WEAK GROTHENDIECK'S THEOREM

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Let $E_n \subset L_1^{2n}$ be the n-dimensional subspace which appeared in Kašin's theorem such that $L_1^{2n} = E_n \oplus E_n^{\perp}$ and the L_1^{2n} and L_2^{2n} norms are universally equivalent on both E_n and $E_n^{\perp}$. In this paper, we introduce and study some properties concerning extension and weak Grothendieck's theorem (WGT). We show that the Schatten space S_p for all $0 < p \leq \infty$ does not verify the theorem of extension. We prove also that S_p fails GT for all $1 \leq p \leq \infty$ and consequently by one result of Maurey does not satisfy WGT for $1 \leq p \leq 2$. We conclude by giving a characterization for spaces verifying WGT.

1. Introduction

This work was inspired by the celebrated theorem of Kašin [5]. We use his decomposition cited in the abstract and which states that L_1^{2n} (this space is of dimension $2n$ and which will be defined in the sequel) can be decomposed into two orthogonal n-dimensional subspaces "respecting" the inner product induced by the norm of L_2^{2n} and on each the norms of L_1^{2n} and L_2^{2n} are universally equivalent on these subspaces. It is interesting to observe that the constants of equivalence are independent of n. Recently this was investigated by Anderson [1] and Schechtman [15]. We will say that a Banach space X verifies weak Grothendieck's theorem if $\pi_2(X, l_2) = B(X, l_2)$. Let $\{\varphi_i\}_{1 \leq i \leq n}$ be a sequence of orthogonal random variables in L_2^{2n}, which generates E_n. Consider $0 < p \leq \infty$. Let $u : E_n \to S_p^n$ be a linear operator and let $\widetilde{u}$ be any extension of u. In this paper we show that $\|\widetilde{u}\| \geq C\sqrt{n}$, where C is an absolute constant. We prove that S_p fails extension theorem for all $1 \leq p \leq \infty$. We also show that S_p does not verify GT for $1 \leq p \leq \infty$ and consequently fails WGT for all $1 \leq p \leq 2$ by using one result of Maurey. We end this work by giving a characterization for operators satisfying WGT.

We start the first section by recalling some necessary notations and definitions such as the definition of cotype q-Kašin as studied in [9] and which is inspired by the Kašin decomposition. We introduce also the property of weak Grothendieck's theorem.

In section two, we recall the Schatten spaces S_p which are the noncommutative analogues of the l_p-spaces and we give some properties concerning these spaces. After this,

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we show that the space S_p fails the property of extension for all $0 < p \leq \infty$ and GT for all $1 \leq p \leq \infty$. We deduce that the space S_p does not verify WGT for all p, $1 \leq p \leq 2$. We do not know if S_p is of cotype 2-Kašin for $1 \leq p \leq 2$ like the classical cotype. We know that the Schatten space S_p is of cotype 2 for $1 \leq p \leq 2$ as the usual l_p-spaces; see [16]. By another method which is not adjustable to our case we have proved in [10] that $L_p([0,1],dx)$ and l_p for $0 < p < 1$ fail the extension property.

In Section 4, we characterize the spaces which satisfy weak Grothendieck's theorem.

2. Notation and preliminaries

Let $0 < p \leq +\infty$. We denote by L_p^n the space $\mathbb{R}^n$ (or $\mathbb{C}^n$) equipped with the norm (and only a p-norm if $0 < p < 1$)

$$||(a_i)||_{L_p^n} = \left(\frac{1}{n}\sum_{i=1}^{n} |a_i|^p\right)^{1/p}, \tag{2.1}$$

and if $p = \infty$, we take $\max|a_i|$.

Recall that a p-norm on a vector space X is a functional

$$\begin{aligned} \|\cdot\| : X &\longrightarrow \mathbb{R}_+, \\ x &\longmapsto \|x\| \end{aligned} \tag{2.2}$$

such that

$$\begin{gathered} \|x\| = 0 \Longleftrightarrow x = 0, \\ \|\lambda x\| = |\lambda|\,\|x\| \quad \forall \lambda \text{ in } \mathbb{C}, \\ \|x+y\| \leq \left(\|x\|^p + \|y\|^p\right)^{1/p} \quad \forall x, y \text{ in } X, \end{gathered} \tag{2.3}$$

X is called a p-normed space if its topology can be defined by a p-norm.

L_p^n is isometric to $L_p^n(\Omega_n,\mathcal{P}(\Omega_n),\mu_n)$ where Ω_n is the set $\{1,2,\dots,n\}$, $\mathcal{P}(\Omega_n)$ the σ- algebra of all subsets $A \subset \Omega_n$ and μ_n the uniform probability on Ω_n (i.e., $\mu_n(i) = 1/n$ for all i in Ω_n). Hence each element in L_p^n can be considered as a random variable which we denote in the sequel by φ and we have for $0 < p \leq q \leq \infty$,

$$\|\varphi\|_{L_p^n} \leq \|\varphi\|_{L_q^n} \leq n^{1/p-1/q}\|\varphi\|_{L_p^n}. \tag{2.4}$$

Moreover, we will denote by $l_p^n(X)$ for any Banach space X (resp., $L_p^n(X)$), the space X^n equipped with the norm if $1 \leq p \leq +\infty$ and the p-norm if $0 < p < 1$:

$$\begin{gathered} ||(x_i)||_{l_p^n(X)} = \left(\sum_{i=1}^{n} ||x_i||_X^p\right)^{1/p}, \\ \left(\text{resp., } ||(x_i)||_{L_p^n t(X)} = \left(\frac{1}{n}\sum_{i=1}^{n} ||x_i||_X^p\right)^{1/p}\right) \end{gathered} \tag{2.5}$$

for all $(x_i)_{1\leq i\leq n} \subset X$. If $p = \infty$, the sums should be replaced by sup.

We will use the following decomposition due to B. S. Kašin (see also [13] and recently [1, 15]), which is the principal inspiration of our idea.

Theorem 2.1 [5]. *Consider p in $\{1,2\}$ and n in $\mathbb{N}$. There are three constants A_p, B_p, and C (C independent of p and n) and a sequence $(\varphi_i)_{1\leq i\leq n}$ of orthogonal random variables in L_2^{2n} such that for all $(a_i)_{1\leq i\leq n}$ in $\mathbb{R}$, there exist*

$$A_p\left(\sum_1^n \|a_i\|^2\right)^{1/2} \leq \left\|\sum_1^n a_i\varphi_i\right\|_{L_p^{2n}} \leq B_p\left(\sum_1^n |a_i|^2\right)^{1/2}, \qquad \sup_{1\leq i\leq n}\|\varphi_i\|_{L_\infty^n} \leq C(\log n)^{1/2}. \tag{2.6}$$

Remark 2.2. It is well known that if X is a finite dimensional space, then, all the norms are equivalent. But what is most remarkable in Theorem 2.1 is that the constants are independent of the dimension n. It is also true for all p in $]0,2]$. We can and do choose the φ_i to be orthonormal, that is what we do in the sequel.

Let E_n be the subspace of L_1^{2n} spanned by the functions $(\varphi_i)_{1\leq i\leq n}$ and let $\mathbf{e}_n : E_n \to L_1^{2n}$ be the natural injection. By the above theorem, E_n is isomorphic to l_2^n, we denote by $\beta_n : l_2^n \to E_n$ the isomorphism which maps e_i onto φ_i, where (e_i) the unit vector basis of l_2^n. We have by (2.6) that $\|\beta_n\| \leq B_1$ and $\|\beta_n^{-1}\| \leq A_1^{-1}$.

Now we give the following definition which is introduced in [9].

Definition 2.3. Let X and Y be Banach spaces and let $u : X \to Y$ be a linear operator. Say that u is of cotype q-Kašin for $2 \leq q < +\infty$, if there is a positive constant K such that for all integer n and for all finite sequence $(x_i)_{1\leq i\leq n}$ in X, there exists

$$\left(\sum_{i=1}^n \|u(x_i)\|^q\right)^{1/q} \leq K\left\|\sum_{i=1}^n \varphi_i x_i\right\|_{L_1^{2n}(X)}. \tag{2.7}$$

Denote by $K_q(u)$ the smallest constant for which this holds. X is of cotype q-Kašin if the identity of X is of cotype q-Kašin.

For example L_p $(1 \leq p \leq 2)$ is of cotype 2-Kašin.

For being complete, we add (see [13, page 115]) that there is an orthonormal basis (φ_n) of $L_2([0,1],\nu)$ (ν is the Lebesgue measure) such that the L_1 and L_2 norms are equivalent on each of the spans of $\{\varphi_n, n \text{ odd}\}$ and $\{\varphi_n, n \text{ even}\}$. Let E_0 be the space spanned by one of these sequences in $L_1([0,1],\nu)$ and let $e : E_0 \to L_1([0,1],\nu)$ be the isometric embedding. We denote also by E_0^n the space spanned by the n first φ_i.

Given two Banach spaces X and Y, denote by $X\hat{\otimes}_\epsilon Y$ their injective tensor product, that is, the completion of $X \otimes Y$ under the cross norm:

$$\left\|\sum_{i=1}^n x_i \otimes y_i\right\|_\epsilon = \sup\left\{\left|\sum_{i=1}^n x_i(\xi)y_i(\eta)\right| : \|\xi\|_{X^*} \leq 1, \|\eta\|_{Y^*} \leq 1\right\}. \tag{2.8}$$

Let $u : X \to Y$ be a linear operator. We will say that u is *absolutely p-summing*, $0 < p < \infty$ (we write $u \in \Pi_p(X,Y)$), if there exists a positive constant C such that for every n in $\mathbb{N}$, the mappings

$$\begin{aligned} I_n \otimes u : l_p^n \otimes_\epsilon X &\longrightarrow l_p^n(Y), \\ \sum_1^n e_i \otimes x_i &\longmapsto \left(u(x_i)\right)_{1\le i\le n} \end{aligned} \tag{2.9}$$

are uniformly bounded by C (i.e., $\|I_n \otimes u\|_{l_p^n\otimes_\epsilon X\to l_p^n(Y)} \le C$).

We define the p-summing norm of an operator u by

$$\pi_p(u) = \sup_n \left|\left|I_n \otimes u\right|\right|_{l_p^n\otimes_\epsilon X\to l_p^n(Y)}. \tag{2.10}$$

The following proposition is a characterization of spaces of cotype 2-Kašin.

Proposition 2.4. *Let C be a positive constant. Then the following properties of a Banach space X are equivalent.*

(i) *The space X^* (X^* is the Banach space dual of X) is of cotype 2-Kašin and $K_2(X^*) \le C$.*

(ii) *For all integers n and for all finite sequences $(x_i)_{1\le i\le n}$ in X, the operator $u : E_n \to X$ defined by $u(\varphi_i) = x_i$ admits an extension $\widetilde{u} : L_1^{2n} \to X$ such that $\widetilde{u}/E_n = u$ and $\|\widetilde{u}\| \le C(\sum_{i=1}^n \|x_i\|^2)^{1/2}$.*

Proof. Let n be a fixed integer. Since X^* is of cotype 2-Kašin, hence for all $(\xi_i)_{1\le i\le n} \subset X^*$ we have

$$\left(\sum_{i=1}^n \left|\left|\xi_i\right|\right|_{X^*}^2\right)^{1/2} \le C\left|\left|\sum_{i=1}^n \varphi_i\xi_i\right|\right|_{L_1^{2n}(X^*)}. \tag{2.11}$$

Let $E = \{\sum_{i=1}^n \varphi_i\xi_i, (\xi_i)_{1\le i\le n} \subset X^*\}$, which is a closed subspace of $L_1^{2n}(X^*)$. We now define the operators

$$\begin{aligned} T : E &\longrightarrow l_2^n(X^*), \\ \sum_{i=1}^n \varphi_i\xi_i &\longmapsto (\xi_i)_{1\le i\le n}. \end{aligned} \tag{2.12}$$

This definition is unambiguous (indeed, $\sum_{i=1}^n \varphi_i\xi_i = \sum_{i=1}^n \varphi_i\eta_i$ implies that $\xi_i = \eta_i$ for all $1 \le i \le n$ because the φ_i are orthogonal and consequently $(\xi_i)_{1\le i\le n} = (\eta_i)_{1\le i\le n}$).

Observe that

$$\|T\| \le C. \tag{2.13}$$

By duality we have

$$\begin{aligned} T^* : l_2^n(X) &\longrightarrow \frac{L_\infty^{2n}(X)}{E^\perp}, \\ (x_i)_{1\le i\le n} &\longmapsto \sum_{i=1}^n x_i\varphi_i + E^\perp, \end{aligned} \tag{2.14}$$

where $E^\perp = \{\sum_{i=n+1}^{2n} \varphi_i x_i', (x_i')_{n+1\le i\le 2n} \subset X\}$ is the subspace of $L_\infty^{2n}(X)$ which is orthogonal to E.

Since $\|T\| = \|T^*\|$ (T^* is the adjoint operator of T), hence we have

$$\inf_{R\in E^\perp} \left\| \sum_{i=1}^{n} x_i\varphi_i + R \right\|_{L_\infty^{2n}(X)} \le C\left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2}. \tag{2.15}$$

If now $\widetilde{u}: L_1^{2n} \to X$ is an extension of u, by Riesz representation theorem then there is Ψ in L_∞^{2n} such that

$$\begin{gathered} \forall \varphi \in L_1^{2n}, \quad \widetilde{u}(\varphi) = \frac{1}{2n}\sum_{i=1}^{2n} \varphi_i \Psi_i, \\ \|\widetilde{u}\| = \|\Psi\|_{L_\infty^{2n}}. \end{gathered} \tag{2.16}$$

Since $\widetilde{u}(\varphi_i) = x_i$, we have

$$\Psi = \sum_{i=1}^{n} x_i\varphi_i + R. \tag{2.17}$$

The correspondence $\widetilde{u} \to \Psi$ is bijective and this implies that

$$\inf \|\widetilde{u}\| = \inf_{R\in E^\perp} \left\| \sum_{i=1}^{n} x_i\varphi_i + R \right\|_{L_\infty^{2n}(X)}. \tag{2.18}$$

This concludes the proof. □

We say now that a Banach space X is of cotype strongly 2-Kašin if there is a positive constant C such that, for all integers n and for all finite sequences $(x_i)_{1\le i\le n}$ in X, we have

$$\pi_2(v) \le C \left\| \sum_{i=1}^{n} \varphi_i x_i \right\|_{L_1^{2n}(X)}, \tag{2.19}$$

where $v: l_2^n \to X$ is the operator defined by $v(e_i) = x_i$ for all $1 \le i \le n$.

We denote by

$$K_2^{\text{strong}}(X) = \inf\{C : (2.19) \text{ holds } \forall (x_i)_{1\le i\le n}, n \ge 1\}. \tag{2.20}$$

Corollary 2.5. *Let X be a Banach space and let C be a positive constant. The following assertions are equivalent.*

(i) *The space X^* is of cotype strongly 2-Kašin and $K_2^{\text{strong}}(X^*) \le C$.*

(ii) *For all integers n and any $u: l_2^n \to X$, u admits an extension $\widetilde{u}$ to L_1^{2n} such that $\widetilde{u}/E_n = u\beta_n^{-1}$ and $\|\widetilde{u}\| \le C\pi_2(u^*)$.*

Proof. Fixed n in $\mathbb{N}$, let $E = \{\sum_{i=1}^{n} \varphi_i \xi_i, (\xi_i)_{1\le i\le n} \subset X^*\}$ which is a closed subspace of $L_1^{2n}(X^*)$. We now define the operators

$$\begin{aligned} T : E &\longrightarrow \pi_2(l_2^n, X^*), \\ \sum_{i=1}^{n} \varphi_i \xi_i &\longmapsto v, \end{aligned} \tag{2.21}$$

where $v : l_2^n \to X^*$ defined by $v(e_i) = \xi_i$.

We have

$$\left\| T\left(\sum_{i=1}^{n} \varphi_i \xi_i\right)\right\| = \pi_2(v) \le C\left\|\sum_{i=1}^{n} \varphi_i \xi_i\right\|. \tag{2.22}$$

By duality, we obtain

$$\begin{aligned} T^* : \pi_2(X^*, l_2^n) &\longrightarrow \frac{L_\infty^{2n}(X)}{E^\perp}, \\ w &\longmapsto \sum_{i=1}^{n} x_i \varphi_i + E^\perp, \end{aligned} \tag{2.23}$$

where $w : X^* \to l_2^n$ is a linear operator defined by $w(\xi) = \langle x_i, \xi\rangle$.

Let $u(e_i) = x_i$. We have

$$\inf_{R\in E^\perp} \left\| \sum_{i=1}^{n} x_i \varphi_i + R \right\|_{L_\infty^{2n}(X)} \le C\pi_2(u^*). \tag{2.24}$$

We conclude directly by using (2.18). □

Remark 2.6. Let X be a Banach space. If X has Gaussian (resp., Rademacher) cotype 2, then (2.19) holds with (g_i) (resp., (r_i)) and conversely. The space X is of cotype strongly 2-Kašin implies that X is of cotype 2-Kašin. We do not know if the converse is true.

Let us introduce the following definition.

Definition 2.7. Let X be a Banach space. Say that X satisfies weak Grothendieck's theorem if there is a positive constant C such that for all n in $\mathbb{N}$ and any linear operator u from X into l_2^n, there exists

$$\pi_2(u) \le C\|u\|. \tag{2.25}$$

Remark 2.8. (1) X satisfies W.G.T. if and only if X^{**} satisfies WGT.

(2) L_1 and L_∞ verify weak Grothendieck's theorem. The spaces S_1 (see below) and $B(l_2)$ (see [8, Corollary 4.2]) fail this.

(3) The classical definition is let X be a Banach space. We will say that X satisfies Grothendieck's theorem if there is a constant C such that, for any linear operator u from X into a Hilbert space H, we have

$$\pi_1(u) \le C\|u\|. \tag{2.26}$$

(4) We can replace H by l_2^n for any integer n (i.e., there is a constant C such that for any integer n and any $u : X \to l_2^n$ we have $\pi_1(u) \leq C\|u\|$). Also, this is equivalent to the dual property (i.e., there is a constant C' such that for every linear operator from X^* into an L_1-space, we have $\pi_2(u) \leq C'\|u\|$). GT implies WGT. If X is of (classical) cotype 2, then we have equivalence between GT and WGT because $\pi_p(X,Y) = \pi_2(X,Y)$ for any Banach space Y and for all $p \leq 2$ (see [7]).

(5) The space L_1 verifies Grothendieck's theorem. In [2] Bourgain proved that L_1/H_1 is of cotype 2 and verifies Grothendieck's theorem (L_1 is the L_1-space relative to the circle group and H_1 the subspace of L_1 spanned by all functions $\{e^{\text{int}}, n \geq 0\}$).

(6) Suppose that X is a subspace of $C(K)$ and that $C(K)/X$ is reflexive. Then every operator with domain X and range a cotype 2 space is 2-summing [6, 11]. As corollary, let X be a reflexive subspace of an L_1. Then, every operator $u : L_1/X \to l_2$ is 1-summing.

(7) For any Banach E of cotype 2, Pisier has constructed in [12] a Banach space X which contains isometrically E such that, X and X^* are both of cotype 2 and verify Grothendieck's theorem.

3. S_p fails WGT for all $1 \leq p \leq 2$

We recall (see [14]) the noncommutative analogues of l_p which is the Schatten class S_p. Let $0 < p < \infty$. We will denote by $B(l_2)$ the space of all bounded linear operators $u : l_2 \to l_2$ and by S_p the subspace of all compact operators such that $\operatorname{tr}|u|^p < \infty$ (where $|u| = (uu^*)^{1/2}$). We equip it with the norm if $1 \leq p < \infty$ and the p-norm if $0 < p < 1$:

$$\|u\|_p = \left(\operatorname{tr}|u|^p\right)^{1/p} \tag{3.1}$$

for which it becomes a Banach space if $1 \leq p < \infty$ and a quasi-Banach if $0 < p < 1$. If $p = \infty$, S_∞ is the subspace of all compact operators on l_2 equipped with operator norm. We have $(S_p)^* = S_q$ for $1 < p \leq \infty$ and $1/p + 1/q = 1$, and also $S_1^* = B(l_2)$. We do not know if the Schatten spaces S_p are of the same cotype Kašin as the usual l_p-spaces for $1 \leq p \leq 2$.

Finally, we denote by S_p^n and $B(l_2^n)$ the finite dimensional version of S_p and $B(l_2)$, respectively.

Let $0 < p \leq q \leq \infty$. We have for $u \in B(l_2^n)$,

$$\|u\|_q \leq \|u\|_p \leq n^{1/p-1/q}\|u\|_q. \tag{3.2}$$

Let R_n denote the subspace of S_p^n consisting of all $n \times n$ matrices u such that $u_{i,j} = 0$ when $i \neq 1$ (first row matrices). Then $a = uu^*$ is the matrix with $a_{1,1} = \sum_{j=1}^n |u_{1,j}|^2 = \|u\|_2^2$ and $a_{i,j} = 0$ when $(i,j) \neq (1,1)$. Hence $|u|$ is the rank one operator $\|u\|_2 e_1 \otimes e_1$. Its norm in all spaces S_p^n, $0 < p \leq \infty$ is equal to $\|u\|_2$. In particular R_n equipped with the S_p^n-norm is isometric to l_2^n. We denote by p_n the natural projection from S_p^n into R_n defined by $p_n(u) = v$ such that $v_{1j} = u_{1j}$ for $1 \leq j \leq n$. We have $\|p_n\| \leq 1$.

The proposition to be proved now is the finite dimensional version of the theorem of extension.

Proposition 3.1. *Suppose that for some $p > 0$, there exits a constant C_p such that for every n and every linear operator u from E_n to S_p^n, there is an extension $\tilde{u} \in B(L_1^{2n}, S_p^n)$ of u with*

$\|\widetilde{u}\| \leq C_p\|u\|$. *Then*

$$C_p \geq C\sqrt{n}, \tag{3.3}$$

where C is an absolute constant.

Proof. Let u_n be the operator sending the n vector basis of E_n to the n vector basis of R_n ($u_n(\varphi_i) = e_{1,i}, 1 \leq i \leq n$). This operator is an isomorphism, by the above remark and (2.6). We have $\|u_n\| \leq B_1$ and $\|u_n^{-1}\| \leq A_1$. Let $\widetilde{u}_n$ be an extension of u_n to an operator from L_1^{2n} to S_p^n, with $\|\widetilde{u}_n\| \leq C_p\|u_n\|$. Consider now the following commutative diagram:

$$\begin{array}{ccccc} L_1^{2n} & \xrightarrow{\widetilde{u}_n} & S_p^n & & \\ \uparrow{\scriptstyle \mathbf{e}_n} & & \downarrow{\scriptstyle p_n} & & \\ E_n & \xrightarrow{u_n} & R_n & \xrightarrow{u_n^{-1}} & E_n \end{array} \tag{3.4}$$

Let $q_n = u_n^{-1}p_n\widetilde{u}_n$. Then q_n is a projection from L_1^{2n} to E_n. Since E_n is A_1B_1-isomorphic to l_2^n (Theorem 2.1), we get by Grothendieck's theorem [4] that q_n is 1-summing with $\pi_1(q_n) \leq A_1K_G\|p_n\widetilde{u}_n\|$. Restricting q_n to E_n we obtain for the identity i_n of E_n the estimation

$$\sqrt{n} = \pi_2(i_n) \leq \pi_2(q_n) \leq \pi_1(q_n) \leq A_1K_G\|\widetilde{u}_n\| \leq A_1K_GC_p\|u_n\| \leq A_1B_1K_GC_p. \tag{3.5}$$

This completes the proof. □

Let now $\mathcal{B}_n$ be the σ-algebra on $[0,1]$ generated by the Rademacher functions $\{r_1,\dots, r_n\}$ ($r_n(t) = \operatorname{sign}(\sin 2^n\pi t)$). The space $L_p([0,1],\mathcal{B}_n,\nu)$, where ν is the Lebesgue measure in $[0,1]$, is isometric to $L_p^{2^n}$.

We denote by G (resp., G_n) the closed linear subspace in $L_1([0,1],\nu)$ (resp., $L_1^{2^n}$) of the Rademacher functions $\{r_n\}_{n\in\mathbb{N}}$ (resp., $\{r_i, 1 \leq i \leq n\}$). Let $g : G \to L_1([0,1],\nu)$ (resp., $g_n : G_n \to L_1^{2^n}$) be the isometric embedding. By Khinchine's inequalities, there are positive constants A_1' and B_1' such that for every (a_n) in l_2 we have

$$A_1'\left(\sum_{n\geq 1}|a_n|^2\right)^{1/2} \leq \left(\int_{[0,1]}\left|\sum_{n\geq 1}a_nr_n(t)\right|d\nu \leq B_1'\left(\sum_{n\geq 1}|a_n|^2\right)^{1/2}. \tag{3.6}$$

Hence G (resp., G_n) is isomorphic to l_2 (resp., l_2^n). We will denote by $\alpha : l_2 \to G$ (resp., $\alpha_n : l_2^n \to G_n$) the isomorphism which maps e_i onto r_i. We have $\|\alpha\| \leq B_1'$, $\|\alpha^{-1}\| \leq A_1'$, and also the same for α_n.

Proposition 3.2. *Suppose that for some $p > 0$, there exits a constant C_p such that for every n and every linear operator u from G_n to S_p^n there is an extension $\widetilde{u} \in B(L_1^{2^n}, S_p^n)$ of u with $\|\widetilde{u}\| \leq C_p\|u\|$. Then*

$$C_p \geq C\sqrt{n}, \tag{3.7}$$

where C is an absolute constant.

Proof. The same proof as in Proposition 3.1. □

THEOREM 3.3. *Let $0 < p \le \infty$. Let $u : G \to S_p$ be a compact linear operator. In general, there is no continuous linear operator $\widetilde{u}$ extending u to $L_1([0,1],\nu)$.*

Proof. Suppose that for any compact linear operator $u : G \to S_p$ there is a bounded linear operator $\widetilde{u} : L_1([0,1],\nu) \to S_p$ extending u. It follows from the open mapping theorem that there is an absolute constant C_p such that

$$\|\widetilde{u}\| \le C_p \|u\| \tag{3.8}$$

for any u. This implies by Proposition 3.2 that $C_p \ge C\sqrt{n}$ for any integer n. This is impossible when n is large enough. □

THEOREM 3.4. *Let $0 < p \le \infty$. Let $u : E_0 \to S_p$ be a compact linear operator. In general, there is no continuous linear operator $\widetilde{u}$ extending u.*

Proof. Using the same proof as in Proposition 3.2 (we take E_0^n instead of G_n) and Theorem 3.3, we show that the extension property concerning $(L_1([0,1],\nu), E_0)$ fails for all $0 < p \le \infty$. □

The following result shows that space S_p fails GT.

THEOREM 3.5. *The space S_p fails GT for all $1 \le p \le \infty$ and consequently WGT for $1 \le p \le 2$.*

Proof. Consider the following diagram:

$$R_n \xrightarrow{i_n} S_p^n \xrightarrow{p_n} R_n, \tag{3.9}$$

where i_n is the canonical injection. We have $\mathrm{id}_{R_n} = p_n \circ i_n$. Since $\sqrt{n} \le \pi_1(\mathrm{id}_{R_n}) \le \pi_1(p_n)$ and $\|p_n\| \le 1$, hence S_p fails GT for all $1 \le p \le \infty$. As S_p is of cotype 2 for $1 \le p \le 2$ then, by one result of Maurey, we have $\pi_1(p_n) \le C\pi_2(p_n)$ for some constant C. This implies the proof. □

Remark 3.6. The space $B(l_2)$ fails weak Grothendieck's theorem because by [8, Corollary 4.2] we have $\pi_2(B(l_2), l_2) \ne B(B(l_2), l_2)$.

4. Characterization of spaces which satisfy WGT

We start this section by recalling some notations and facts. We denote by $l_p^{\omega}(X)$ (resp., $l_p^{n\omega}(X)$) the space of all sequences (x_i) (resp., $(x_i)_{1 \le i \le n}$) in X with the norm

$$\begin{gathered} \|(x_i)\|_{l_p^{\omega}(X)} = \sup_{\|\xi\|_{X^*}=1} \left(\sum_1^{\infty} |\langle x_i, \xi\rangle|^p \right)^{1/p} < \infty, \\ \left(\text{resp., } \|(x_i)\|_{l_p^{n\omega}(X)} = \sup_{\|\xi\|_{X^*}=1} \left(\sum_1^{n} |\langle x_i, \xi\rangle|^p \right)^{1/p} \right). \end{gathered} \tag{4.1}$$

We know (see [3]) that $l_p(X) = l_p^\omega(X)$ for some $1 \le p < \infty$ if and only if $\dim(X)$ is finite. If $p = \infty$, we have $l_\infty(X) = l_\infty^\omega(X)$. We have also if $1 < p \le \infty$, $l_p^\omega(X) \equiv B(l_{p^*}, X)$, and $l_1^\omega(X) \equiv B(c_O, X)$ isometrically (where p^* is the conjugate of p, i.e., $1/p + 1/p^\star = 1$). In other words, let $v : l_{p^*} \to X$ be a linear operator such that $v(e_i) = x_i$ (namely, $v = \sum_1^\infty e_j \otimes x_j$, e_j denotes the unit vector basis of l_p), then

$$\|v\| = \big\|(x_i)\big\|_{l_p^\omega(X)} = \left\|\sum_1^\infty e_j \otimes x_j\right\|_{l_p \hat{\otimes}_\epsilon X}. \tag{4.2}$$

We prove in the following theorem that the spaces which satisfy WGT and which happen to be also of cotype strongly 2-Kašin can be characterized by an extension property.

Theorem 4.1. *The following properties of a Banach space X are equivalent:*

(i) *the space X^* is of cotype strongly 2-Kašin and verifies WGT;*

(ii) *there is a positive constant C such that for every $n \in \mathbb{N}$ and every $u : E_n \to X$, then u admits an extension $\widetilde{u} : L_1^{2n} \to X$ such that $\widetilde{u}/E_n = u$ and $\|\widetilde{u}\| \le C\|u\|$.*

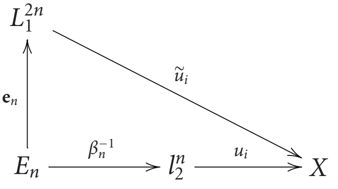
Proof. We prove that (ii) ⇒ (i). Let $v : l_2^n \to X$ be a linear operator. Consider $u = v\beta_n^{-1} : E_n \to X$, then u admits an extension $\widetilde{u} : L_1^{2n} \to X$ such that

$$\|\widetilde{u}\| \le C\|u\| \le C\big\|\beta_n^{-1}\big\|\|v\| \le C/A_1\pi_2(v^*). \tag{4.3}$$

From Corollary 2.5, we obtain that X^* is of cotype strongly 2-Kašin and $K_2^{\text{strong}}(X^*) \le C/A_1$. Let now $u : X^* \to l_2^n$ be an operator. First, we notice that $B(l_2^n, X^{**}) \equiv B(l_2^n, X)^{**} \equiv B(X^*, l_2^n)$ isometrically. Since $u : X^* \to l_2^n$ is in $B(l_2^n, X)^{**}$, then by Goldstine's theorem, there is a net of operators $u_i^* : X^* \to l_2^n$ which are w^*-continuous with $\|u_i\| \le \|u\|$ for all i and $\{u_i^*\}$ converges to u in w^*-topology of $B(l_2^n, X)^{**}$. As u_i^* is 2-summing this implies that u is 2-summing and $\pi_2(u) = \lim_i \pi_2(u_i^*)$. Indeed,

$$\begin{aligned}
\pi_2(u) &= \sup\{\operatorname{Tr}(uv), v : l_2^n \longrightarrow X^{***}\ \pi_2(v) \le 1\} \\
&= \sup\{\lim_i \operatorname{Tr}(u_i^* v), v : l_2^n \longrightarrow X^{***}\ \pi_2(v) \le 1\} \\
&= \lim_i \sup\{\operatorname{Tr}(u_i^* v), v : l_2^n \longrightarrow X^{***}\ \pi_2(v) \le 1\} \\
&= \lim_i \pi_2(u_i^*).
\end{aligned} \tag{4.4}$$

Let us consider the following commutative diagram:



(4.5)

by duality, we have

$$\begin{array}{ccccc} & & & & L_\infty^{2n} \\ & & \nearrow^{\widetilde{u}_i^*} & & \downarrow \mathbf{e}_n^* \\ X^* & \xrightarrow{u_i^*} & l_2^n & \xrightarrow{(\beta_n^{-1})^*} & E_n^* \end{array} \tag{4.6}$$

hence

$$\begin{aligned} \pi_2(u_i^*) &= \pi_2(\beta_n^*(\beta_n^{-1})^* u_i^*) \leq \|\beta_n^*\| \pi_2((\beta_n^{-1})^* u_i^*) \\ &\leq \|\beta_n^*\| \|\widetilde{u}_i^*\| \pi_2(\mathbf{e}_n^*) \leq \|\beta_n^*\| \|\beta_n^{-1}\| \|u_i\| \pi_2(\mathbf{e}_n^*) \\ &\leq A_1^{-1} B_1 \|u_i\| \pi_2(\mathbf{e}_n^*). \end{aligned} \tag{4.7}$$

Thus

$$\lim_i \pi_2(u_i^*) \leq A_1^{-1} B_1 \pi_2(\mathbf{e}_n^*) \lim_i \|u_i\| \leq A_1^{-1} B_1 \pi_2(\mathbf{e}_n^*) \|u\|. \tag{4.8}$$

Consequently

$$\pi_2(u) \leq A_1^{-1} B_1 \pi_2(\mathbf{e}_n^*) \|u\|. \tag{4.9}$$

This shows that X has WGT because the numbers $\pi_2(\mathbf{e}_n^*)$ are uniformly bounded by Maurey's theorem [7].

(i) ⇒ (ii). The space X^* is of cotype strongly 2-Kašin which implies by Corollary 2.5 that for any $u : l_2^n \to X$, u admits an extension $\widetilde{u}$ to L_1^{2n} such that $\widetilde{u}/E_n = u\beta_n^{-1}$ and $\|\widetilde{u}\| \leq K_2^{\text{strong}}(X^*)\pi_2(u^*)$. As X^* verifies WGT, then $\pi_2(u^*) \leq C' \|u\|$ and hence

$$\|\widetilde{u}\| \leq C' K_2(X^*) \|u\| \leq C\|u\| \, (C = C' K_2(X^*)) \tag{4.10}$$

which gives the extension. □

We end this paper by the following remark.

Remark 4.2. We do not know if S_p for $1 \leq p \leq 2$ is of cotype 2-Kašin.

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