'(z))/z]^{\mu}\in\mathcal{H}[q(0),1]\cap Q$, and*

$$1+\gamma\mu\left\{\frac{zf'(z)+\beta z^2f''(z)}{(1-\beta)f(z)+\beta zf'(z)}-1\right\} \tag{4.10}$$

is univalent in Δ. If

$$1+\gamma\frac{zq'(z)}{q(z)}\prec 1+\gamma\mu\left\{\frac{zf'(z)+\beta z^2f''(z)}{(1-\beta)f(z)+\beta zf'(z)}-1\right\}, \tag{4.11}$$

then

$$q(z)\prec\left[\frac{(1-\beta)f(z)+\beta zf'(z)}{z}\right]^{\mu} \tag{4.12}$$

and q is the best subordinant.

Theorem 4.4. *Let q be convex univalent in $\Delta,\gamma\neq 0$, $\delta,\alpha\in\mathbb{C}$, and let $0\leq\beta\leq 1$. Let $f\in\mathcal{A}$. Suppose q satisfies*

$$\Re\left\{\frac{\alpha}{\gamma}q'(z)\right\}>0. \tag{4.13}$$

If

$$\alpha q(z)+\delta+\gamma zq'(z) \prec \left[\frac{(1-\beta)f(z)+\beta zf'(z)}{z}\right]^{\mu}\left\{\alpha+\gamma\mu\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta zf'(z)}-1\right\}\right\}+\delta, \tag{4.14}$$

then

$$q(z) \prec \left[\frac{(1-\beta)f(z)+\beta zf'(z)}{z}\right]^{\mu} \tag{4.15}$$

and q is the best subordinant.

Proof. Define the function $p(z)$ by

$$p(z) := \left[\frac{(1-\beta)f(z)+\beta zf'(z)}{z}\right]^{\mu}. \tag{4.16}$$

Then a computation shows that

$$\mu\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta zf'(z)}-1\right\}=\frac{zp'(z)}{p(z)}, \tag{4.17}$$

and hence

$$\mu p(z)\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta zf'(z)}-1\right\}=zp'(z). \tag{4.18}$$

By setting

$$\vartheta(\omega) := \alpha\omega+\delta, \qquad \phi(\omega) := \gamma, \tag{4.19}$$

it can be easily observed that both $\theta(\omega)$ and $\phi(\omega)$ are analytic in $\mathbb{C}$. Now,

$$\Re\left(\frac{\vartheta'(q(z))}{\varphi(q(z))}\right)=\Re\left\{\frac{\alpha q'(z)}{\gamma}\right\}>0, \tag{4.20}$$

by the hypothesis (4.13) of Theorem 4.4. Thus, by applying Theorem 2.5, our proof of Theorem 4.4 is completed. □

5. Sandwich results

Combining the results of differential subordination and superordination, we state the following "sandwich results."

Theorem 5.1. *Let q_1 be convex univalent and let q_2 be univalent in Δ, $\lambda \in \mathbb{C}$, and $0 < \alpha < 1$. Suppose q_1 satisfies (4.1) and q_2 satisfies (3.1). If $0 \neq (z/f(z))^{\alpha} \in \mathcal{H}[q(0),1] \cap Q$, $(1+\lambda)(z/f(z))^{\alpha}-\lambda f'(z)(z/f(z))^{1+\alpha}$ is univalent in Δ, and*

$$q_1(z)+\frac{\lambda}{\alpha}zq_1'(z) \prec (1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha}-\lambda f'(z)\left(\frac{z}{f(z)}\right)^{1+\alpha} \prec q_2(z)+\frac{\lambda}{\alpha}zq_2'(z), \tag{5.1}$$

then

$$q_1(z) \prec \left(\frac{z}{f(z)}\right)^{\alpha} \prec q_2(z) \tag{5.2}$$

and q_1 and q_2 are, respectively, the best subordinant and best dominant.

Theorem 5.2. *Let q_1 be convex univalent and let q_2 be univalent in Δ, $\gamma \neq 0 \in \mathbb{C}$, $\mu \neq 0 \in \mathbb{C}$, $0 \leq \beta \leq 1$, and q_2 satisfies (3.11). Let $f \in \mathcal{A}$. Suppose $0 \neq [((1-\beta)f(z)+\beta z f'(z)/z)]^{\mu} \in \mathcal{H}[q(0),1] \cap Q$,*

$$1+\gamma\mu\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta z f'(z)}-1\right\} \tag{5.3}$$

is univalent in Δ. If

$$1+\gamma\frac{zq_1'(z)}{q_1(z)} \prec 1+\gamma\mu\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta z f'(z)}-1\right\} \prec 1+\gamma\frac{zq_2'(z)}{q_2(z)}, \tag{5.4}$$

then

$$q_1(z) \prec \left[\frac{(1-\beta)f(z)+\beta z f'(z)}{z}\right]^{\mu} \prec q_2(z) \tag{5.5}$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Theorem 5.3. *Let q_1 be convex univalent and let q_2 be univalent in Δ, $\gamma \neq 0 \in \mathbb{C}$, $\mu \neq 0 \in \mathbb{C}$ and $0 \leq \beta \leq 1$. Suppose q_1 satisfies (4.13), q_2 satisfies (3.28), and $[((1-\beta)f(z)+\beta z f'(z))/z]^{\mu} \in \mathcal{H}[q(0),1] \cap Q$. Let*

$$\left[\frac{(1-\beta)f(z)+\beta z f'(z)}{z}\right]^{\mu}\left\{\alpha+\gamma\mu\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta z f'(z)}-1\right\}\right\}+\delta \tag{5.6}$$

be univalent in Δ. If

$$\begin{aligned} &\alpha q_1(z)+\delta+\gamma z q_1'(z) \\ &\quad \prec \left[\frac{(1-\beta)f(z)+\beta z f'(z)}{z}\right]^{\mu}\left\{\alpha+\gamma\mu\left\{\frac{zf'(z)+\beta z^2 f''(z)}{(1-\beta)f(z)+\beta z f'(z)}-1\right\}\right\}+\delta \\ &\quad \prec \alpha q_2(z)+\delta+\gamma z q_2'(z), \end{aligned} \tag{5.7}$$

then

$$q_1(z) \prec \left[\frac{(1-\beta)f(z)+\beta z f'(z)}{z}\right]^{\mu} \prec q_2(z) \tag{5.8}$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

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