NULL CONTROLLABILITY OF A NONLINEAR POPULATION DYNAMICS PROBLEM

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Received 23 November 2005; Revised 8 August 2006; Accepted 11 October 2006

We establish a null controllability result for a nonlinear population dynamics model. In our model, the birth term is nonlocal and describes the recruitment process in newborn individuals population. Using a derivation of Leray-Schauder fixed point theorem and Carleman inequality for the adjoint system, we show that for all given initial density, there exists an internal control acting on a small open set of the domain and leading the population to extinction.

1. Introduction

For a given positive real function F, we consider in this paper the following nonlinear population dynamics model:

$$
\begin{aligned}
&\frac{\partial y}{\partial t}+\frac{\partial y}{\partial a}-\Delta y+\mu y=v1_{\omega} \quad \text{in } (0,T)\times(0,A)\times\Omega,\\
&y(t,a,\sigma)=0 \quad \text{on } (0,T)\times(0,A)\times\partial\Omega,\\
&y(0,a,x)=y_0(a,x) \quad \text{in } (0,T)\times(0,A)\times\Omega,\\
&y(t,0,x)=F\left(\int_0^A \beta(t,a,x)y(t,a,x)da\right) \quad \text{on } (0,T)\times\Omega,
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N\geq 1$ with a smooth boundary $\partial\Omega$, $\sigma\in\partial\Omega$, T is a positive real and ω an open subset such that $\overline{\omega}\subset\Omega$. Here $y(t,a,x)$ is the distribution of individuals of age a at time t and location $x\in\Omega$, 1_{ω} is the characteristic function of ω, A is the maximal live expectancy, Δ the Laplacian with respect to the spatial variable, $\beta(t,a,x)$ and $\mu(t,a,x)$ denote, respectively, the natural fertility and the natural death rate of individuals of age a at time t and location x. Thus, the formula $\int_0^A \beta(t,a,x)y(t,a,x)da$ denotes the distribution of newborn individuals at time t and location x. In an oviparus species it denotes the total eggs at time t and position x. Therefore, the quantity $F(\int_0^A \beta(t,a,x)y(t,a,x)da)$ is the distribution of eggs that hatches at time t and position x.

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 49279, Pages 1–20
DOI 10.1155/IJMMS/2006/49279

System (1.1) describes the evolution of an internal controlled age and space structured population under inhospitable boundary conditions in the case that the flux of individuals has the form $-\nabla y(t,a,x)$.

The purpose of this paper is to prove a null controllability result for (1.1) at any time T. This means more precisely that there exists a control $v \in L^2((0,T)\times(0,A)\times\omega)$ such that the associated solution of (1.1) verifies

$$y(T,a,x) = 0 \quad \text{a.e. in } (0,A)\times\Omega. \tag{1.2}$$

In our knowledge the first controllability result for an age and space structured population dynamics model was established by Ainseba and Langlais in [4]: they proved that a set of profiles is approximately reachable. In [2] a local exact controllability result was proved for a linear population dynamics. More precisely, in [2] the authors proved that if the initial distribution is small enough, one can find a control that leads the population to extinction. The method used there is different from ours. In fact in [2] the adjoint system was taken as a collection of parabolic equations along characteristic lines. This allowed the authors to use Carleman inequality for parabolic equation. Ainseba and Iannelli in [3] proved a null controllability result for a nonlinear population dynamics model. In [3] the natural rates depend on the total population $P = \int_0^A y(t,a,x)da$. The method in [3] used Kakutani fixed point theorem. Therefore, crucial assumptions were made: first, the natural rates were supposed to be globally Lipschitz with respect to the variable P, secondly in order to perform key estimates, the death rate μ verified the following growth condition: $0 \le \mu\exp(\int_0^a \mu(s)ds) \le \zeta$ where ζ is a positive constant.

In the case we study here, the above results cannot be applied. Indeed, since the birth process is not globally Lipschitz with respect to the variable P and, without the previous growth condition on μ one cannot use the method of [3]. On the other hand, the nonlinearity excludes the use of the result of [2]. In what follows, using a Carleman inequality for an adjoint system we establish a null controllability result for the nonlinear population dynamics models stated in (1.1) when the initial distribution is in $L^2((0,A)\times\Omega)$. Roughly, in our method we first study a null controllability result for a population in which the birth process is given by a fixed function. Afterwards, we prove the null controllability result for the system (1.1) by means of a derivation of Leray-Shauder theorem.

The remainder of this paper is as follows: in Section 2, we state assumptions and we provide the main result. In Section 3 we study a null controllability result for some associated model. Section 4 is devoted to the proof of the main result.

2. Assumptions and main result

For the sequel we assume that the following assumptions hold:

$$H_1 \begin{cases} \mu(t,a,x) = \mu_0(a) + \mu_1(t,a,x) \quad \text{a.e. in } (0,T)\times(0,A)\times\Omega, \\ \mu \ge 0 \quad \text{a.e. in } (0,T)\times(0,A)\times\Omega, \\ \mu_1 \in L^\infty((0,T)\times(0,A)\times\Omega); \quad \mu_1(t,a,x) \ge 0 \quad \text{a.e. in } (0,T)\times(0,A)\times\Omega, \\ \mu_0 \in L^1_{\text{loc}}(0,A), \quad \lim_{a\to A}\displaystyle\int_0^a \mu_0(s)ds = +\infty, \end{cases}$$

$$H_2 \begin{cases} \beta \in C^2([0,T]\times[0,A]\times\overline{\Omega}), \\ \beta(t,a,x) \geq 0 \quad \text{in } [0,T]\times[0,A]\times\overline{\Omega}, \\ \exists 0<a_0<a_1<A \qquad \text{such that } \beta(t,a,x)=0 \text{ in } [0,T]\times((0,a_0)\bigcup(a_1,A)\times\Omega). \end{cases} \tag{2.1}$$

H_3 F defined on $\mathbb{R}$ is a positive continuous function and there exist positive constants C_0 and C_1 such that $F(t) \leq C_0 + C_1|t|$, for all $t \in \mathbb{R}$.

Remark 2.1. Since μ and β are natural rates, the second assumptions of H_1 and H_2 are natural. The third assumption of H_2 is also natural, since it means that older and younger individuals are not fertile. The fourth assumption in H_1 is also a standard one, it means that all individual dies before the age A. In [3] the model did not take explicitly into account the death of newborns. Indeed the birth process there has the form $y(t,0,x) = \int_0^A \beta(t,a,x,P(t,x))y(t,a,x)da$ where $P(t,x) = \int_0^A y(t,a,x)da$. We present here a quite different model. In fact our model addresses both supply and death of newborns. Moreover in the case $F(t) = kt$ with k a fixed positive constant, one obtains from (1.1) a linear population dynamics problem.

Assume now that the function F is a globally Lipschitz one and verifies $F(0) = 0$. Then, one can rewrite F as $F(t) = t\Phi(t)$ for a.e. $t \in \mathbb{R}$. Therefore, the fourth equation of (1.1) becomes $y(t,0,x) = \int_0^A \beta y\, da \Phi(\int_0^A \beta y\, da)$. Hence, one obtains the system considered with Neumann boundary conditions in [8, 10] where existence of solution was studied.

From now we set $Q = (0,T)\times(0,A)\times\Omega$; $q = (0,T)\times(0,A)\times\omega$; $Q_A = (0,A)\times\Omega$; $Q_T = (0,T)\times\Omega$; $\Sigma = (0,T)\times(0,A)\times\partial\Omega$ and $C_\beta = \|\beta\|_{C^2(\overline{Q})}$.

For $\alpha \geq 0$ we set $S_\alpha(t,a) = \exp(-\alpha t + \int_0^a \mu_0(s)ds)$, $X_\alpha = \{z \in L^2(Q_A);\ S_\alpha(t,a)z \in L^2(Q_A)\}$, and $Y_\alpha = \{v \in L^2(q);\ S_\alpha(t,a)v \in L^2(q)\}$. It is obvious that $\alpha_1 \geq \alpha_2$ implies $X_{\alpha_1} \subset X_{\alpha_2}$ and $Y_{\alpha_1} \subset Y_{\alpha_2}$.

In the sequel, ν will denote the unit outward normal vector to $\partial\Omega$ and $C(\Omega,T,A,\dots)$ will denote positive constant that depends only on $\Omega,T,A,\dots$.

We are now ready to state the main result of this paper.

Theorem 2.2. *For any $\gamma > 0$ assumed to be small enough, there exists a control $v \in Y_0$ such that the associated solution of (1.1) satisfies*

$$y(T,a,x) = 0 \quad \text{a.e. in } (\gamma,A)\times\Omega \tag{2.2}$$

for all $y_0 \in X_0$.

Remark 2.3. In the proof, it will appear clearly that such a control depends essentially on γ.

Let us denote by λ_0 a positive constant which will be fixed later. We make the following standard changes: $\hat{y} = S_{\lambda_0}(t,a)y$, $\hat{v} = S_{\lambda_0}(t,a)v$, $\widehat{\beta} = S_{\lambda_0}^{-1}(0,a)\beta$ and $\widehat{y_0} = S_{\lambda_0}(t,a)y_0$. Then it follows that $\widehat{y}$ solves the following system:

$$\begin{aligned}
&\frac{\partial \widehat{y}}{\partial t}+\frac{\partial \widehat{y}}{\partial a}-\Delta \widehat{y}+(\mu_1+\lambda_0)\widehat{y}=\widehat{v}1_\omega \quad \text{in } Q,\\
&\widehat{y}(t,a,\sigma)=0 \quad \text{on } \Sigma,\\
&\widehat{y}(0,a,x)=\widehat{y}_0(a,x) \quad \text{in } Q_A,\\
&\widehat{y}(t,0,x)=e^{-\lambda_0 t}F\left(e^{\lambda_0 t}\int_0^A \widehat{\beta}(t,a,x)\widehat{y}(t,a,x)da\right) \quad \text{in } Q_T.
\end{aligned} \tag{2.3}$$

The null controllability problem of Theorem 2.2 is now reduced to find $\widehat{v}$ in $L^2(q)$ such that $\widehat{y}$ verifies (2.2). In fact after the previous change we obtain a system involving bounded coefficients and this allows one to establish a global Carleman inequality. In the sequel for the sake of simplicity, we will consider only the previous system without hats and in addition we will write μ instead of $\mu_1+\lambda_0$.

3. Null controllability for some linearized model

3.1. An observability inequality result. We recall here that there exists a function $\Psi \in C^2(\overline{\Omega})$ such that $\Psi(x)=0$, for all $x\in\partial\Omega$; $\Psi(x)>0$, for all $x\in\Omega$ and $\nabla\Psi(x)\neq 0$, for all $x\in\overline{\Omega-\widetilde{\omega}}$ where $\widetilde{\omega}$ is an open set such that $\overline{\widetilde{\omega}}\subset\omega\subset\Omega$. (See [6] for the existence of Ψ.)

Let us consider the following system:

$$\begin{aligned}
&-\frac{\partial w}{\partial t}-\frac{\partial w}{\partial a}-\Delta w+\mu w=f \quad \text{in } Q,\\
&w(t,a,\sigma)=0 \quad \text{on } \Sigma,\\
&w(T,a,x)=g(a,x) \quad \text{in } Q_A,\\
&w(t,A,x)=0 \quad \text{in } Q_T.
\end{aligned} \tag{3.1}$$

Setting for all positive real λ, $\eta(t,a,x)=(e^{2\lambda\|\Psi\|_\infty}-e^{\lambda\Psi(x)})/at(T-t)$ and $\varphi(t,a,x)=e^{\lambda\Psi(x)}/at(T-t)$ one can prove easily by adapting the method of [6] or [9] the following.

Proposition 3.1. *There exist positive constants $s_1\geq 1$ and $\lambda_1\geq 1$ and there exists a positive constant C such that for all $s\geq s_1$, $\lambda\geq\lambda_1$, and for all solution of (3.1), the following inequality holds:*

$$\int_Q e^{-2s\eta}s^3\varphi^3\lambda^4w^2\,dt\,da\,dx\leq C\left(\int_Q e^{-2s\eta}f^2\,dt\,da\,dx+\int_q e^{-2s\eta}s^3\varphi^3\lambda^4w^2\,dt\,da\,dx\right). \tag{3.2}$$

Remark 3.2. The proof of Proposition 3.1 is absolutely similar to those of global Carleman inequality for the linear heat equation proposed in [9] or in [6]. Roughly, for the proof of (3.2), one makes the change of variable: $u=e^{-s\eta}w$ in order to get from the definition of η the following:

$$u(0,a,x)=u(T,a,x)=u(t,0,x)=0. \tag{3.3}$$

Subsequently, one derives estimates on u and after return to w. We will prove first (3.2) for a function $w\in C^2(\overline{Q})$ and after the result for $w\in L^2(Q)$ will follow by density arguments.

Proof of Proposition 3.1. We suppose that the function $w \in C^2(\overline{Q})$ and verifies (3.1) and we make the following change of variables $u = e^{-s\eta}w$. Then immediately it follows by using the definition of η and (3.1) that

$$u(0,a,x) = u(T,a,x) = 0 \quad \text{in } (0,A)\times\Omega, \tag{3.4}$$

$$u(t,0,x) = u(t,A,x) = 0 \quad \text{in } (0,T)\times\Omega, \tag{3.5}$$

$$u(t,a,\sigma) = 0 \quad \text{in } (0,T)\times(0,A)\times\partial\Omega. \tag{3.6}$$

Notice that

$$\nabla\eta = -\lambda\varphi\nabla\Psi, \tag{3.7}$$

$$\nabla\varphi = \lambda\varphi\nabla\Psi. \tag{3.8}$$

Using once again the definitions of η and φ, we deduce that there exist positive constants denoted by C such that $|\partial\eta/\partial a| \le C\varphi^2$, $|\partial\eta/\partial t| \le C\varphi^2$, $|\partial^2\eta/\partial a\partial t| \le C\varphi^3$, and $|\partial\eta/\partial a^2| \le C\varphi^3$.

Similarly we get

$$\left|\frac{\partial\varphi}{\partial a}\right| \le C\varphi^2, \qquad \left|\frac{\partial\varphi}{\partial t}\right| \le C\varphi^2, \qquad \left|\frac{\partial^2\varphi}{\partial a\partial t}\right| \le C\varphi^3, \qquad \left|\frac{\partial\varphi}{\partial a^2}\right| \le C\varphi^3. \tag{3.9}$$

We have

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -s\left(\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial a}\right)u + e^{-s\eta}\left(\frac{\partial w}{\partial t} + \frac{\partial w}{\partial a}\right). \tag{3.10}$$

From (3.7) and (3.8) we get

$$\Delta u = s\lambda\Delta\Psi\varphi u + s\lambda^2|\nabla\Psi|^2\varphi u - s^2\lambda^2|\nabla\Psi|^2\varphi^2 u + 2s\lambda\varphi\nabla\Psi\cdot\nabla u + e^{-s\eta}\Delta w. \tag{3.11}$$

Therefore

$$\begin{aligned} &-\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a}\right) - \Delta u + \mu u \\ &\quad = e^{-s\eta}f - s\lambda^2 u\varphi|\nabla\Psi|^2 - 2s\lambda\varphi\nabla\Psi\cdot\nabla u + s^2\lambda^2\varphi^2|\nabla\Psi|^2 u + s\left(\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial a}\right)u - s\lambda\varphi u\Delta\Psi. \end{aligned} \tag{3.12}$$

This equation can be rewritten as

$$P_1 u + P_2 u = g_s, \tag{3.13}$$

where

$$\begin{aligned} P_1 u &= -\frac{\partial u}{\partial t} - \frac{\partial u}{\partial a} + 2s\lambda\varphi\nabla\Psi\cdot\nabla u + 2s\lambda^2 u\varphi|\nabla\Psi|^2, \\ P_2 u &= -\Delta u - s\left(\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial a}\right)u - s^2\lambda^2\varphi^2|\nabla\Psi|^2 u, \\ g_s &= e^{-s\eta}f + s\lambda^2 u\varphi|\nabla\Psi|^2 - \mu u - s\lambda u\varphi\Delta\Psi. \end{aligned} \tag{3.14}$$

Taking the square of (3.13) and integrating the result over Q yield

$$\int_Q |P_1 u|^2 \, dt\, da\, dx + \int_Q |P_2 u|^2 \, dt\, da\, dx + 2\int_Q P_2 u P_1 u \, dt\, da\, dx = \int_Q g_s^2 \, dt\, da\, dx. \tag{3.15}$$

Let us compute $K = \int_Q P_2 u P_1 u \, dt\, da\, dx$.

We obtain

$$\begin{aligned} K = &\int_Q \Big(-\frac{\partial u}{\partial t} - \frac{\partial u}{\partial a} + 2s\lambda\varphi\nabla\Psi\cdot\nabla u + 2s\lambda^2 u\varphi|\nabla\Psi|^2\Big)\Big(-\Delta u - s\Big(\frac{\partial \eta}{\partial a} + \frac{\partial \eta}{\partial t}\Big)u\Big) dt\, da\, dx \\ &- \int_Q \Big(-\frac{\partial u}{\partial t} - \frac{\partial u}{\partial a} + 2s\lambda\varphi\nabla\Psi\cdot\nabla u + 2s\lambda^2 u\varphi|\nabla\Psi|^2\Big) s^2\lambda^2\varphi^2|\nabla\Psi|^2 u \, dt\, da\, dx. \end{aligned} \tag{3.16}$$

This computation gives twelve terms denoted by $I_{i,j}$, $i = 1,\dots,4$, $j = 1,2,3$.

We have by integration by parts

$$I_{1,1} = \int_Q \frac{\partial u}{\partial t}\Delta u \, dt\, da\, dx = \int_\Sigma \frac{\partial u}{\partial t}\frac{\partial u}{\partial \nu} dt\, da\, d\sigma - \frac{1}{2}\int_Q \frac{\partial}{\partial t}|\nabla u|^2 \, dt\, da\, dx. \tag{3.17}$$

Hence using (3.4) and (3.6) it follows that

$$\begin{gathered} I_{11} = 0, \\ I_{1,2} = s\int_Q \frac{\partial u}{\partial t}\Big(\frac{\partial \eta}{\partial a} + \frac{\partial \eta}{\partial a}\Big) u \, dt\, da\, dx. \end{gathered} \tag{3.18}$$

An integration by parts leads to

$$\begin{gathered} I_{1,2} = -\frac{s}{2}\int_Q |u|^2 \frac{\partial}{\partial t}\Big(\frac{\partial \eta}{\partial a} + \frac{\partial \eta}{\partial a}\Big) dt\, da\, dx, \\ I_{1,3} = s^2\lambda^2 \int_Q \frac{\partial u}{\partial t}\varphi^2 u|\nabla\Psi|^2 \, dt\, da\, dx. \end{gathered} \tag{3.19}$$

This gives

$$I_{1,3} = \frac{s^2\lambda^2}{2}\int_Q \frac{\partial |u|^2}{\partial t}\varphi^2|\nabla\Psi|^2 \, dt\, da\, dx. \tag{3.20}$$

Keeping in mind (3.4), an integration by parts with respect to the variable t yields

$$I_{1,3} = -s^2\lambda^2 \int_Q |u|^2 \frac{\partial \varphi}{\partial t}\varphi|\nabla\Psi|^2 \, dt\, da\, dx. \tag{3.21}$$

Similarly, one gets easily that

$$
\begin{aligned}
&I_{21} = 0, \\
&I_{2,2} = -\frac{s}{2}\int_Q |u|^2 \frac{\partial}{\partial a}\left(\frac{\partial \eta}{\partial a} + \frac{\partial \eta}{\partial a}\right) dt\,da\,dx, \\
&I_{2,3} = -s^2\lambda^2 \int_Q \varphi |u|^2 \frac{\partial \varphi}{\partial a} |\nabla \Psi|^2\,dt\,da\,dx.
\end{aligned}
\tag{3.22}
$$

Now, we are concerned by the term $I_{3,j}$.

We have

$$
I_{3,1} = -2s\lambda \int_Q \varphi \nabla \Psi \cdot \nabla u \Delta u\,dt\,da\,dx. \tag{3.23}
$$

Then we have by an integration by parts

$$
I_{3,1} = -2s\lambda \int_\Sigma \varphi \nabla \Psi \cdot \nabla u \frac{\partial u}{\partial \nu}\,dt\,da\,d\sigma + 2s\lambda \int_Q \nabla u \cdot \nabla(\varphi \nabla \Psi \cdot \nabla u)dt\,da\,dx. \tag{3.24}
$$

From the definition of Ψ and since (3.6) is fulfilled we see that for all $\sigma \in \partial\Omega$ we have $\nabla u(t,a,\sigma) = (\nabla u(t,a,\sigma) \cdot \nu(\sigma))\nu(\sigma)$ and $\nabla \Psi(\sigma) = (\nabla \Psi(\sigma) \cdot \nu(\sigma))\nu(\sigma)$.

Therefore it follows, using also (3.8), that

$$
\begin{aligned}
I_{3,1} = &-2s\lambda \int_\Sigma \varphi(\nabla \Psi \cdot \nu)|\nabla u \cdot \nu|^2\,dt\,da\,d\sigma + 2s\lambda^2 \int_Q |\nabla u \cdot \nabla \Psi|^2 \varphi\,dt\,da\,dx \\
&+ 2s\lambda \Sigma_{i,j=1}^N \left(\int_Q \varphi \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial \Psi}{\partial x_j}\,dt\,da\,dx + \int_Q \varphi \frac{\partial u}{\partial x_i}\frac{\partial^2 \Psi}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_j}\,dt\,da\,dx \right).
\end{aligned}
\tag{3.25}
$$

We have

$$
\begin{aligned}
&2s\lambda \Sigma_{i,j=1}^N \int_Q \varphi \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial \Psi}{\partial x_j}\,dt\,da\,dx \\
&\quad = s\lambda \int_\Sigma \varphi(\nabla \Psi \cdot n)|\nabla u \cdot n|^2\,dt\,da\,d\sigma - s\lambda^2 \int_Q |\nabla u|^2 |\nabla \Psi|^2 \varphi\,dt\,da\,dx \\
&\qquad - s\lambda \int_Q \varphi |\nabla u|^2 \Delta \Psi\,dt\,da\,dx.
\end{aligned}
\tag{3.26}
$$

Therefore

$$
\begin{aligned}
I_{3,1} = &-s\lambda \int_\Sigma \varphi(\nabla \Psi \cdot n)|\nabla u \cdot \nu|^2\,dt\,da\,d\sigma + 2s\lambda^2 \int_Q |\nabla u \cdot \nabla \Psi|^2 \varphi\,dt\,da\,dx \\
&- s\lambda^2 \int_Q |\nabla u|^2 |\nabla \Psi|^2 \varphi\,dt\,da\,dx - s\lambda^2 \int_Q |\nabla u|^2 |\nabla \Psi|^2 \varphi\,dt\,da\,dx \\
&- s\lambda \int_Q \varphi |\nabla u|^2 \Delta \Psi\,dt\,da\,dx + 2s\lambda \Sigma_{i,j=1}^N \int_Q \varphi \frac{\partial^2 \Psi}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\,dt\,da\,dx
\end{aligned}
\tag{3.27}
$$

$$
I_{3,2} = -2s^2\lambda \int_Q \varphi \nabla \Psi \cdot \nabla u \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial a}\right) u\,dt\,da\,dx.
$$

Classical computations give

$$I_{3,2} = -s^2\lambda^2 \int_Q \varphi|\nabla\psi|^2|u|^2\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)dt\,da\,dx + s^2\lambda\int_Q \varphi|u|^2\nabla\cdot\left(\nabla\Psi\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)\right),$$
$$I_{3,3} = -2s^3\lambda^3\int_Q \varphi^3\nabla\Psi\cdot\nabla u|\nabla\Psi|^2 u\,dt\,da\,dx. \tag{3.28}$$

Equality (3.8) and an integration by part give

$$I_{3,3} = 3s^3\lambda^4\int_Q \varphi^3u^2|\nabla\Psi|^4\,dt\,da\,dx + s^3\lambda^3\int_Q \varphi^3|u|^2\nabla\cdot\left(\nabla\Psi|\nabla\Psi|^2\right)dt\,da\,dx. \tag{3.29}$$

Now we compute the terms $I_{4,j}$

$$I_{4,1} = -2s\lambda^2\int_Q \varphi u|\nabla\Psi|^2\Delta u\,dt\,da\,dx = 2s\lambda^2\int_Q \nabla\left(\varphi u|\nabla\Psi|^2\right)\cdot\nabla u\,dt\,da\,dx. \tag{3.30}$$

Therefore

$$I_{4,1} = 2s\lambda^3\int_Q \varphi u\nabla\Psi\cdot\nabla u|\nabla\Psi|^2 dt\,da\,dx + 2s\lambda^2\int_Q \varphi|\nabla u|^2|\nabla\Psi|^2\nabla u\,dt\,da\,dx$$
$$+2s\lambda^2\int_Q \varphi u\nabla u\cdot\nabla\left(|\nabla\Psi|^2\right)dt\,da\,dx. \tag{3.31}$$

Directly, we have

$$I_{42} = -2s^2\lambda^2\int_Q \varphi|\nabla\Psi|^2\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)|u|^2\,dt\,da\,dx, \tag{3.32}$$

$$I_{43} = -2s^3\lambda^4\int_Q \varphi^3|\nabla\Psi|^4u^2\,dt\,da\,dx. \tag{3.33}$$

Grouping all the terms $I_{i,j}$ and using the boundeness of the derivatives of φ and η one can write

$$2\int_Q P_1uP_2u\,dt\,da\,dx = X_1 + X_2 - 2s\lambda\int_\Sigma \varphi\nabla\Psi\cdot\nu|\nabla u\cdot\nu|^2\,dt\,da\,d\sigma$$
$$+4s\lambda^2\int_Q \varphi|\nabla u\cdot\nabla\Psi|^2\,dt\,da\,dx$$
$$+2s\lambda^2\int_Q \varphi|\nabla u|^2|\nabla\Psi|^2\,dt\,da\,dx + 2s^3\lambda^4\int_Q \varphi^3u^2|\nabla\Psi|^4\,dt\,da\,dx, \tag{3.34}$$

where X_1 and X_2 verify

$$X_1 \le C(s\lambda+\lambda^2)\int_Q \varphi|\nabla u|^2\,dt\,da\,dx,$$
$$X_2 \le C(s^2\lambda^4+s^3\lambda^3)\int_Q \varphi^3|u|^2\,dt\,da\,dx. \tag{3.35}$$

Note that ν is the outward normal vector to $\partial\Omega$. So, using the fact that $\Psi(x) > 0$ for all $x \in \Omega$ and $\Psi(\sigma) = 0$ for all $\sigma \in \partial\Omega$ we infer that $\nabla\Psi \cdot \nu < 0$. Therefore, (3.34) yields

$$\begin{aligned} 2\int_Q P_1 u P_2 u \, dt\, da\, dx \geq X_1 + X_2 + 2s\lambda^2 \int_Q \varphi |\nabla u|^2 |\nabla\Psi|^2 \, dt\, da\, dx \\ + 2s^3\lambda^4 \int_Q \varphi^3 u^2 |\nabla\Psi|^4 \, dt\, da\, dx. \end{aligned} \tag{3.36}$$

Note also that $\Psi \in C^2(\overline{\Omega})$ and $|\nabla\Psi| \neq 0$ in $\overline{\Omega - \widetilde{\omega}}$. Consequently, there exists a positive constant δ such that $|\nabla\Psi| > \delta$ in $\overline{\Omega - \widetilde{\omega}}$. Therefore (3.36) gives

$$\begin{aligned} 2\int_Q P_1 u P_2 u \, dt\, da\, dx + 2s\lambda^2\delta^2 \int_{\widetilde{q}} \varphi |\nabla u|^2 \, dt\, da\, dx + 2s^3\lambda^4\delta^4 \int_{\widetilde{q}} \varphi^3 u^2 \, dt\, da\, dx \\ \geq X_1 + X_2 + 2s\lambda^2\delta^2 \int_Q \varphi |\nabla u|^2 \, dt\, da\, dx + 2s^3\lambda^4\delta^4 \int_Q \varphi^3 u^2 \, dt\, da\, dx, \end{aligned} \tag{3.37}$$

where $\widetilde{q} = (0,T) \times (0,A) \times \widetilde{\omega}$. Furthermore, we have

$$\int_Q g_s^2 \, dt\, da\, dx \leq \int_Q e^{-2s\eta} f^2 \, dt\, da\, dx + X_1 + X_2. \tag{3.38}$$

Then, it follows from (3.15) and (3.37) that

$$\begin{aligned} \int_Q e^{-2s\eta} f^2 \, dt\, da\, dx + X_1 + X_2 + 2s^3\lambda^4\delta^4 \int_{\widetilde{q}} \varphi^3 |u|^2 \, dt\, da\, dx + 2s\lambda^2\delta^2 \int_{\widetilde{q}} \varphi |\nabla u|^2 \, dt\, da\, dx \\ \geq \int_Q |P_1 u|^2 \, dt\, da\, dx + \int_Q |P_2 u|^2 \, dt\, da\, dx + 2s\lambda^2\delta^2 \int_Q \varphi |\nabla u|^2 \, dt\, da\, dx \\ + 2s^3\lambda^4\delta^4 \int_Q \varphi^3 |u|^2 \, dt\, da\, dx. \end{aligned} \tag{3.39}$$

We can choose s and λ sufficiently large so that

$$s\lambda^2\delta^2 \int_Q \varphi |\nabla u|^2 \, dt\, da\, dx + s^3\lambda^4\delta^4 \int_Q \varphi^3 |u|^2 \, dt\, da\, dx \geq X_1 + X_2. \tag{3.40}$$

This means more precisely that there exists positive constants $s_1 > 1$ and $\lambda_1 > 1$ such that for $s \geq s_1$ and $\lambda \geq \lambda_1$ (3.39) yields

$$\begin{aligned} \int_Q e^{-2s\eta} f^2 \, dt\, da\, dx + 2s^3\lambda^4\delta^4 \int_{\widetilde{q}} \varphi^3 |u|^2 \, dt\, da\, dx + 2s\lambda^2\delta^2 \int_{\widetilde{q}} \varphi |\nabla u|^2 \, dt\, da\, dx \\ \geq \int_Q |P_1 u|^2 \, dt\, da\, dx + \int_Q |P_2 u|^2 \, dt\, da\, dx + s\lambda^2\delta^2 \int_Q \varphi |\nabla u|^2 \, dt\, da\, dx \\ + s^3\lambda^4\delta^4 \int_Q \varphi^3 |u|^2 \, dt\, da\, dx. \end{aligned} \tag{3.41}$$

We want now to eliminate the term

$$2s\lambda^2\delta^2\int_{\tilde{q}}\varphi|\nabla u|^2\,dt\,da\,dx \tag{3.42}$$

in (3.41). For this aim, we introduce a cut-off function α such that $\alpha\in C_0^\infty(\omega)$; $0\leq\alpha\leq 1$; and $\alpha=1$ on $\tilde{\omega}$.

Multiplying P_2u by $\varphi\alpha^2 u$ and integrating the result over Q leads to

$$\begin{aligned}&\int_Q\varphi\alpha^2uP_2u\,dt\,da\,dx\\&\quad=-s\int_Q\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)u^2\varphi\alpha^2\,dt\,da\,dx-s^2\lambda^2\int_Q u^2\varphi^3\alpha^2|\Psi|^2\,dt\,da\,d-\int_Q u\Delta u\varphi\alpha^2\,dt\,da\,dx.\end{aligned} \tag{3.43}$$

Note that

$$\begin{aligned}&\int_Q u\Delta u\varphi\alpha^2\,dt\,da\,dx\\&\quad=-\int_Q|\nabla u|^2\varphi\alpha^2\,dt\,da\,dx-\lambda\int_Q u\nabla u\cdot\nabla\Psi\varphi\alpha^2\,dt\,da\,dx-2\int_Q u\nabla u\cdot\nabla\alpha\varphi\alpha\,dt\,da\,dx.\end{aligned} \tag{3.44}$$

Therefore

$$\begin{aligned}\int_Q\varphi\alpha^2uP_2u\,dt\,da\,dx=&-s\int_Q\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)u^2\varphi\alpha^2\,dt\,da\,dx\\&-s^2\lambda^2\int_Q u^2\varphi^3\alpha^2|\Psi|^2\,dt\,da\,dx+\int_Q|\nabla u|^2\varphi\alpha^2\,dt\,da\,dx\\&+\lambda\int_Q u\nabla u\cdot\nabla\Psi\varphi\alpha^2\,dt\,da\,dx+2\int_Q u\nabla u\cdot\nabla\alpha\varphi\alpha\,dt\,da\,dx.\end{aligned} \tag{3.45}$$

This gives

$$\begin{aligned}\int_Q|\nabla u|^2\varphi\alpha^2\,dt\,da\,dx=&\int_Q\varphi\alpha^2uP_2u\,dt\,da\,dx+s\int_Q\left(\frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial a}\right)u^2\varphi\alpha^2\,dt\,da\,dx\\&+s^2\lambda^2\int_Q u^2\varphi^3\alpha^2|\Psi|^2\,dt\,da\,dx-\lambda\int_Q u\nabla u\cdot\nabla\Psi\varphi\alpha^2\,dt\,da\,dx\\&-2\int_Q u\nabla u\cdot\nabla\alpha\varphi\alpha\,dt\,da\,dx.\end{aligned} \tag{3.46}$$

Note that

$$-\lambda\int_Q u\nabla u\cdot\nabla\Psi\varphi\alpha^2\,dt\,da\,dx\leq C\lambda^2\int_Q|u|^2\varphi\alpha^2\,dt\,da\,dx+\frac{1}{2}\int_Q|\nabla u|^2\varphi\alpha^2\,dt\,da\,dx, \tag{3.47}$$

where C is a positive constant. As $\varphi \leq C\varphi^3$ with C a positive constant, using now the properties of α and Ψ we deduce

$$\begin{aligned}
&\int_{\tilde{q}} |\nabla u|^2 \varphi\alpha^2 \, dt\, da\, dx \\
&\qquad \leq C\int_Q \varphi\alpha^2 u P_2 u \, dt\, da\, dx + Cs^2\lambda^2 \int_Q u^2\varphi^3\alpha^2 \, dt\, da\, dx + C\int_Q u\varphi^{1/2}|\nabla u|\varphi^{1/2}\alpha \, dt\, da\, dx.
\end{aligned} \tag{3.48}$$

Therefore we deduce from the previous estimate that

$$2s\lambda^2\delta^2 \int_{\tilde{q}} |\nabla u|^2 \varphi \, dt\, da\, dx \leq \frac{1}{2}\int_Q |P_2 u^2| \, dt\, da\, dx + Cs^2\lambda^2 \int_q u^2\varphi^3 \, dt\, da\, dx, \tag{3.49}$$

where C is a positive constant.

Combining (3.41) and (3.49) we get

$$\begin{aligned}
&C\left(\int_Q e^{-2s\eta} f^2 \, dt\, da\, dx + s^3\lambda^4 \int_q \varphi^3 u^2 \, dt\, da\, dx\right) \\
&\qquad \geq \int_Q |P_1 u|^2 \, dt\, da\, dx + \int_Q |P_2 u|^2 \, dt\, da\, dx + s\lambda^2 \int_Q \varphi|\nabla u|^2 \, dt\, da\, dx \\
&\qquad\quad + s^3\lambda^4 \int_Q \varphi^3 u^2 \, dt\, da\, dx.
\end{aligned} \tag{3.50}$$

We want now to turn back to the variable w.

Note that $u = e^{-s\eta} w$. Then, we have

$$\begin{aligned}
\int_Q \varphi^3 |u|^2 \, dt\, da\, dx &= \int_Q e^{-2s\eta}\varphi^3 |w|^2 \, dt\, da\, dx, \\
\int_q \varphi^3 |u|^2 \, dt\, da\, dx &= \int_q e^{-2s\eta}\varphi^3 |w|^2 \, dt\, da\, dx.
\end{aligned} \tag{3.51}$$

Therefore one gets from (3.50)

$$s^3\lambda^4 \int_Q \varphi^3 e^{-2s\eta} w^2 \, dt\, da\, dx \leq C\int_Q e^{-2s\eta} f^2 \, dt\, da\, dx + Cs^3\lambda^4 \int_q e^{-2s\eta}\varphi^3 w^2 \, dt\, da\, dx. \tag{3.52}$$

This ends the proof. □

Remark 3.3. (i) Indeed one can prove that there exist positive constants $s_1 \geq 1$ and $\lambda_1 \geq 1$ and there exists a positive constant $C > 0$ such that for all $s \geq s_1$, $\lambda \geq \lambda_1$ and for all solution of (3.1) the following inequality holds:

$$\int_Q \frac{e^{-2s\eta}}{s\varphi}\left(\left|\frac{\partial w}{\partial t}+\frac{\partial w}{\partial a}\right|^2+|\Delta w|^2\right)dx\,da\,dt+\int_Q e^{-2s\eta}s^3\varphi^3\lambda^4 w^2\,dt\,da\,dx$$
$$+\int_Q e^{-2s\eta}s\lambda\varphi|\nabla w|^2\,dt\,da\,dx\le C\left(\int_Q e^{-2s\eta}f^2\,dt\,da\,dx+\int_q e^{-2s\eta}s^3\varphi^3\lambda^4 w^2\,dt\,da\,dx\right). \tag{3.53}$$

It is sufficient to use (3.50) and to turn back to the variable w by using the explicit expression of P_1u and P_2u.

(ii) In [1] the author tried to prove a Carleman inequality for the system (3.1) with $\beta w(t,0,x)$ instead of f. The problem there is more complex: after the change of variable $u=e^{-2s\eta}w$ the right term becomes $e^{-2s\eta}w(t,0,x)$ and cannot be written in terms of the variable u. Unfortunately, see [1, system (6) page 566], this term was ignored in the computations.

In the sequel we take $f=0$ in order to avoid this situation.

Our observability inequality is as follows.

Proposition 3.4. *Assume that*

$$f=0 \tag{3.54}$$

and that there exists a real $\gamma\ge 0$ such that

$$g(a,x)=0 \quad \text{a.e. in } (0,\gamma)\times\Omega. \tag{3.55}$$

Then, there exists a positive constant C_γ such that the following inequality holds:

$$\int_{Q_A} w^2(0,a,x)da\,dx+\int_{Q_T} w^2(t,0,x)dt\,dx\le C_\gamma\int_q w^2(t,a,x)dt\,da\,dx \tag{3.56}$$

for all solution w of (3.1).

Let γ be small enough so that $\gamma\le\min(T,A)$. We define now two subsets of $(0,T)\times(0,A)$:

$$\begin{aligned} N_1&=\{(t,a)\in(0,T)\times(0,A);\ t\ge a+T-\gamma\},\\ N_2&=\{(t,a)\in(0,T)\times(0,A);\ t\le a+\gamma-A\}, \end{aligned} \tag{3.57}$$

and we formulate a *lemma* which will be used in the proof of Proposition 3.4.

Lemma 3.5. *If (3.54) and (3.55) hold, then all solutions of (3.1) verify*

$$w(t,a,x)=0 \quad \text{a.e. in } (N_1\cup N_2)\times\Omega. \tag{3.58}$$

Proof of Lemma 3.5. We will prove that $w = 0$ on almost every characteristic line in $N_1 \cup N_2$.

Let $(t_0, a_0) \in N_1$. Then we have $t_0 = a_0 + T - \gamma + d$ with $0 \leq d \leq \gamma$. Therefore, $a_0 \leq \gamma - d$.

Let $S(d) = \{(t_0 + s, a_0 + s),\ s \in (0, \gamma - d - a_0)\}$ be a characteristic line of (3.1). Setting $z(s,x) = w(t_0 + s, a_0 + s, x)$ and $\overline{\mu}(s,x) = \mu(t_0 + s, a_0 + s, x)$ from (3.1), we deduce that z solves

$$\begin{gathered} -\frac{\partial z}{\partial s} - \triangle z + \overline{\mu} z = 0 \quad \text{in } (0, \gamma - d - a_0) \times \Omega, \\ z(s,x) = 0 \quad \text{on } (0, \gamma - d - a_0) \times \partial\Omega, \\ z(\gamma - d - a_0, x) = w(T, \gamma - d, x) = g(\gamma - d, x) \quad \text{in } \Omega. \end{gathered} \tag{3.59}$$

Then from (3.55) for almost all $d \in (0, \gamma)$, standard results on heat equation imply that $z = 0$. Thus, for almost all $d \in (0, \gamma)$, $w = 0$ on $S(d)$. Therefore, $w = 0$ in $N_1 \times \Omega$. The same argument and the fact that $w(t, A, x) = 0$ in $(0, T) \times \Omega$ allow us to prove that $w = 0$ in $N_2 \times \Omega$. □

Now, let us prove Proposition 3.4.

Proof of Proposition 3.4. We set

$$\begin{aligned} D_1 &= \left\{(t,a) \in (0,T) \times (0,A),\ t \leq -\frac{T - \gamma/2}{A - \gamma/2} a + T - \frac{\gamma}{2}\right\}, \\ D_2 &= \left\{(t,a) \in (0,T) \times (0,A),\ a \geq -\frac{A - \gamma/2}{T - \gamma/2} t + A - \frac{\gamma(\gamma - 2A)}{2(2T - \gamma)}\right\}, \\ D_3 &= (0,T) \times (0,A) - (D_1 \cup D_2), \\ D_4 &= \{(t,a) \in D_3; (t,a) \notin (N_1 \cup N_2)\}, \quad \text{(cf. Figure 3.1)}. \end{aligned} \tag{3.60}$$

Consider now $\theta \in C_0^\infty(\mathbb{R}^2)$ a cut-off function such that $\theta = 1$ on D_1; $\theta = 0$ on D_2. Setting $\widetilde{w} = \theta w$, it follows that $\widetilde{w}$ solves

$$\begin{gathered} -\frac{\partial \widetilde{w}}{\partial t} - \frac{\partial \widetilde{w}}{\partial a} - \Delta \widetilde{w} + \mu \widetilde{w} = -\left(\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial a}\right) w \quad \text{in } Q, \\ \widetilde{w}(t,a,x) = 0 \quad \text{on } \Sigma, \\ \widetilde{w}(T,a,x) = 0 \quad \text{in } Q_A, \\ \widetilde{w}(t,A,x) = 0 \quad \text{in } Q_T. \end{gathered} \tag{3.61}$$

Multiplying (3.61) by $\widetilde{w}$ and integrating over Q yield after minor majoration

$$\int_0^{T-\gamma/2} \int_\Omega w^2(t,0,x)\,dx\,dt + \int_0^{A-\gamma/2} \int_\Omega w^2(0,a,x)\,dx\,da \leq -2 \int_Q \left(\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial a}\right) \theta w^2\,dt\,da\,dx. \tag{3.62}$$

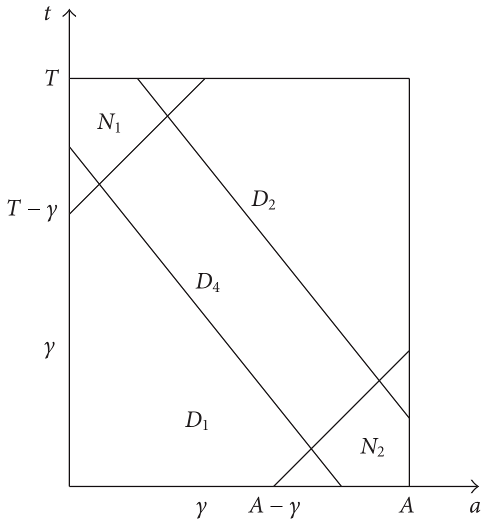

Figure 3.1

Using Lemma 3.5 and the definition of θ, we deduce that $(\partial\theta/\partial t + \partial\theta/\partial a)\theta w = 0$ almost every where outside of $D_4 \times \Omega$. Note that η and φ are bounded on $D_4 \times \Omega$ by strictly positive reals. Hence there exists a positive constant $\overline{C}_\gamma > 0$ such that

$$-2\int_Q \left(\frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial a}\right)\theta w^2\, dt\, da\, dx \le \overline{C}_\gamma \int_Q \varphi^2 e^{-2s\eta} w^2\, dt\, da\, dx. \tag{3.63}$$

Therefore (3.62) yields

$$\int_0^{T-(\gamma/2)}\int_\Omega w^2(t,0,x)\,dx\,dt + \int_0^{A-(\gamma/2)}\int_\Omega w^2(0,a,x)\,dx\,da \le \overline{C}_\gamma \int_Q \varphi^2 e^{-2s\eta} w^2\, dt\, da\, dx, \tag{3.64}$$

where $\overline{C}_\gamma$ is a positive constant depending on γ. Using now (3.2), (3.58) and the fact that $\varphi^2 e^{-2s\eta} \le 1$ for λ and s sufficiently large we deduce (3.56). □

Remark 3.6. A careful calculation for $s \ge s_1$ and $\lambda \ge \lambda_1$ leads to the following estimate of C_γ:

$$C_\gamma \ge C(T)\gamma^2 \exp\left(\frac{C(\Psi, s, \lambda)}{\gamma^3 A T}\right), \tag{3.65}$$

where $C(\Psi, s, \lambda)$ and $C(T)$ are positive constants.

3.2. A null controllability result. In this section, for a given function $b \in L^2(Q_T)$ we consider the following system:

$$\begin{aligned}
&\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - \triangle y + \mu y = v 1_\omega \quad \text{in } Q, \\
&y(t,a,\sigma) = 0 \quad \text{on } \Sigma, \\
&y(0,a,x) = y_0(a,x) \quad \text{in } Q_A, \\
&y(t,0,x) = b(t,x) \quad \text{in } Q_T.
\end{aligned} \tag{3.66}$$

For all $\epsilon > 0$ we introduce the functional

$$J_\epsilon(v) = \frac{1}{2\epsilon} \int_\gamma^A \int_\Omega y^2(T,a,x)\,dx\,da + \frac{1}{2} \int_q v^2(t,a,x)\,dx\,da\,dt. \tag{3.67}$$

It follows easily that J_ϵ is continuous, convex, and coercive. Hence, J_ϵ admits a unique minimizer v_ϵ and we have

$$v_\epsilon(t,a,x) = -w_\epsilon(t,a,x) 1_\omega(x) \quad \text{in } Q, \tag{3.68}$$

where w_ϵ is the solution of the following system:

$$\begin{aligned}
&-\frac{\partial w_\epsilon}{\partial t} - \frac{\partial w_\epsilon}{\partial a} - \Delta w_\epsilon + \mu w_\epsilon = 0 \quad \text{in } Q, \\
&w_\epsilon(t,a,\sigma) = 0 \quad \text{on } \Sigma, \\
&w_\epsilon(T,a,x) = \frac{1}{\epsilon} y_\epsilon(T,a,x) 1_{(\gamma,A)}(a) \quad \text{in } Q_A, \\
&w_\epsilon(t,A,x) = 0 \quad \text{in } Q_T,
\end{aligned} \tag{3.69}$$

and y_ϵ is the solution of (3.66) associated to v_ϵ.

Multiplying (3.69) by y_ϵ and integrating on Q give

$$\begin{aligned}
&-\frac{1}{\epsilon} \int_\gamma^A \int_\Omega y_\epsilon^2(T,a,x)\,dx\,da + \int_0^A \int_\Omega w_\epsilon(0,a,x) y_0(a,x)\,dx\,da \\
&\quad + \int_0^T \int_\Omega w_\epsilon(t,0,x) b(t,x)\,dx\,dt + \int_q v_\epsilon w_\epsilon\,dt\,da\,dx = 0.
\end{aligned} \tag{3.70}$$

Using (3.68) we obtain

$$\begin{aligned}
&\int_0^A \int_\Omega w_\epsilon(0,a,x) y_0(a,x)\,dx\,da + \int_0^T \int_\Omega w_\epsilon(t,0,x) b(t,x)\,dx\,dt \\
&\quad = \frac{1}{\epsilon} \int_\gamma^A \int_\Omega y_\epsilon^2(T,a,x)\,dx\,da + \int_q v_\epsilon^2\,dt\,da\,dx.
\end{aligned} \tag{3.71}$$

On the other hand, Young inequality gives

$$\begin{aligned}
&\int_0^A\int_\Omega w_\epsilon(0,a,x)y_0(a,x)\,dx\,da+\int_0^T\int_\Omega w_\epsilon(t,0,x)b(t,x)\,dx\,dt\\
&\qquad\le\frac{1}{2C_\gamma}\left(\int_0^A\int_\Omega w_\epsilon^2(0,a,x)\,dx\,da+\int_0^T\int_\Omega w_\epsilon^2(t,0,x)\,dt\,dx\right)\\
&\qquad\quad+2C_\gamma\left(\int_0^A\int_\Omega y_0^2(a,x)\,dx\,da+\int_0^T\int_\Omega b^2(t,x)\,dx\,dt\right).
\end{aligned}\tag{3.72}$$

Therefore Proposition 3.4 and inequality (3.72) imply

$$\begin{aligned}
&\frac{1}{\epsilon}\int_\gamma^A\int_\Omega y_\epsilon^2(T,a,x)\,dx\,da+\frac{1}{2}\int_q v_\epsilon^2\,dt\,da\,dx\\
&\qquad\le 2C_\gamma\left(\int_0^A\int_\Omega y_0^2(a,x)\,dx\,da+\int_0^T\int_\Omega b^2(t,x)\,dx\,dt\right).
\end{aligned}\tag{3.73}$$

Consequently

$$\begin{aligned}
||v_\epsilon||^2_{L^2(q)}&\le 4C_\gamma\left(\|b\|^2_{L^2(Q_T)}+||y_0||^2_{L^2(Q_A)}\right),\\
\int_\Omega y_\epsilon^2(T,a,x)\,dx\,da&\le 2\epsilon C_\gamma\left(\|b\|^2_{L^2(Q_T)}+||y_0||^2_{L^2(Q_A)}\right).
\end{aligned}\tag{3.74}$$

Then, one can extract subsequences also denoted by v_ϵ and y_ϵ such that $v_\epsilon\to v$ weakly in $L^2(q)$ and $y_\epsilon\to y$ weakly in $L^2((0,T)\times(0,A),H_0^1(\Omega))$.

Moreover y is the unique solution of (3.66) and verifies (2.2). Notice also that v verifies (2.2).

Therefore, we have proved the following null controllability result.

Proposition 3.7. *For any given positive real γ small enough, there exists a control $v\in L^2(q)$ that verifies (3.74), such that the associated solution y of (3.66) verifies (2.2).*

Remark 3.8. (i) This result is quite similar to what was proved in [7] for a so-called "linearized crocco-type equation." More precisely, it was proved in [7] that there exists a control v acting on $(x_0,x_1)\times\omega$, with $0<x_0<x_1<A$ such that the corresponding solution of (3.66) with $\Omega\subset\mathbb{R}$ verifies

$$y(T,a,x)=0\quad\text{in }(x_0+\delta,L)\times\Omega,\tag{3.75}$$

where

$$L=\begin{cases}x_1+T-\delta & \text{if } 0<T<A-x_1+\delta,\\ A & \text{if } T>A-x_1+\delta.\end{cases}\tag{3.76}$$

See [7, page 710].

The method in [7] uses the fact that $0<x_0<A$, energy estimates, and Carleman estimates for parabolic equation along characteristic lines of (3.66). Therefore one cannot use the result of [7] for the case $x_0=0$ and $x_1=A$ which is studied here.

(ii) System (3.13) describes in fact the evolution of a controlled age and space structured population in which the birth process is given by a function regardless of the distribution of individuals of age $a > 0$. That explains why it seems impossible to eradicate individuals of age close to 0.

4. Proof of the main result

For $\theta \in L^2(Q_T)$, letting $b = e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta)$, we derive from Proposition 3.7 that there exists a control v that verifies (3.74) so that the corresponding solution of (3.66) verifies (2.2). Then for all $\theta \in L^2(Q_T)$ we define by $\Lambda(\theta)$ the nonempty set of all $\int_0^A \beta y\,da$ where y verifies (2.2), solves (3.66) with $v \in L^2(q)$ that verifies (3.74). The problem is now reduced to find a fixed point for Λ. In order to apply a generalization of the Leray-Schauder fixed point theorem stated in [5], we define the set $N = \{\theta \in L^2(Q_T),\ (\exists)\zeta \in (0,1),\ \theta \in \zeta\Lambda(\theta)\}$. Thus doing the existence of a fixed point is a obvious consequence of the following.

Proposition 4.1. *(i) Λ is a compact multivalued mapping of $L^2(Q_T)$.*
(ii) For all $\theta \in L^2(Q_T)$, $\Lambda(\theta)$ is a nonempty closed convex subset of $L^2(Q_T)$.
(iii) N is bounded in $L^2(Q_T)$.
(iv) Λ is upper semicontinuous on $L^2(Q_T)$.

Proof of Proposition 4.1. (i) We prove the compactness of Λ. Let $\theta \in L^2(Q_T)$ such that $\|\theta\| \le r$, $r > 0$. We have to prove that $\Lambda(\theta)$ is compact in $L^2(Q_T)$. Consider $(\rho_n)_n \subset \Lambda(\theta)$. From the definition of Λ, for all n there exists a pair $(v_n, y_n) \in L^2(q) \times L^2(Q)$ such that $\rho_n = \int_0^A \beta y_n\,da$, v_n verifies (3.74) and y_n, the associated solution of (3.66) with $b = e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta)$ verifies (2.2).

Using (3.74) we deduce that

$$||v_n||^2_{L^2(q)} \le 4C_\gamma\Big(||e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta)||^2_{L^2(Q_T)} + ||y_0||^2_{L^2(Q_A)}\Big). \tag{4.1}$$

Then we get via H_3

$$||v_n||^2_{L^2(q)} \le C_\gamma\Big(C(F,\Omega,T,r) + ||y_0||^2_{L^2(Q_A)}\Big). \tag{4.2}$$

Multiplying (3.66) with $e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta)$ instead of b by y_n and integrating over Q, we obtain

$$||\nabla y_n||^2_{L^2(Q)} + \frac{\lambda_0}{2}||y_n||^2_{L^2(Q)} \le \frac{2}{\lambda_0}||v_n||^2_{L^2(q)} + \frac{1}{2}||y_0||^2_{L^2(Q_A)} + \frac{1}{2}||e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta)||^2_{L^2(Q_T)}. \tag{4.3}$$

Therefore, for $\lambda_0 \ge 2$ we get

$$\|\nabla y\|^2_{L^2(Q)} + \|y\|^2_{L^2(Q)} \le (C_\gamma + 1)\Big(C(F,\Omega,r,T) + ||y_0||^2_{L^2(Q_A)}\Big). \tag{4.4}$$

Moreover, using H_2 we deduce that $\rho_n = \int_0^A \beta y_n da$ solves the system

$$\frac{\partial \rho_n}{\partial t} - \Delta\rho_n + \int_0^A \beta\mu y_n\, da = z_n(t,x) \quad \text{in } Q_T,$$
$$\rho(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{4.5}$$
$$\rho_n(0,x) = \int_0^A \beta(0,a,x)y_0(a,x)da \quad \text{in } \Omega,$$

where $z_n(t,x) = \int_0^A \beta v_n da 1_\omega + \int_0^A y_n(\partial\beta/\partial t + \partial\beta/\partial a - \Delta\beta)da + \int_0^A \nabla y_n \nabla\beta\, da$.

Notice that

$$||z_n||^2_{L^2(Q_T)} \le 3C_\beta^2 A\Big(||v_n||^2_{L^2(q)} + ||y_n||^2_{L^2(Q)} + ||\nabla y_n||^2_{L^2(Q)}\Big). \tag{4.6}$$

This implies via (4.2) and (4.4) that

$$||z_n||^2_{L^2(Q_T)} \le (C_y+1)C(\beta,A)\Big(C(F,\Omega,r,T) + ||y_0||^2_{L^2(Q_A)}\Big). \tag{4.7}$$

Now let us multiply (4.5) by ρ_n, we obtain after an integration by parts and minor changes that

$$||\nabla\rho_n||^2_{L^2(Q_T)} + \frac{\lambda_0}{2}||\rho_n||^2_{L^2(Q_T)} \le \frac{2}{\lambda_0}||z_n||^2_{L^2(Q_A)}. \tag{4.8}$$

Consequently, ρ_n is bounded in $L^2((0,T),H_0^1(\Omega))$ and standard arguments allow us to see that $\partial\rho_n\partial t$ is also bounded in $L^2((0,T),H_0^{-1}(\Omega))$. Hence, using Lions-Aubin lemma we conclude the proof of (i).

We address now the proof of (ii).

First, it is obvious that for all $\theta \in L^2(Q_T)$, $\Lambda(\theta)$ is a nonempty convex set. Let $(\rho_n)_n \subset \Lambda(\theta)$ such that $\rho_n \to \rho$ in $L^2(Q_T)$. We have to prove that $\rho \in \Lambda(\theta)$. For all n there exists v_n that verifies (3.74) such that $\rho_n = \int_0^A \beta y_n da$ where y_n is the corresponding solution of (3.66) with $e^{\lambda_0 t}F(e^{\lambda t}\theta)$ instead of b, and y_n verifies also (2.2). Then, from (4.2) and (4.4) we deduce that one can extract subsequences also denoted by v_n and y_n converging weakly to v and y, respectively, in $L^2(q)$ and $L^2((0,T)\times(0,A),H_0^1(\Omega))$. Standard device implies that $\int_0^A \beta y\, da = \rho$. In addition, it follows that y is the associated solution of (3.66) with $b = e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta)$. In addition v verifies (3.74) and y verifies (2.2). Therefore, the definition of Λ yields that $\rho \in \Lambda(\theta)$.

Let us perform now the proof of (iii). Let $\theta \in N$, then there exists $\zeta \in (0,1)$ such that $(1/\zeta)\ \theta \in \Lambda\theta$. As a consequence, there exists a pair $(v,y) \in L^2(q)\times L^2(Q)$ such that $\theta = \zeta\int_0^A \beta y\, da$, v verifies (3.74) and y is the associated solution of (3.66) with $b = e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta)$. This implies on one hand that

$$\|\theta\|^2_{L^2(Q_T)} \le C(\beta,A)\|y\|^2_{L^2(Q)}. \tag{4.9}$$

By (4.1) and H_3 we deduce

$$\|v\|^2_{L^2(q)} \le 8C_y\Big(C(C_0,\Omega,T) + C_1^2\|\theta\|^2_{L^2(Q_T)} + ||y_0||^2_{L^2(Q_A)}\Big) \tag{4.10}$$

and consequently, (4.3) yields

$$\|y\|^2_{L^2(Q)} \le \frac{16}{\lambda_0}\Big(C(T,\Omega,C_0)+||y_0||^2_{L^2(Q_A)}\Big)+\frac{(16C_\gamma+1)C_1^2}{\lambda_0}\|\theta\|^2_{L^2(Q_T)}. \tag{4.11}$$

Taking now $\lambda_0 > \max(2,(16C_\gamma+1)C_1^2)$ and combining (4.9) and (4.11) we get

$$\|\theta\|^2_{L^2(Q_T)} \le C\Big(A,T,\Omega,F,\gamma,||y_0||^2_{L^2(Q_A)}\Big) \tag{4.12}$$

that achieves the proof of (iii).

It remains to check that Λ is upper semicontinuous on $L^2(Q_T)$. This is equivalent to prove that for any closed subset G of $L^2(Q_T)$, $\Lambda^{-1}(G)$ is closed in $L^2(Q_T)$. Let $\theta_n \in \Lambda^{-1}(G)$ such that θ_n converges towards θ in $L^2(Q_T)$. Then, θ_n is bounded and for all n there exists $\rho_n \in G$ such that $\rho_n \in \Lambda(\theta_n)$. Therefore, from the definition of Λ there exists a pair $(v_n,y_n) \in L^2(q)\times L^2(Q)$ such that $\rho_n = \int_0^A \beta y_n\,da$, v_n verifies (3.74), y_n the corresponding solution of (3.66) with $e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta_n)$ instead of b verifies (2.2), so that v_n verifies (4.2) and y_n (4.4). Consequently (v_n,y_n) is bounded in $L^2(q)\times L^2(Q)$. Thus, there exists a subsequence still denoted by (v_n,y_n) that converges weakly to (v,y) in $L^2(q)\times L^2(Q)$. Since F is continuous, it follows that $e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta_n)$ converges strongly towards $e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta)$. Now, by standard device we see that v verifies (3.74), $\rho = \int_0^A \beta y\,da$, y solves (3.66) with $e^{-\lambda_0 t}F(e^{\lambda_0 t}\theta)$ instead of b and y verifies in addition (2.2). This implies obviously that

$$\rho \in \Lambda(\theta). \tag{4.13}$$

On the other hand, thanks to (4.8) and Lions-Aubin lemma once again, one can extract a subsequence also denoted by ρ_n that converges strongly towards the function ρ in $L^2(Q_T)$. Since G is closed we deduce that $\rho \in G$. Finally, from (4.13) we deduce that $\theta \in \Lambda^{-1}(G)$. This completes the proof of Proposition 4.1. □

Acknowledgments

This work was improved at the Université de Versailles. The author is grateful to Professor J.-P. Puel for helpful suggestions.

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