OSCILLATION OF SOLUTIONS OF IMPULSIVE NEUTRAL DIFFERENCE EQUATIONS WITH CONTINUOUS VARIABLE

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We obtain sufficient conditions for oscillation of all solutions of the neutral impulsive difference equation with continuous variable $\Delta_\tau(y(t)+P(t)y(t-m\tau))+Q(t)y(t-l\tau)=0$, $t\geq t_0-\tau$, $t\neq t_k$, $y(t_k+\tau)-y(t_k)=b_ky(t_k)$, $k\in\mathbb{N}(1)$, where Δ_τ denotes the forward difference operator, that is, $\Delta_\tau z(t)=z(t+\tau)-z(t)$, $P(t)\in C([t_0-\tau,\infty),\mathbb{R})$, $Q(t)\in C([t_0-\tau,\infty),(0,\infty))$, m, l are positive integers, $\tau>0$ and b_k are constants, $0\leq t_0<t_1<t_2<\cdots<t_k<\cdots$ with $\lim_{k\to\infty}t_k=\infty$.

1. Introduction

Let $\mathbb{R}$ denote the set of all real numbers. For any $a\in\mathbb{R}$, define $\mathbb{N}(a)=\{a,a+1,a+2,\dots\}$. For any $t,\tau\in\mathbb{R}$, $r\in\mathbb{N}(1)$, define $\mathbb{N}(t-r\tau,t-\tau)=\{t-r\tau,t-(r-1)\tau,\dots,t-\tau\}$.

Consider the neutral impulsive difference equation with continuous variable

$$\begin{gathered}\Delta_\tau\big(y(t)+P(t)y(t-m\tau)\big)+Q(t)y(t-l\tau)=0,\quad t\geq t_0-\tau,\ t\neq t_k,\\ y(t_k+\tau)-y(t_k)=b_ky(t_k),\quad k\in\mathbb{N}(1),\end{gathered}\tag{1.1}$$

where Δ_τ denotes the foreward difference operator, that is, $\Delta_\tau z(t)=z(t+\tau)-z(t)$, $P(t)\in C([t_0-\tau,\infty),\mathbb{R})$, $Q(t)\in C([t_0-\tau,\infty),(0,\infty))$, m, l are positive integers, $\tau>0$ and b_k are constants, $0\leq t_0<t_1<t_2<\cdots<t_k<\cdots$ with $\lim_{k\to\infty}t_k=\infty$. Set $l_0=\max\{m,l\}$. For any $t_0\geq 0$, let $\phi_{t_0}=\{\varphi:[t_0-(l_0+1)\tau,t_0-\tau]\to\mathbb{R}\mid\varphi(t)$ is piecewise continuous on $[t_0-(l_0+1)\tau,t_0-\tau]$, $\varphi(t)$ is finite for every $t\in[t_0-(l_0+1)\tau,t_0-\tau]$, the right and left limits $\varphi(t^+)$ and $\varphi(t^-)$ of $\varphi(t)$ exist for every $t\in(t_0-(l_0+1)\tau,t_0-\tau)$, and $\varphi((t_0-(l_0+1)\tau)^+)$ and $\varphi((t_0-\tau)^-)$ exist$\}$.

Definition 1.1. For given $t_0\geq 0$ and $\varphi\in\phi_{t_0}$, a real-valued function $x(t)$ is said to be a solution of (1.1) satisfying the initial value condition

$$x(t)=\varphi(t),\quad t\in\big[t_0-(l_0+1)\tau,t_0-\tau\big],\tag{1.2}$$

if $x(t)$ is defined on $[t_0-(l_0+1)\tau,\infty)$ and satisfies (1.1) and (1.2).

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For given $t_0 \geq 0$ and $\varphi \in \phi_{t_0}$, by means of the method of steps, the solution of (1.1) exists and is unique.

Definition 1.2. A solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

When $\{t_k\} = \phi$, that is, $\{t_k\}$ is an empty set, (1.1) reduces to the neutral difference equation

$$\Delta_\tau\big(y(t)+P(t)y(t-m\tau)\big)+Q(t)y(t-l\tau)=0, \quad t \geq t_0-\tau. \tag{1.3}$$

The oscillatory behavior of difference equations with continuous variable without impulses has been investigated by some authors, see, for example, [1–3, 5, 6]. However, to the present time, there exists no literature on impulsive delay difference equations with continuous variable. The purpose of this note is to study the oscillatory behavior of solutions of (1.1). If there is a sequence $\{m_k\}$ of positive integers such that $m_k \to \infty$ as $k \to \infty$ and $b_{m_k} \leq -1$, then it is easily seen that every solution of (1.1) is oscillatory. Therefore, we always assume that $b_k > -1$ for all $k \in \mathbb{N}(1)$. Throughout this note, we will use the convention

$$\prod_{\{t_k\}\cap\mathbb{N}(t-r\tau,t-\tau)=\phi}(1+b_k)^{-1} \equiv 1, \qquad \prod_{\{t_k\}\cap\mathbb{N}(t-r\tau,t-\tau)=\phi}(1+b_k) \equiv 1, \tag{1.4}$$

where ϕ is an empty set and $r \in \mathbb{N}(1)$.

2. Main results

We first introduce two lemmas due to [4]. We give the following hypothesis:

(H) r is an integer, $p(n) \geq 0$, $n = 0,1,2,\ldots$, $b_k > -1$, $k = 1,2,3,\ldots,\{n_k\}$ is an infinite subset of $\mathbb{N}(1)$ satisfying $n_1 < n_2 < \cdots < n_k < \cdots$ with $\lim_{k\to\infty} n_k = \infty$.

Lemma 2.1. *Let (H) hold. Assume that*

(i)

$$\limsup_{n\to\infty} \prod_{n-r\leq n_k\leq n-1}(1+b_k)^{-1} < \infty, \tag{2.1}$$

(ii)

$$\liminf_{n\to\infty} \sum_{\substack{i=n-r,\\ i\notin\{n_k\}}}^{n-1} p(i) \prod_{n-r\leq n_k\leq n-1}(1+b_k)^{-1} > \left(\frac{r}{r+1}\right)^{r+1}. \tag{2.2}$$

Then the discrete impulsive difference inequality

$$\begin{aligned} y(n+1)-y(n)+p(n)y(n-r) &\leq 0, \quad n \in \mathbb{N}(0),\ n \neq n_k, \\ y(n_k+1)-y(n_k) &\leq b_k y(n_k), \quad k \in \mathbb{N}(1), \end{aligned} \tag{2.3}$$

has no eventually positive solution.

Lemma 2.2. *Let (H) hold and $r \geq 2$. Assume that*

(i)

$$\liminf_{n\to\infty} \prod_{n+1\leq n_k\leq n+r-1} (1+b_k) > 0, \tag{2.4}$$

(ii)

$$\liminf_{n\to\infty} \prod_{n+1\leq n_k\leq n+r-1} (1+b_k) < \infty, \tag{2.5}$$

(iii)

$$\liminf_{n\to\infty} \sum_{\substack{i=n+1\\ i\notin\{n_k\}}}^{n+r-1} p(i) \prod_{n+1\leq n_k\leq n+r-1} (1+b_k) > \left(\frac{r-1}{r}\right)^r. \tag{2.6}$$

Then the discrete impulsive difference inequality

$$\begin{gathered} y(n+1)-y(n)-p(n)y(n+r) \leq 0, \quad n \in \mathbb{N}(0),\ n \neq n_k, \\ y(n_k+1)-y(n_k) \leq b_k y(n_k), \quad k \in \mathbb{N}(1), \end{gathered} \tag{2.7}$$

has no eventually negative solution.

Theorem 2.3. *Let $t_{k+1}-t_k = m\tau$, $b_k > 0$, and $(1+b_k)P(t_k) = (1+b_{k-1})P(t_k+\tau)$ for $k = 1,2,3,\ldots, -1 < P(t) < 0$ with $\inf_{t\in[t_0-\tau,\infty)} P(t) > -1$, $Q(t) \in C([t_0-\tau,\infty),(0,\infty))$. If*

$$\liminf_{t\to\infty} \sum_{\substack{i\in\mathbb{N}(t-l\tau,t-\tau)\\ i\notin\{t_k\}}} Q(i) \prod_{t_k\in\mathbb{N}(t-l\tau,t-\tau)} (1+b_k)^{-1} > \left(\frac{l}{l+1}\right)^{l+1}, \tag{2.8}$$

then every solution of (1.1) oscillates.

Proof. Suppose, on the contrary, there is a solution $y(t)$ of (1.1) which is eventually nonoscillatory. If $y(t)$ is a solution of (1.1), then $-y(t)$ is a solution of (1.1). Without loss of generality, we assume that $y(t) > 0$ for $t \geq t_N - (l_0+1)\tau \geq t_0 - \tau$, where N is some positive integer. Let

$$z(t) = y(t) + P(t)y(t-m\tau), \quad t \geq t_N - \tau. \tag{2.9}$$

For any $t \geq t_N - \tau$, by (1.1) and $(1+b_k)P(t_k) = (1+b_{k-1})P(t_k+\tau)$ for $k = 1,2,3,\ldots$, we have

$$\Delta_\tau z(t) = -Q(t)y(t-l\tau) < 0 \quad \text{for } t \notin \{t_k\}, \tag{2.10}$$

$$z(t_k+\tau) - z(t_k) = b_k z(t_k) \quad \text{for } k = N, N+1, N+2, \ldots. \tag{2.11}$$

From (2.10), it follows that $z(t)$ strictly decreases on $\{t_k+\tau, t_k+2\tau, \ldots, t_k+(m-1)\tau, t_{k+1}\}$ $(k = N, N+1, N+2, \ldots)$, and noting (2.11), the seqence $\{z(t_N+n\tau)\}_{n=1}^{\infty}$ has only two cases: eventually positive or eventually negative. If $\{z(t_N+n\tau)\}_{n=1}^{\infty}$ is eventually negative, noticing

$$z(t_N+n\tau)=y(t_N+n\tau)+P(t_N+n\tau)y(t_N+(n-m)\tau),\quad n=1,2,3,\dots,\tag{2.12}$$

then $y(t_N+n\tau)<-P(t_N+n\tau)y(t_N+(n-m)\tau)$ eventually holds for n. It follows that $0<y(t_N+(n+jm)\tau)<-P(t_N+(n+jm)\tau)y(t_N+(n+(j-1)m)\tau)<P(t_N+(n+jm)\tau)P(t_N+(n+(j-1)m)\tau)y(t_N+(n+(j-2)m)\tau)<\cdots<(-1)^j\prod_{i=1}^{j}P(t_N+(n+i)\tau)y(t_N+n\tau)$. By condition $-1<P(t)<0$ with $\inf_{t\in[t_0-\tau,\infty)}P(t)>-1$, we have $y(t_N+(n+jm)\tau)\to 0(j\to\infty)$, that is, $\lim_{j\to\infty}y(t_N+jm\tau)=0$. Noticing (2.12), we get $\lim_{j\to\infty}z(t_N+jm\tau)=0$. This is a contradiction with the condition that $\{z(t_N+n\tau\}_{n=1}^{\infty}$ is eventually negative and strictly decreases. If $\{z(t_N+n\tau)\}_{n=1}^{\infty}$ is eventually positive, then

$$y(t_N+n\tau)>z(t_N+n\tau),\quad z(t_N+n\tau)>0 \text{ for large } n.\tag{2.13}$$

Let $t_{N+j}=t_N+n_j\tau$, $j=1,2,3,\dots$. By (1.1), (2.9), and (2.11)–(2.13), we conclude that $\{z(t_N+n\tau\}_{n=1}^{\infty}$ is an eventually positive solution of the following impulsive difference inequality:

$$\begin{gathered}\Delta z(t_N+n\tau)+Q(t_N+n\tau)z(t_N+(n-l)\tau)<0,\quad n\geq 1,\ n\neq n_j,\\ z(t_N+(n_j+1)\tau)-z(t_N+n_j\tau)=b_{N+j}z(t_N+n_j\tau),\quad j=1,2,3,\dots,\end{gathered}\tag{2.14}$$

where Δ is the forward difference operator with respect to n. On the other hand, by condition (2.8), we have

$$\liminf_{n\to\infty}\sum_{\substack{i=n-l\\ i\notin\{n_j\}}}^{n-1}Q(t_N+i\tau)\prod_{n-l\leq n_j\leq n-1}(1+b_j)^{-1}>\left(\frac{l}{l+1}\right)^{l+1}.\tag{2.15}$$

Employing Lemma 2.1, we conclude that (2.14) has no eventually positive solution. This is a contradiction. Thus, the proof is complete. □

Theorem 2.4. *Let* $t_{k+1}-t_k\equiv m\tau$, $b_k>0$, $(1+b_k)P(t_k)=(1+b_{k-1})P(t_k+\tau)$ *for* $k=1,2,3,\dots$, $m>l+1$, $P(t)\leq -1$, $Q(t)\in C([t_0-\tau,\infty),(0,\infty))$. *If*

$$\liminf_{t\to\infty}\sum_{\substack{i\in\mathbb{N}(t-l\tau,t-\tau)\\ i\notin\{t_k\}}}Q(i)\prod_{t_k\in\mathbb{N}(t-l\tau,t-\tau)}(1+b_k)^{-1}>\left(\frac{l}{l+1}\right)^{l+1},\tag{2.16}$$

$$\liminf_{t\to\infty}\sum_{\substack{i\in\mathbb{N}(t+\tau,t+(m-l-1)\tau)\\ i\notin\{t_k\}}}\frac{Q(i)}{-P(i+(m-l)\tau)}\prod_{t_k\in\mathbb{N}(t+\tau,t+(m-l-1)\tau)}(1+b_k)>\left(\frac{m-l-1}{m-l}\right)^{m-l},\tag{2.17}$$

then every solution of (1.1) oscillates.

Proof. Suppose, on the contrary, there is a solution $y(t)$ of (1.1) which is eventually nonoscillatory. If $y(t)$ is a solution of (1.1), then $-y(t)$ is a solution of (1.1). Without loss of generality, we assume that $y(t) > 0$ for $t \geq t_N - (l_0+1)\tau \geq t_0 - \tau$, where N is some positive integer. Set

$$z(t) = y(t) + P(t)y(t - m\tau), \quad t \geq t_N - \tau. \tag{2.18}$$

For any $t \geq t_N - \tau$, by (1.1), we get

$$\Delta_\tau z(t) = -Q(t)y(t - l\tau) < 0 \quad \text{for } t \notin \{t_k\}, \tag{2.19}$$

$$z(t_k + \tau) - z(t_k) = b_k z(t_k), \quad k = N, N+1, \dots. \tag{2.20}$$

From (2.19), it follows that $z(t)$ strictly decreases on $\{t_k+\tau, t_k+2\tau, \dots, t_k+(m-1)\tau, t_{k+1}\}$ $(k = N, N+1, N+2, \dots)$, and noting (2.20), the sequence $\{z(t_N+n\tau)\}_{n=1}^{\infty}$ has only two cases: eventually positive or eventually negative. If $\{z(t_N+n\tau)\}_{n=1}^{\infty}$ is eventually positive, noticing

$$z(t_N+n\tau) = y(t_N+n\tau) + P(t)y(t_N+(n-m)\tau), \quad n = 1,2,3,\dots, \tag{2.21}$$

then

$$y(t_N+n\tau) > z(t_N+n\tau), \quad z(t_N+n\tau) > 0 \text{ for large } n. \tag{2.22}$$

Let $t_{N+j} = t_N + n_j\tau$, $j = 1,2,3,\dots$. By (1.1), (2.18), and (2.20)–(2.22), we conclude that $\{z(t_N+n\tau)\}_{n=1}^{\infty}$ is an eventually positive solution of the following impulsive difference inequality:

$$\begin{gathered}\Delta z(t_N+n\tau) + Q(t_N+n\tau)z(t_N+(n-l)\tau) < 0, \quad n \geq 1,\ n \neq n_j,\\ z(t_N+(n_j+1)\tau) - z(t_N+n_j\tau) = b_{N+j}z(t_N+n_j\tau), \quad j = 1,2,3,\dots,\end{gathered} \tag{2.23}$$

where Δ is the forward difference operator with respect to n. On the other hand, by condition (2.16), we have

$$\liminf_{n\to\infty} \sum_{\substack{i=n-l\\ i\notin\{n_j\}}}^{n-1} Q(t_N+i\tau) \prod_{n-l\leq n_j\leq n-1} (1+b_j)^{-1} > \left(\frac{l}{l+1}\right)^{l+1}. \tag{2.24}$$

Employing Lemma 2.1, we conclude that (2.23) has no eventually positive solution. This is a contradiction. If $\{z(t_N+n\tau)\}_{n=1}^{\infty}$ is eventually negative, by simple calculation, we find that $z(t)$ satisfies

$$\begin{gathered}\Delta_\tau z(t) + P(t-l\tau)\frac{Q(t)}{Q(t-m\tau)}\Delta_\tau z(t-m\tau) + Q(t)z(t-l\tau) = 0, \quad t \geq m\tau,\ t \neq t_k,\\ z(t_k+\tau) - z(t_k) = b_{N+j}z(t_k), \quad k = 2,3,\dots.\end{gathered} \tag{2.25}$$

Let $t_{N+j} = t_N + n_j\tau$ for $j = 1,2,3,\ldots$, then $\{z(t_N + n\tau)\}_{n=1}^{\infty}$ satisfies

$$\begin{gathered}\Delta z(t_N+n\tau)+P(t_N+n\tau-l\tau)\frac{Q(t_N+n\tau)}{Q(t_N+(n-m)\tau)}\Delta z(t_N+(n-m)\tau)\\ +Q(t_N+n\tau)z(t_N+(n-l)\tau)=0,\quad n\geq m,\ n\neq n_j,\\ z(t_N+(n_j+1)\tau)-z(t_N+n_j\tau)=b_{N+j}z(t_N+n_j\tau),\quad j=2,3,\ldots,\end{gathered} \tag{2.26}$$

where Δ is the forward difference operator with respect to n. Noticing $\Delta z(t_N + n\tau) < 0$ for $n \neq n_j$, $\{z(t_N + n\tau)\}_{n=1}^{\infty}$ eventually satisfies the inequality

$$\begin{gathered}P(t_N+(n-l)\tau)\frac{Q(t_N+n\tau)}{Q(t_N+(n-m)\tau)}\Delta z(t_N+(n-m)\tau)\\ +Q(t_N+n\tau)z(t_N+(n-l)\tau)>0,\quad n\geq m,\ n\neq n_j,\\ z(t_N+(n_j+1)\tau)-z(t_N+n_j\tau)=b_{N+j}z(t_N+n_j\tau),\quad j=2,3,\ldots.\end{gathered} \tag{2.27}$$

And (2.27) is equivalent to the inequality

$$\begin{gathered}\Delta z(t_N+n\tau)+\frac{Q(t_N+n\tau)}{P(t_N+(n+m-l)\tau)}z(t_N+(n+m-l)\tau)<0,\quad n\geq m,\ n\neq n_j,\\ z(t_N+(n_j+1)\tau)-z(t_N+n_j\tau)=b_{N+j}z(t_N+n_j\tau),\quad j=2,3,\ldots.\end{gathered} \tag{2.28}$$

So $\{z(t_N + n\tau)\}_{n=1}^{\infty}$ is an eventually negative solution of (2.28). On the other hand, noticing the condition (2.17), we have

$$\liminf_{t\to\infty}\sum_{\substack{i=n+1\\ i\notin\{n_j\}}}^{n+m-l-1}\frac{Q(t_N+i\tau)}{-P(t_N+(i+m-l)\tau)}\prod_{n+1\leq n_j\leq n+m-l-1}(1+b_j)>\left(\frac{m-l-1}{m-l}\right)^{m-l}. \tag{2.29}$$

By Lemma 2.2, (2.28) has no eventually negative solution. This is a contradiction. Thus, the poof is complete. □

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