MULTIPLIERS AND OPERATORS ON THE TEMPERED ULTRADISTRIBUTION SPACES OF ROUMIEU TYPE FOR THE DISTRIBUTIONAL HANKEL-TYPE TRANSFORMATION SPACES

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The tempered ultradistribution space of Roumieu type for the space $H_{\mu,\nu}$ is defined, which is a subspace of the Hausdörff locally convex topological linear space. Further, results are obtained for the multipliers and operators on the tempered ultradistribution spaces for the distributional Hankel-type transformation spaces.

1. Introduction

During the course of time researchers have defined ultradistributions as the duals of various types of ultradifferentiable functions of Roumieu type and of Buerling type, Roumieu type ultradifferentiable functions have been considered in the present paper. Three types of spaces of rapidly decreasing ultradifferentiable functions of Roumieu type have been defined, which can be considered as subspaces of the Hausdörff locally convex topological linear space $H_{\mu,\nu}$ (cf. [3]), as the set of all infinitely differentiable complex-valued functions $\varphi(x)$ on an interval I $(0,\infty)$ with the norm

$$\Gamma_{i,j}^{\mu,\nu}(\varphi) = \sup_{x\in I} \left| x^i (x^{-1}D)^j x^{-\mu-\nu-1}\varphi(x) \right| < \infty, \tag{1.1}$$

where i, j are a pair of nonnegative integers and μ,ν are real numbers which will be an auxiliary to obtain duals of Roumieu-type tempered ultradistributions.

The a_is and b_js, wherever they appear ($i, j = 0,1,2,\ldots$), are to be considered as sequences of positive real numbers on which the constraints imposed are (cf. [2, page 66])

$$\begin{aligned} a_i^2 &\le a_{i-1}\,a_{i+1} \quad \forall i \in \mathbb{N},\\ b_j^2 &\le b_{j-1}\,b_{j+1} \quad \forall j \in \mathbb{N}. \end{aligned} \tag{1.2}$$

$S, S_1 > 0$ and $T, T_1 > 1$ are constants such that

$$\begin{aligned} a_i &\le ST^i \min_{0\le k\le i} a_k a_{i-k}, \quad i \in \mathbb{N}_0,\\ b_j &\le S_1 T_1^j \min_{0\le k\le j} b_k b_{j-k}, \quad j \in \mathbb{N}_0. \end{aligned} \tag{1.3}$$

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As a consequence of (1.2), we have (cf. [1])

$$a_i a_k \le a_0 a_{i+k}, \quad i,k \in \mathbb{N}_0,$$
$$b_j b_k \le b_0 b_{j+k}, \quad j,k \in \mathbb{N}_0. \tag{1.4}$$

Definition 1.1. (I) $H^{\nu}_{\mu,a_i,A}$ is the space of all those functions, defined on interval I, and are infinitely smooth, for which

$$\left| x^i (x^{-1}D)^j x^{-\mu-\nu-1}\varphi(x) \right| \le C_j^{\mu,\nu}(A+\alpha)^i a_i, \quad i,j \in \mathbb{N}_0, \tag{1.5}$$

where A and $C_j^{\mu,\nu}$ are constants depending on φ and α is an arbitrary positive constant.

(II) Define $\varphi \in H_{\mu}^{\nu,b_j,B}$ whenever φ is smooth and the inequality

$$\left| x^i (x^{-1}D)^j x^{-\mu-\nu-1}\varphi(x) \right| \le C_j^{\mu,\nu}(B+\beta)^j b_j \tag{1.6}$$

is true for all $i,j \in \mathbb{N}_0$.

(III) For φ being a smooth function, $\varphi(x) \in H^{\nu,b_j,B}_{\mu,a_i,A}$, which satisfies the inequality

$$\left| x^i (x^{-1}D)^j x^{-\mu-\nu-1}\varphi(x) \right| \le C^{\mu,\nu}(A+\alpha)^i(B+\beta)^j a_i b_j, \tag{1.7}$$

where $i,j \in \mathbb{N}_0$ (arbitrary constants) and $\alpha,\beta > 0$, A, B, and $C^{\mu,\nu}$ are positive constants depending on φ.

The norms on the spaces $H^{\nu}_{\mu,a_i,A}$, $H_{\mu}^{\nu,b_j,B}$, and $H^{\nu,b_j,B}_{\mu,a_i,A}$ are defined as

$$\vartheta^{\mu,\nu}_{j,\alpha}(\varphi) = \sup_{\substack{x\in(0,\infty)\\ i\in\mathbb{N}_0}} \frac{\left| x^i (x^{-1}D)^j x^{-\mu-\nu-1}\varphi(x) \right|}{(A+\alpha)^i a_i},$$
$$\vartheta^{\mu,\nu}_{i,\beta}(\varphi) = \sup_{\substack{x\in(0,\infty)\\ i\in\mathbb{N}_0}} \frac{\left| x^i (x^{-1}D)^j x^{-\mu-\nu-1}\varphi(x) \right|}{(B+\beta)^j b_j}, \tag{1.8}$$

where $i,j = 0,1,2,3,\dots$ and $\alpha,\beta = 1,1/2,1/3,\dots$, and

$$\vartheta^{\mu,\nu}_{\alpha,\beta}(\varphi) = \sup_{\substack{x\in(0,\infty)\\ i,j\in\mathbb{N}_0}} \frac{\left| x^i (x^{-1}D)^j x^{-\mu-\nu-1}\varphi(x) \right|}{(A+\alpha)^i(B+\beta)^j a_i b_j}, \tag{1.9}$$

for the conditions already mentioned.

The duals of the spaces, defined earlier, are denoted by ${H'}^{\nu}_{\mu,a_i,A}$, ${H'}_{\mu}^{\nu,b_j,B}$, and ${H'}^{\nu,b_j,B}_{\mu,a_i,A}$, respectively, which, indeed, are Roumieu-type ultradifferentiable functions. Whereas the duals are the tempered ultradistributions of Roumieu type.

The spaces $H^{\nu}_{\mu,a_i,A}$, $H_{\mu}^{\nu,b_j,B}$, and $H^{\nu,b_j,B}_{\mu,a_i,A}$ being the subspaces of the space $H_{\mu,\nu}$, indeed, $H'_{\mu,\nu}$ will be a subspace of their duals.

2. Multipliers for ultradifferentiable functions

This section deals with results, which justify certain mapping to be linear on the three spaces defined earlier. The following definition will make sense in the analysis of the results.

Definition 2.1. Denote by $\mathscr{E}_{a_i,A_0}, \mathscr{E}^{b_j,B_0}$, and $\mathscr{E}_{a_i,A_0}^{b_j,B_0}$ the set of all complex-valued infinitely differentiable functions (C^∞-functions) $\theta\ (x)$ on the interval $(0,\ \infty)$, which satisfy the following:

$$\begin{aligned} \left|(x^{-1}D)^i\theta(x)\right| &\le CA_0^i a_i(1+x^r), \\ \left|(x^{-1}D)^j\theta(x)\right| &\le CB_0^j b_j(1+x^r), \\ \left|(x^{-1}D)^j\theta(x)\right| &\le EA_0^i B_0^j a_j b_j(1+x^r), \end{aligned} \tag{2.1}$$

where C, D, E are positive constants and $r>0$ is an arbitrary integer.

Theorem 2.2. *Let $\varphi \in H_{\mu,a_i,A}^{\nu}, H_{\mu}^{\nu,b_j,B}$, and $H_{\mu,a_i,A}^{\nu,b_j,B}$, respectively. Then*

$$\begin{gathered} \vartheta_{j,1/n}^{\mu,\nu}(\varphi) \le \vartheta_{j,1/(n+1)}^{\mu,\nu}(\varphi), \qquad \vartheta_{i,1/n}^{\mu,\nu}(\varphi) \le \vartheta_{i,1/(n+1)}^{\mu,\nu}(\varphi), \\ \vartheta_{1/n}^{\mu,\nu}(\varphi) \le \vartheta_{1/(n+1),1/(n+1)}^{\mu,\nu}(\varphi), \quad \textit{respectively.} \end{gathered} \tag{2.2}$$

Proofs of the above inequalities are straightforward consequences of (1.8) and (1.9), respectively. The details are thus avoided.

Theorem 2.3. *Let $\theta \in \mathscr{E}_{a_i,A_0}$ and $\varphi \in H_{\mu,a_i,A}^{\nu}$. Then*

$$\vartheta_{j,\alpha}^{\mu,\nu}(\theta\varphi) \le 2CA_0^j a_j \sum_{k=0}^{j} \binom{j}{k} \vartheta_{j-k,\alpha}^{\mu,\nu}(\varphi), \tag{2.3}$$

that is, the mapping $\varphi \to \theta\varphi$ is a continuous linear mapping on $H_{\mu,a_i,A}^{\nu}$ into itself.

Proof. By virtue of Leibniz theorem and employing (1.5) and Definition 2.1, we obtain

$$\begin{aligned} &\left|x^i(x^{-1}D)^j x^{-\mu-\nu-1}(\theta\varphi)(x)\right| \\ &\quad = \left|\sum_{k=0}^{j}\binom{j}{k}\frac{(x^{-1}D)^k\theta(x)}{(1+x^r)}x^i(1+x^r)(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)\right| \\ &\quad \le \sum_{k=0}^{j}\binom{j}{k}CA_0^j a_j\left\{\left|x^i(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)\right|\right. \\ &\qquad\qquad \left.+\left|x^{i+r}(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)\right|\right\}. \end{aligned} \tag{2.4}$$

By multiplying by $1/(A+\alpha)^i a_i$ and considering the supremum over all integers $i>0$, for all $x\in(0,\infty)$, the above relation is expressed as

$$\begin{aligned}\sup_{\substack{i\in\mathbb{N}_0\\ x\in(0,\infty)}} &\frac{|x^i(x^{-1}D)^j x^{-\mu-\nu-1}(\theta\varphi)(x)|}{(A+\alpha)^i a_i}\\ &\le \sum_{k=0}^{j}\binom{j}{k} CA_0^j a_j\Bigg[\sup_{\substack{i\in\mathbb{N}_0\\ x\in(0,\infty)}}\frac{|x^i(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)|}{(A+\alpha)^i a_i}\\ &\qquad + \sup_{\substack{i\in\mathbb{N}_0\\ x\in(0,\infty)}}\frac{|x^{i+\nu}(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)|}{(A+\alpha)^i a_i}\Bigg],\end{aligned} \tag{2.5}$$

that is,

$$\vartheta^{\mu,\nu}_{j,\alpha}(\theta\varphi)\le CA_0^j a_j\sum_{k=0}^{j}\binom{j}{k}\vartheta(j-k,\alpha^{\mu,\nu}(\varphi)+)\vartheta^{\mu,\nu}_{j-k,\alpha}(\varphi), \tag{2.6}$$

that is,

$$\vartheta^{\mu,\nu}_{j,\alpha}(\theta\varphi)\le 2CA_0^j a_j\sum_{k=0}^{j}\binom{j}{k}\vartheta^{\mu,\nu}_{j-k,\alpha}(\varphi). \tag{2.7}$$

This completes the proof of the theorem. □

Theorem 2.4. *If $\mathscr{E}^{b_j,B_0}$ and b_j satisfy the relation (1.4), then for all $\varphi\in H_\mu^{\nu,b_j,B}$, the map $\varphi\to\theta\varphi$ is a continuous linear map on the space $H_\mu^{\nu,b_j,B}$ into itself, that is,*

$$\vartheta^{\mu,\nu}_{j,\alpha}(\theta\varphi)\le 2Db_0B_0^j b_j\sum_{k=0}^{j}\binom{j}{k}\vartheta^{\mu,\nu}_{j-k,\beta}(\varphi). \tag{2.8}$$

Proof. Employing Definition 2.1 to (2.8) and multiplying it by $1/(B+\beta)^j b_j$, we have

$$\begin{aligned}&\frac{|x^i(x^{-1}D)^j x^{-\mu-\nu-1}(\theta\varphi)(x)|}{(B+\beta)^j b_j}\\ &\quad\le \sum_{k=0}^{j}\binom{j}{k} DB_0^j b_j\big(|x^i(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)|\\ &\qquad + |x^{i+\nu}(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)|\big)\frac{1}{(B+\beta)^j b_j}.\end{aligned} \tag{2.9}$$

Now by virtue of (1.4), $1/b_j\le b_0/b_k b_{j-k}$. Owing to $j-k>0$, $x^i\le x^{i+r}$ for all $i,r>0$, from (2.9) we have

$$\frac{|x^i(x^{-1}D)^j x^{-\mu-\nu-1}(\theta\varphi)(x)|}{(B+\beta)^j b_j} \le \sum_{k=0}^{j}\binom{j}{k} DB_0^j \frac{b_j b_0}{b_k(B+\beta)^k}\left(\frac{|x^i(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)|}{(B+\beta)^{j-k}b_{j-k}} + \frac{|x^{i+r}(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)|}{(B+\beta)^{j-k}b_{j-k}}\right) \tag{2.10}$$

for all integers $j>0$, and for all $x\in(0,\infty)$, we consider the supremum to obtain

$$\vartheta_{i,\beta}^{\mu,\nu}(\theta\varphi) \le 2Db_0B_0^jb_j\sum_{k=0}^{j}\binom{j}{k}\vartheta_{i+r,\beta}^{\mu,\nu}(\varphi). \tag{2.11}$$

The theorem is thus completely proved. □

Theorem 2.5. *If $\theta\in\mathscr{E}_{a_i,A_0}^{b_j,B_0}$, $\varphi\in H_{\mu,a_i,A}^{\nu,b_j,B}$, and b_j satisfy (1.4), then*

$$\vartheta_{i,\beta}^{\mu,\nu}(\theta\varphi) \le C\sum_{k=0}^{j}\frac{\binom{j}{k}}{b_k}\vartheta_{\alpha,\beta}^{\mu,\nu}(\varphi), \tag{2.12}$$

that is, the mapping $\varphi\to\theta\varphi$ is a continuos linear mapping on $H_{\mu,a_i,A}^{\nu,b_j,B}$ into itself, where $C=EA_0^jB_0^ja_ib_jb_0$.

Proof. By virtue of (2.8) and Definition 1.1, we are led to

$$|x^i(x^{-1}D)^j x^{-\mu-\nu-1}(\theta\varphi)(x)| \le \sum_{k=0}^{j}\binom{j}{k}EA_0^jB_0^ja_jb_j\,|(x^i+x^{i+r})(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)|. \tag{2.13}$$

Now multiplying this relation by $1/(A+\alpha)^i(B+\beta)^ja_ib_j$, we conclude, indeed, that $1/(B+\beta)^k\le 1$, that is,

$$\frac{|x^i(x^{-1}D)^j x^{-\mu-\nu-1}(\theta\varphi)(x)|}{(A+\alpha)^i(B+\beta)^ja_ib_j} \le \sum_{k=0}^{j}\binom{j}{k}EA_0^jB_0^ja_jb_j\frac{b_{j-k}}{b_j}\left|\frac{(x^i+x^{i+r})(x^{-1}D)^{j-k}x^{-\mu-\nu-1}\varphi(x)}{(A+\alpha)^i(B+\beta)^{j-k}a_ib_{j-k}}\right|. \tag{2.14}$$

Further, employing (1.4) and as usual taking the supremum over all $x \in (0,\infty)$, $i, j \in \mathbb{N}_0$, we have

$$\vartheta^{\mu,\nu}_{i,\beta}(\theta\varphi) \le C\sum_{k=0}^{j} \frac{\binom{j}{k}}{b_k}\vartheta^{\mu,\nu}_{\alpha,\beta}(\varphi), \tag{2.15}$$

where C has the usual meaning. Thus, the theorem is completely proved. □

3. Operators for the duals of tempered ultradistribution

In this section, a lemma is proved to establish the isomorphism between the spaces which are infinitely smooth and defined on $(0,\infty)$, described in Section 1 and further, a mapping is proved to be an isomorphism between duals of these spaces.

Lemma 3.1. *For k being a nonnegative integer, the mapping $\varphi \to x^k\varphi$ is an isomorphism from the spaces $H^{\nu}_{\mu,a_i,A}$, $H^{\nu,b_j,B}_{\mu}$, and $H^{\nu,b_j,B}_{\mu,a_i,A}$ onto $H^{\nu+k}_{\mu,a_i,A}$, $H^{\nu+k,b_j,B}_{\mu}$, and $H^{\nu,b_j,B}_{\mu,a_i,A}$, respectively.*

Proof. Let $\varphi \in H^{\nu}_{\mu,a_i,A}, H^{\nu,b_j,B}_{\mu}$, and $H^{\nu,b_j,B}_{\mu,a_i,A}$, respectively.

In the case $\varphi \in H^{\nu}_{\mu,a_i,A}$, we write

$$\left|x^i(x^{-1}D)^j x^{-\mu-(\nu+k)-1}(x^k\varphi(x))\right| = \left|x^i(x^{-1}D)^j x^{-\mu-\nu-1}\varphi(x)\right|. \tag{3.1}$$

Now multiplying (3.1) by $1/(A+\alpha)^i a_i$ and taking supremum over all $x \in (0,\infty)$, for $j \in \mathbb{N}_0$, we obtain

$$\vartheta^{\mu,\nu+k}_{j,\alpha}(x^k\varphi) = \vartheta^{\mu,\nu}_{j,\alpha}(\varphi). \tag{3.2}$$

Now, when $\varphi \in H^{\nu,b_j,B}_{\mu}$, we multiply (3.1) by $1/(B+\beta)^j b_j$, and following the above lines of the proof, we arrive at

$$\vartheta^{\mu,\nu+k}_{i,\beta}(x^k\varphi) = \vartheta^{\mu,\nu}_{i,\beta}(\varphi). \tag{3.3}$$

Finally, for the case $\varphi \in H^{\nu,b_j,B}_{\mu,a_i,A}$, similarly, we conclude to write

$$\vartheta^{\mu,\nu+k}_{\alpha,\beta}(x^k\varphi) = \vartheta^{\mu,\nu}_{\alpha,\beta}(\varphi). \tag{3.4}$$

The lemma is completely proved. □

As an obvious consequence of Lemma 3.1 and the result (cf. [4, Theorem 1.10-2, page 29]), for $k \ge 0$, it is shown that the mapping $f \to x^k f$ is an isomorphism from the duals $H'^{\nu+k}_{\mu,a_i,A}$, $H'^{\nu+k,b_j,B}_{\mu}$, and $H'^{\nu+k,b_j,B}_{\mu,a_i,A}$, respectively, onto $H'^{\nu}_{\mu,a_i,A}$, $H'^{\nu,b_j,B}_{\mu}$, and $H'^{\nu,b_j,B}_{\mu,a_i,A}$, respectively.

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