ON THE COMMUTATOR LENGTHS OF CERTAIN CLASSES OF FINITELY PRESENTED GROUPS

H. DOOSTIE AND P. P. CAMPBELL

Received 23 June 2004; Revised 4 July 2005; Accepted 28 March 2006

For a finite group $G = \langle X \rangle$ $(X \neq G)$, the least positive integer $\mathrm{ML}_X(G)$ is called the maximum length of G with respect to the generating set X if every element of G may be represented as a product of at most $\mathrm{ML}_X(G)$ elements of X. The maximum length of G, denoted by $\mathrm{ML}(G)$, is defined to be the minimum of $\{\mathrm{ML}_X(G) \mid G = \langle X \rangle, X \neq G, X \neq G - \{1_G\}\}$. The well-known commutator length of a group G, denoted by $c(G)$, satisfies the inequality $c(G) \leq \mathrm{ML}(G')$, where G' is the derived subgroup of G. In this paper we study the properties of $\mathrm{ML}(G)$ and by using this inequality we give upper bounds for the commutator lengths of certain classes of finite groups. In some cases these upper bounds involve the interesting sequences of Fibonacci and Lucas numbers.

1. Introduction

For an abstract group G the commutator length $c(G)$ is defined to be $c(G) = \mathrm{Sup}\{\lambda(g) \mid g \in G'\}$, where $\lambda(g)$ is the minimal number of commutators of which g is the product. This notion has been studied by many authors during the years and the results of the combinatorial methods estimate or calculate the commutator lengths of abstract groups which are mainly infinite (one may see [2, 6–10], e.g.). For a finite group $G = \langle X \rangle$ we examine the effect of the generating set on the evaluation of this number by considering the following definitions, this gives us a method to calculate upper bounds for $c(G)$. Let $G = \langle X \rangle$ $(X \neq G)$ be a finite group. Then the following hold.

Definition 1.1. $\mathrm{ML}_X(G)$, the maximum length of G relative to the generating set X, is defined to be $\max\{\lambda(g) \mid g \in G\}$, where $\lambda(g)$ is the minimum number of the elements of X of which g is the product.

Definition 1.2. The maximum length of a group G, denoted by $\mathrm{ML}(G)$, is defined to be the minimum of all numbers $\mathrm{ML}_X(G)$, for all generating sets X $(X \neq G)$ of G.

Our notations are fairly standard, we use $[x]$ for the integer part of the real number x, $[a,b] = a^{-1}b^{-1}ab$ is defined to be the commutator of the elements a and b of a group,

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 74981, Pages 1–9
DOI 10.1155/IJMMS/2006/74981

the usual notation $N \times_\varphi H$ is used for the semidirect product of the group N by H, where $\varphi : H \to \mathrm{Aut}(N)$ is a homomorphism such that $h\varphi = \varphi_h$ and $\varphi_h : N \to N$ is an element of $\mathrm{Aut}(N)$, and the Reidemeister-Schreier algorithm in the form given in [1] will be used to find presentations of subgroups. In the following sections we study certain classes of finite groups for their maximal lengths and find upper bounds for the commutator lengths. The groups studied here are the dihedral groups $D_{2n} = \langle a,b \mid a^2 = b^n = (ab)^2 = 1\rangle$, $n \geq 3$, the quaternion groups $Q_{2^n} = \langle a,b \mid a^{2^{n-1}} = 1,\ b^2 = a^{2^{n-2}},\ b^{-1}ab = a^{-1}\rangle$, $n \geq 3$, the semidirect products $D_{2n} \times_\varphi Z_{2m}$, and $Q_{2^n} \times_\varphi Z_{2m}$, $(n,m \geq 3)$ (where if $Z_{2m} = \langle c\rangle$, then $\varphi : Z_{2m} \to \mathrm{Aut}(D_{2n})$ is such that $c\varphi = \varphi_c$; and $\varphi_c : D_{2n} \to D_{2n}$ is defined by $a\varphi_c = a$ and $b\varphi_c = b^{-1}$, a similar φ exists for $Q_{2^n} \times_\varphi Z_{2m}$), the direct products $D_{2n} \times Z_{2m}$ and $Q_{2^n} \times Z_{2m}$, and the following classes of groups:

$$\begin{gathered} G_1 = \langle a,b \mid a^2 = b^n = abab^{-1}ab^2ab^{-1}ab^{-2}ab = 1\rangle, \quad n \geq 3, \\ G_2 = \langle a,b \mid a^2 = b^n,\ (ab^2)^2(ab^{-1})^2 = b^{2n}\rangle, \quad n \neq 0, \\ G_3 = \langle a,b \mid a^2 = b^n = 1,\ abab^{-2}ab^3ab^{-2}ab^{-1}ab = 1\rangle, \quad n \geq 3, \\ G_4 = \langle a,b \mid a^2 = b^5 = 1,\ [a,b]^k[a,b^3] = 1\rangle, \quad k \geq 1. \end{gathered} \tag{1.1}$$

The groups G_i $(i = 1,2,3,4)$ are soluble and have been studied for their structures and orders in [3–5]. These groups are generalizations of the well-known Coxeter groups.

We will use the Fibonacci and Lucas numbers:

$$\begin{gathered} f_0 = f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 2, \\ g_0 = 2, \quad g_1 = 1, \quad g_n = g_{n-1} + g_{n-2}, \quad n \geq 2, \end{gathered} \tag{1.2}$$

which are related to each other via the relation $g_n = f_{n-2} + f_n$. In Section 2 we study the notion of the maximum length for the mentioned direct and semidirect products, and by using the inequality $c(G) \leq \mathrm{ML}(G')$ which holds for every nonabelian finite group G, we give upper bounds for the commutator lengths of the groups G_i $(i = 1,2,3,4)$, in Sections 3 and 4.

2. The groups D_{2n}, Q_{2^n}, $D_{2n} \times Z_{2m}$, $Q_{2^n} \times Z_{2m}$, $D_{2n} \times_\varphi Z_{2m}$, $Q_{2^n} \times_\varphi Z_{2m}$

Let $m,n \geq 3$ be integers. By the definitions of the direct and semidirect products, we get the following presentations:

$$\begin{gathered} D_{2n} \times Z_{2m} = \langle a,b,c \mid a^2 = b^n = (ab)^2 = c^{2m} = [a,c] = [b,c] = 1\rangle, \\ Q_{2^n} \times Z_{2m} = \langle a,b,c \mid a^{2^{n-1}} = 1,\ b^2 = a^{2^{n-2}},\ b^{-1}aba = c^{2m} = [a,c] = [b,c] = 1\rangle, \\ D_{2n} \times_\varphi Z_{2m} = \langle a,b,c \mid a^2 = b^n = (ab)^2 = c^{2m} = 1,\ c^{-1}aca = 1,\ c^{-1}bcb = 1\rangle, \\ Q_{2^n} \times_\varphi Z_{2m} = \langle a,b,c \mid a^{2^{n-1}} = 1,\ b^2 = a^{2^{n-2}},\ b^{-1}aba = c^{2m} = 1,\ c^{-1}aca = 1,\ c^{-1}bcb = 1\rangle, \end{gathered} \tag{2.1}$$

where φ is the homomorphism defined in the last section.

As an immediate result of the definitions we have the following.

Lemma 2.1. *For every finite groups* $G_1 = \langle X \mid R\rangle$ *and* $G_2 = \langle Y \mid S\rangle$,

$$\mathrm{ML}_{\{X,Y\}}\left(G_1 \times G_2\right) \leq \mathrm{ML}_X\left(G_1\right) + \mathrm{ML}_Y\left(G_2\right). \tag{2.2}$$

Proof. Obviously, $G_1 \times G_2 = \langle X, Y \mid R, S, [X,Y]\rangle$, where $[X,Y] = \{[x,y] \mid x \in X,\ y \in Y\}$. There exists an element $g \in G_1 \times G_2$ such that $\mathrm{ML}_{\{X,Y\}}(G_1 \times G_2) = \lambda(g)$, and $g = (x_1x_2\cdots x_m)(y_1y_2\cdots y_n)$, where $x_i \in X$ and $y_i \in Y$ (for $[x_i, y_j] = 1$ holds for every i and j). Let $g_1 = x_1x_2\cdots x_m$ and $g_2 = y_1y_2\cdots y_n$. The definition of $\lambda(g)$ then yields

$$\mathrm{ML}_{\{X,Y\}}\left(G_1 \times G_2\right) = \lambda(g) = m + n \leq \lambda\left(g_1\right) + \lambda\left(g_2\right) \leq \mathrm{ML}_X\left(G_1\right) + \mathrm{ML}_Y\left(G_2\right). \tag{2.3}$$

□

Proposition 2.2. $m, n \geq 3$ *are integers, then*

(i) *for a nonabelian metacyclic group* G, $c(G) \leq |G'| - 1$;

(ii) *for a metabelian group* G, *where* $|G'| = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ *(here* $p_1, p_2, \ldots$ *are different primes)* $c(G) \leq (p_1^{\alpha_1} + \cdots + p_k^{\alpha_k}) - k$;

(iii) $\mathrm{ML}(D_{2n}) \leq [n/2] + 1$ *and* $\mathrm{ML}(Q_{2^n}) \leq 2^{n-3} + 3$;

(iv) $\mathrm{ML}(D_{2n} \times Z_{2m}) \leq [n/2] + 2m$ *and* $\mathrm{ML}(Q_{2^n} \times Z_{2m}) \leq 2^{n-3} + 2m + 2$;

(v) $\mathrm{ML}(D_{2n} \times_\varphi Z_{2m}) \leq [n/2] + 2m$;

(vi) $\mathrm{ML}(Q_{2^n} \times_\varphi Z_{2m}) \leq 2^{n-3} + 2m$.

Proof. For a nonabelian metacyclic group G, the derived subgroup G' is a cyclic group, so $\mathrm{ML}(G') = |G'| - 1$ and (i) follows at once.

For a metabelian group G, where $|G'| = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, we may use the direct decomposition of G' to get (ii).

We give proof of the first part of (iii), the second part is similar. The elements of D_{2n} are of the form a^ib^j, where $i = 0, 1$ and $j = 0, 1, \ldots, n-1$. Consider two cases for n and for every $g \in D_{2n}$ compute $\lambda(g)$, the minimum number of the elements of $X = \langle a, b\rangle$ of which g is the product. We see that the relations $(ab)^2 = a^2 = b^n = 1$ of D_{2n} yield the relations $b^{n-k} = b^{-k} = ab^ka$, for every integer k.

Case 1. n is even. It is easy to see that

$$\lambda\left(b^{n-k}\right) = \begin{cases} k+2 & \text{if } 1 \leq k \leq \dfrac{n}{2} - 1, \\ n-k & \text{if } \dfrac{n}{2} \leq k \leq n-1. \end{cases} \tag{2.4}$$

Also,

$$\lambda\left(ab^k\right) = \begin{cases} k+1 & \text{if } 1 \leq k \leq \dfrac{n}{2}, \\ n-k+1 & \text{if } \dfrac{n}{2} + 1 \leq k \leq n-1. \end{cases} \tag{2.5}$$

So, in this case $\mathrm{ML}_X(D_{2n}) = \max\{\lambda(g) \mid g \in D_{2n}\} = n/2 + 1$.

Case 2. n is odd. In this case similar classification of the elements gives us

$$\lambda(b^{n-k}) = \begin{cases} k+2 & \text{if } 1 \leq k \leq \dfrac{n-3}{2}, \\ n-k & \text{if } \dfrac{n-1}{2} \leq k \leq n-1. \end{cases} \tag{2.6}$$

Also,

$$\lambda(ab^k) = \begin{cases} k+1 & \text{if } 1 \leq k \leq \dfrac{n-1}{2}, \\ n-k+1 & \text{if } \dfrac{n+1}{2} \leq k \leq n-1, \end{cases} \tag{2.7}$$

and in this case $\mathrm{ML}_X(D_{2n}) = \max\{\lambda(g) \mid g \in D_{2n}\} = (n+1)/2$. Consequently, for every $n \geq 3$, $\mathrm{ML}(D_{2n}) \leq \mathrm{ML}_X(D_{2n}) = [n/2]+1$. A similar proof for Q_{2^n} yields $\mathrm{ML}_X(Q_{2^n}) = 2^{n-3}+3$, where $X = \langle a,b\rangle$ is the generating set of Q_{2^n}, and $\mathrm{ML}(Q_{2^n}) \leq \mathrm{ML}_X(Q_{2^n}) = 2^{n-3}+3$.

The inequalities of (iv) are the results of Lemma 2.1 and the calculations of part (iii), where we know that $\mathrm{ML}_{\{a,b\}}(D_{2n}) = [n/2]+1$, $\mathrm{ML}_{\{a,b\}}(Q_{2^n}) = 2^{n-3}+3$, and $\mathrm{ML}_{\{c\}}(Z_{2m}) = 2m-1$.

To prove (v) we see that up to the relations $ca = ac$, $cb = b^{-1}c$, and $ba = ab^{-1}$ of the group $D_{2n} \times_\varphi Z_{2m}$ the $4mn$ elements of this group may be considered as the union of the following sets:

$$\begin{aligned} S_1 &= \{c^i \mid 1 \leq i \leq 2m-1\}, \\ S_2 &= \{a^i b^j \mid 0 \leq i \leq 1,\ 0 \leq j \leq n-1\}, \\ S_3 &= \{ac^i \mid 1 \leq i \leq 2m-1\}, \\ S_4 &= \{b^i c^j \mid 1 \leq i \leq n-1,\ 1 \leq j \leq 2m-1\}, \\ S_5 &= \{ab^i c^j \mid 1 \leq i \leq n-1,\ 1 \leq j \leq 2m-1\}. \end{aligned} \tag{2.8}$$

Since the relations $b^{n-k} = ab^k a$, $b^{n-k} = cb^k c$, and $c^k = ac^k a$ hold in the group, for every integer k, then the minimum length $\lambda(g)$ is definitely acceptable for every $g \in D_{2n} \times_\varphi Z_{2m}$, in the similar way as in (iii) and we get

$$\begin{aligned} \max\{\lambda(g) \mid g \in S_1\} &= 2m-1, \\ \max\{\lambda(g) \mid g \in S_2\} &= 1+\left[\frac{n}{2}\right], \\ \max\{\lambda(g) \mid g \in S_3\} &= 2m, \\ \max\{\lambda(g) \mid g \in S_4\} &= \left[\frac{n}{2}\right]+2m-1, \\ \max\{\lambda(g) \mid g \in S_5\} &= \left[\frac{n}{2}\right]+2m. \end{aligned} \tag{2.9}$$

Consequently, $\mathrm{ML}(D_{2n} \times_\varphi Z_{2m}) \leq \mathrm{ML}_{\{a,b,c\}}(D_{2n} \times_\varphi Z_{2m}) = [n/2]+2m$.

The proof of (vi) is similar to the above proof and one may get the result by considering the $m2^{n+1}$ elements of the group $Q_{2^n} \times_\varphi Z_{2m}$ as the union of the following sets:

$$\begin{gathered} S_1 = \{a^i b^j \mid 0 \leq i \leq 2^{n-1} - 1,\ 0 \leq j \leq 1\}, \\ S_2 = \{a^i c^j \mid 0 \leq i \leq 2^{n-1} - 1,\ 1 \leq j \leq 2m - 1\}, \\ S_3 = \{bc^i \mid 1 \leq i \leq 2m - 1\}, \\ S_4 = \{a^i bc^j \mid 1 \leq i \leq 2^{n-1} - 1,\ 1 \leq j \leq 2m - 1\}. \end{gathered} \tag{2.10}$$

And this completes the proof. □

3. The groups G_1 and G_2

The groups G_1 and G_2 are finite and nonmetabelian soluble groups for many values of n (see [4, 5]). The following propositions are our main results on the commutator lengths of these groups.

Proposition 3.1. *For every $n \geq 3$, where g.c.d $(n,3) \neq 1$,*

$$c(G_1) \leq \begin{cases} g_n & \text{if } n \equiv \pm 3 \pmod{12}, \\ 4 f_{n/2-1} & \text{if } n \equiv 0 \pmod{12}, \\ 2 g_{n/2} & \text{if } n \equiv 6 \pmod{12}. \end{cases} \tag{3.1}$$

Proposition 3.2. *For every $n \geq 4$,*

$$c(G_2) \leq \begin{cases} 2n - 1 & \text{if } n \equiv \pm 1 \pmod 3, \\ \dfrac{n+7}{2} & \text{if } n \equiv 3 \pmod 6, \\ \dfrac{n+8}{2} & \text{if } n \equiv 0 \pmod 6. \end{cases} \tag{3.2}$$

Proof of Proposition 3.1. Let $n \equiv \pm 3 \pmod{12}$. Using the results of [5] gives us here the following presentation for G_1':

$$G_1' = \left\langle x, y \mid (xy)^2 = (yx)^2, [x^2, y] = [y^2, x] = 1,\ xy = y^{-f_{n-2}} x^{f_{n-3}},\ x^{1+f_{n-2}} = y^{f_{n-1}} \right\rangle. \tag{3.3}$$

First we show that the relations $y^{g_n} = x^{g_n} = 1$ hold in G_1'.

The relation $xy = y^{-f_{n-2}} x^{f_{n-3}}$ gives us

$$x y^{f_{n-1}} x^{-1} = y^{-f_{n-2} f_{n-3}} x^{(-1+f_{n-3}) f_{n-1}}. \tag{3.4}$$

For f_{n-3} is odd, then $[y^{-f_{n-2}}, x^{-1+f_{n-3}}] = 1$. This relation together with the last relation of G_1' yields

$$y^{f_{n-1}} = y^{-f_{n-2} f_{n-3}} x^{(-1+f_{n-3}) f_{n-1}}. \tag{3.5}$$

And substituting for $y^{f_{n-1}}$ yields

$$x^{(1+f_{n-2})^2-f_{n-1}(-1+f_{n-3})}=1. \tag{3.6}$$

By an inductive method we may show that $g_n=(1+f_{n-2})^2-f_{n-1}(-1+f_{n-3})$ and then $x^{g_n}=1$. A similar proof exists for $y^{g_n}=1$ and since G_1' is of order $2g_n$ (see [5]), g_n is the order of y.

Now consider the relations $[x^2,y]=[y^2,x]=1$ and conclude that the words xy^{-1+g_n} (or $y^{-1+g_n}x$) are the words of largest length; that is, $c(G_1)\le \mathrm{ML}(G_1')\le \mathrm{ML}_{\{x,y\}}(G_1')=g_n$.

Let $n\equiv 6(\mathrm{mod}\,12)$. In this case G_1' may be presented as

$$\left\langle x,y\mid (xy)^2=(yx)^2,[x^2,y]=[y^2,x]=1,\ x^{1+f_{n-3}}y^{-1+f_{n-2}},\ x^{-1+f_{n-2}}=y^{f_{n-1}}\right\rangle. \tag{3.7}$$

This group is of order $2(-2+g_n)$ (see [5]). First we show that the relations $x^{g_{n/2}}=y^{g_{n/2}}=1$ hold in G_1'. Combining the last two relations of G_1' yields $x^{f_{n-4}-f_2}=y^{f_{n-3}+f_1}$. Again combining this relation with $x^{1+f_{n-3}}=y^{-1+f_{n-2}}$ gives us $x^{f_{n-5}+f_3}=y^{f_{n-4}-f_2}$. We repeat this method and get the relations

$$x^{f_{n-2-i}-(-1)^i f_i}=y^{f_{n-1-i}+(-1)^i f_{i-1}},\quad i=2,3,\dots. \tag{3.8}$$

For $i=(n-4)/2$ and $i=(n-6)/2$ we get $x^t=y^t$ and $x^t=y^{2t}$, respectively, where $t=f_{n/2}+f_{(n-4)/2}$. Consequently, $y^t=1$ holds in G_1'. Now it is easy to see that $t=g_{n/2}$. The word $yx^{-1+g_{n/2}}y^{-1+g_{n/2}}x$ is one of the words of maximum length. So $c(G_1)\le \mathrm{ML}(G_1')\le \mathrm{ML}_{\{x,y\}}(G_1')=2g_{n/2}$ as required in this case.

The remained case is $n\equiv 0(\mathrm{mod}\,12)$. In this case G_1' has the previous case's presentation. We first show that the relations

$$x^{f_{n/2-1}}=y^{3f_{n/2-1}},\qquad y^{5f_{n/2-1}}=1 \tag{3.9}$$

hold in G_1'. As well as in the last case we get the relations

$$x^{f_{n-2-i}-(-1)^i f_i}=y^{f_{n-1-i}+(-1)^i f_{i-1}},\quad i=2,3,\dots. \tag{3.10}$$

If we let $i=-2+n/2$ then, after a simplification we get

$$x^{f_{n/2-1}}=y^{3f_{n/2-1}}. \tag{3.11}$$

Raising both sides to the power 3 yields $x^{3f_{n/2-1}}=y^{9f_{n/2-1}}$. Also for the value $i=-3+n/2$ we get $x^{3f_{n/2-1}}=y^{4f_{n/2-1}}$. Consequently, $y^{5f_{n/2-1}}=1$, and these relations together with the relations $[x^2,y]=[y^2,x]=1$ show that the word

$$y^{-1+5f_{n/2-1}}x^{-1+f_{n/2-1}}yx \tag{3.12}$$

is of the largest length in G_1'. However, by considering the relation $x^{f_{n/2-1}} = y^{3f_{n/2-1}}$, this word will be reduced to

$$y^{-1+2f_{n/2-1}}x^{-1+2f_{n/2-1}}yx, \tag{3.13}$$

then $c(G_1) \le \mathrm{ML}(G_1') \le \mathrm{ML}_{\{x,y\}}(G_1') = 4f_{n/2-1}$, as required. This completes the proof. □

Proof of Proposition 3.2. For every $n \equiv \pm 1 (\mathrm{mod}\, 3)$, G_2 is a metabelian group and as a result of the computations of [4], G_2' is a cyclic group of order $2n$. So $c(G_2) \le 2n-1$ comes from the results of Section 2.

Let $n \equiv 3(\mathrm{mod}\, 6)$. Using the Todd-Coxeter coset enumeration algorithm gives us the presentation

$$G_2' = \langle x, y \mid (xy)^2 = y^n,\ (yx)^2 = x^n,\ x^3y^3 = 1\rangle. \tag{3.14}$$

It is easy to show the validity of the relations $x^n = y^n$ and $x^{2n} = y^{2n} = 1$ in G_2'. Let $n = 6k+3$. We claim that the words $w_1 = x^2y^{3k+1}xy^2$ and $w_2 = y^2x^{3k+1}yx^2$ in G_1' are of the largest length $3k+5$. Indeed, the relations $x^3y^3 = 1$, $xyx = y^{n-1}$, and $yxy = x^{n-1}$ show that any word with maximal length could not contain the subwords x^2y^2 and y^2x^2. The remained words which have to be examined are indeed $w_3 = y^{3k+1}xy^2x$ and $w_4 = x^{3k+1}yx^2y$. These words are of length $3k+4$, for we have

$$w_3 = y^{3k}(yxy)yx = y^{3k}x^{6k+2}yx = x^{3k+2}yx. \tag{3.15}$$

Similarly, w_4 is of length $3k+4$. To complete the proof we now show that w_1 and w_2 are of length $3k+5$. By using the relations we get

$$w_1 = x^2y^{3k}(yxy)y = x^2y^{3k}x^{6k+2}y = x^2(y^{3k}x^{3k})x^{3k+2}y = x^{3k+4}y, \tag{3.16}$$

and in a similar way, w_2 will be reduced to $y^{3k+4}x$. So $c(G_2) \le \mathrm{ML}(G_2') \le \mathrm{ML}_{\{x,y\}}(G_2') = 3k+5$.

Let $n \equiv 0(\mathrm{mod}\, 6)$. We use the Todd-Coxeter coset enumeration algorithm to find a presentation for G_2'. In two different cases, $n \equiv 0(\mathrm{mod}\, 12)$ and $n \equiv 6(\mathrm{mod}\, 12)$, we get the following presentations:

$$\begin{aligned} G_2' &= \left\langle x, y \mid [x^2, y] = [x, y^3] = x^4 = y^n = R_1 = R_2 = 1,\ (xy)^{n/2} = x^2 \right\rangle, \\ G_2' &= \left\langle x, y \mid [x^2, y] = [x, y^3] = x^4 = y^n = R_1 = R_2 = 1,\ (xy)^{n/2} = y^{n/2} \right\rangle, \end{aligned} \tag{3.17}$$

respectively, where $R_1 = y^{2+n/2}x^3y^{-1}xy^{-1}x^3$ and $R_2 = yxy^{-1}xyx^3y^{-1}x$. The largest power of x in every word of G_2' is equal to 3, however, the largest power of y is $n/2+1$, for $R_1 = 1$ yields

$$y^{2+n/2} = xyx^3yx. \tag{3.18}$$

This relation also gives us the relation $(xy)^2 = y^{2+n/2}x$. The relation $R_2 = 1$ yields $yxy^{n-1}xy = x^3yx$ and hereby we deduce that the only words of maximum length must be among the following words:

$$w_1 = x^3y^{1+n/2}, \qquad w_2 = y^{1+n/2}x^3, \qquad w_3 = x^3y^{1+n/2}xy, \qquad w_4 = xyx^3y^{1+n/2}. \tag{3.19}$$

Obviously the words w_1 and w_2 are of length $4+n/2$, however w_4 is of length 5, for we see that

$$w_4 = (xyx^3y)y^{n/2} = (y^{2+n/2}x^{-1})y^{n/2} = y^{2+n/2}x^3y^{n/2} = y^{n+2}x^3 = y^2x^3. \tag{3.20}$$

Finally w_3 is also of length $4+n/2$, for

$$w_3 = y^{n/2}(x^3yx)y = y^{n/2}(yxy^{n-1}xy)y = y^n(yxy^{-1+n/2}xy^2) = yxy^{-1+n/2}xy^2. \tag{3.21}$$

Consequently, $c(G_2) \leq \mathrm{ML}(G_2') \leq \mathrm{ML}_{\{x,y\}}(G_2') = 4+n/2$. This completes the proof. □

4. The groups G_3 and G_4

These groups are examples of metacyclic groups and we have the following result.

Proposition 4.1. *For every $n \geq 3$, the groups G_3, and for every $k \geq 1$, the groups G_4 are finite. Moreover, $c(G_3) \leq (2^n - 3 - (-1)^n)/3$ and $c(G_4) \leq k(k^3+2k^2+4k+3)$.*

Proof. The subgroup $H = \langle x_i = [a^{-1},b^{-1}]^{b^{-i}} : 0 \leq i \leq n-1\rangle$ of G_3 is indeed the derived subgroup of G_3 (one may easily check that $|G_3 : H| = |G_3 : G_3'| = 2n$ and $H \subseteq G_3'$). A presentation for H may be given as

$$H = \left\langle x_0,\dots,x_{n-1} \mid x_ix_{i+1}^2 = 1,\ x_{n-1}x_0^2 = 1,\ 0 \leq i \leq n-2\right\rangle. \tag{4.1}$$

Obviously this group is finite and cyclic of order $(2^n-(-1)^n)/3$, and $c(G_3) \leq \mathrm{ML}_{\{x,y\}}(G_3') = |G_3'| - 1 = (2^n - 3 - (-1)^n)/3$ is a result of Section 2. In a similar way we consider the subgroup

$$K = \left\langle [a^{-1},b],[b^{-1},a]^{b^{-i}} : 0 \leq i \leq 3\right\rangle \tag{4.2}$$

of G_4. K is the derived subgroup of G_4 and may be presented by

$$K = \langle a_1,\dots,a_5 \mid a_1a_2a_3a_4a_5 = 1,\ a_i^k = a_{i+1}a_{i+2},\ 1 \leq i \leq 5\rangle, \tag{4.3}$$

(where indices are reduced modulo 5). An almost easy simplification of the relations shows that K is a cyclic group of order $k^4+2k^3+4k^2+3k+1$, and $c(G_4) \leq \mathrm{ML}_{\{x,y\}}(G_4') = |G_4'| - 1 = k(k^3+2k^2+4k+3)$ is a result of Section 2. This completes the proof. □

References

[1] M. J. Beetham and C. M. Campbell, *A note on the Todd-Coxeter coset enumeration algorithm*, Proceedings of the Edinburgh Mathematical Society. Series II **20** (1976), no. 1, 73–79.

[2] J. L. Brenner, *Covering theorems for finasigs. X. The group $G = \mathrm{PSL}(n,q)$ had a class C such that $CC = G$*, Ars Combinatoria **16** (1983), 57–67.

[3] C. M. Campbell and R. M. Thomas, *On $(2,n)$-groups related to Fibonacci groups*, Israel Journal of Mathematics **58** (1987), no. 3, 370–380.

[4] H. Doostie, *Non-metabelian soluble groups involving Lucas numbers*, Journal of Sciences, Islamic Republic of Iran **3** (1992), 63–67.
[5] H. Doostie and A. R. Jamali, *A class of deficiency zero soluble groups of derived length* 4, Proceedings of the Royal Society of Edinburgh. Section A **121** (1992), no. 1-2, 163–168.
[6] V. V. Kursov, *The commutant of the general linear group over a field*, Doklady Akademii Nauk BSSR **23** (1979), no. 10, 869–871, 955.
[7] ______, *Commutators in real Chevalley groups*, Doklady Akademii Nauk BSSR **28** (1984), no. 3, 208–210.
[8] O. Ore, *Some remarks on commutators*, Proceedings of the American Mathematical Society **2** (1951), 307–314.
[9] R. C. Thompson, *Commutators in the special and general linear groups*, Transactions of the American Mathematical Society **101** (1961), 16–33.
[10] H. Q. Zhang and H. You, *Commutators in symplectic unimodular groups*, Chinese Annals of Mathematics. Series A **12** (1991), no. 2, 238–248.

H. Doostie: Mathematics Department, Teacher Training University, 49 Mofateh Avenue,
Tehran 15614, Iran
E-mail address: Doostih@saba.tmu.ac.ir

P. P. Campbell: Institute of Mathematics, St. Andrews University, St. Andrews, KY16 9SS,
Scotland, UK
E-mail address: peterc@st-andrews.ac.uk