TAUBERIAN-TYPE THEOREMS WITH APPLICATION TO THE STIELTJES TRANSFORMATION

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In the first part, we define the space $L'(r)$ and the modified Stieltjes transformation introduced by Lavoine and Misra (1979) and Marichev (1983), respectively. In the second part of the paper, we extend Tauberian-type theorems for the distributional Stieltjes transformations to the distributional modified Stieltjes transformations.

1. Introduction

The Abelian and Tauberian-type theorems were introduced by Stanković [7] and Pilipović et al. [5]. In the first part of this paper, we give the definition of the quasiasymptotic expansion at 0^+ and the quasiasymptotic behaviour of distributions at infinity from S'_+ introduced in [1]. In this paper, we give the definition of space $L'(r)$, classical Stieltjes transformation, modified Stieltjes transformation, and generalized modified Stieltjes transformation. This enables us to obtain, in the second part of the paper, the Tauberian-type theorems of quasiasymptotic behaviour of distributions at infinity. We give sufficient conditions under which the behaviour at infinity of the modified Stieltjes transformation $\Gamma(r+1)(T_{r+1}f)(x)$, $r \in \mathbb{R}\backslash(-\mathbb{N})$, $f \in L'(r)$ determines the quasiasymptotic behaviour of f at infinity.

Notation 1.1. As usually $\mathbb{R}, \mathbb{C}, \mathbb{N}$ are the spaces of real, complex, and natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. D is the space of infinitely differentiable functions with compact support. S'_+ denotes the space of tempered distributions with support in the $[0,\infty)$. The space of rapidly decreasing functions is denoted as S, S' is the space of all distributions of slow growth. T_{r+1} denotes the modified Stieltjes transformation with index r.

A positive continuous function L defined on $(0,\infty)$ is called slowly varying function at ∞ if for every $k > 0$,

$$\lim_{k\to\infty} \frac{L(kx)}{L(k)} = 1. \tag{1.1}$$

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We denote by $\sum_\infty$ the set of all slowly varying functions at ∞. For the properties of slowly varying functions, we refer the reader to [6].

If L is a slowly varying function at ∞, then for every $\varepsilon > 0$, there is $A_\varepsilon > 0$ such that

$$x^{-\varepsilon} < L(x) < x^{\varepsilon} \quad \text{when } x > A_\varepsilon. \tag{1.2}$$

Recall that for $\alpha > -1$, $x_+^\alpha = H(x)x^\alpha$, where H is Heaviside's function. The following scale of distributions from S'_+ has been used in the investigations of the quasiasymptotic behaviour of distributions:

$$f_{\alpha+1} = \begin{cases} \dfrac{Ht^\alpha}{\Gamma(\alpha+1)}, & \alpha > -1, \\ D^n f_{\alpha+n+1}, & \alpha \leq -1,\ \alpha+n > -1, \end{cases} \tag{1.3}$$

where D is the distributional derivative.

2. Definitions

2.1. Definition of the quasiasymptotic behaviour (q.a.b.)

Definition 2.1. The quasiasymptotic behaviour of distribution (q.a.b.) at infinity.

If T is a distribution from S'_+ such that the distributional limit

$$\lim_{k\to\infty} \frac{T(kx)}{\rho(k)} = \gamma(x) \tag{2.1}$$

exists in S' ($\gamma(x) \neq 0$), then T is called the quasiasymptotic behaviour at infinity related to the regular function $\rho(k) = k^\alpha L(k)$ with the limit γ; write this as

$$T \overset{q}{\sim} \gamma \quad \text{in } S' \text{ as } x \longrightarrow \infty. \tag{2.2}$$

Here ρ is regularly varying at infinity and the limit γ from S'_+ is of the form

$$\gamma(x) = \mathbb{C} f_{\alpha+1}(x). \tag{2.3}$$

We will repeat in this section some well-known facts about the quasiasymptotic behaviour from [8].

Let $f \in S'_+$. It is said that f has the q.a.b. at ∞ with the limit $g \neq 0$ with respect to

$$\begin{gathered} k^\alpha L(k), \quad L \in \sum_\infty \left(\left(\frac{1}{k}\right)^\alpha L\left(\frac{1}{k}\right),\ L \in \sum_0 \right),\ \alpha \in \mathbb{R}, \\ \text{if } \lim_{k\to\infty} \left\langle \frac{f(kt)}{k^\alpha L(k)}, \phi(t) \right\rangle = \langle g(t),\ \phi(t) \rangle, \quad \phi \in S. \end{gathered} \tag{2.4}$$

Definition 2.2. A function $\rho : (a,\infty) \to \mathbb{R}$, $a \in \mathbb{R}$, is called regularly varying function at infinity if it is positive, measurable, and there exists a real number α such that for each

$$x > 0, \quad \lim_{k\to\infty} \frac{\rho(kx)}{\rho(k)} = x^{\alpha}; \tag{2.5}$$

where the number α is called the index of ρ.

2.2. Space $L'(r)$. We extend the definition of the space $I'(r)$ given in [4] and using the same idea, we give the definition of space $L'(r)$.

$L'(r)$, $r \in \mathbb{R}\backslash(-\mathbb{N})$, the space of all distributions $f \in S'_{+}(\mathbb{R})$ such that there exist $k \in \mathbb{N}_0$ and locally integrable function F, $\operatorname{supp} F \subset [0,\infty)$, so that f is of the form

$$f = t^{-r} D^k F \tag{2.6}$$

and there exist $C = C(F)$ and $\varepsilon = \varepsilon\,(F) > 0$ such that

$$|F(x)| \leq C(1+x)^{r+k-\varepsilon}, \quad x \geq 0. \tag{2.7}$$

The *Stieltjes transformation* $S_r(f)(s)$, $r \in \mathbb{R}\backslash(-\mathbb{N})$, is complex-valued function, defined by

$$S_r(f(t))(s) = \int_0^{\infty} \frac{f(t)}{(s+t)^{r+1}} dt \quad s \in \mathbb{C}\backslash(-\infty,0],\ 0 < t < \infty,\ r \in \mathbb{R}\backslash(-\mathbb{N}). \tag{2.8}$$

Modified Stieltjes transformation introduced by Marichev is defined as

$$T_{\alpha}(f(x)) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \left(1+\frac{x}{y}\right)^{-\alpha} \cdot \frac{1}{y} f(y)dy, \quad x \in \mathbb{C}\backslash(-\infty,0],\ 0 < y < \infty,\ r \in \mathbb{R}\backslash(-\mathbb{N}). \tag{2.9}$$

It can be written as

$$T_{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \frac{y^{\alpha-1} f(y)}{(x+y)^{\alpha}} dy, \quad x \in \mathbb{C}\backslash(-\infty,0]. \tag{2.10}$$

Setting $r = \alpha - 1$, $f(t) = y^{\alpha-1} f(y)$ in (2.8), we get

$$S_{\alpha-1}(y^{\alpha-1} f(y))(x) = \int_0^{\infty} \frac{y^{\alpha-1} f(y)}{(x+y)^{\alpha}} dy, \quad x \in \mathbb{C}\backslash(-\infty,0]. \tag{2.11}$$

Using (2.10) and (2.11), we obtain the relationship between (2.8) and (2.10),

$$T_{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} S_{\alpha-1}(y^{\alpha-1} f(y))(x), \quad x \in \mathbb{C}\backslash(-\infty,0]. \tag{2.12}$$

By changing x by z and α by $r+1$, it follows

$$\Gamma(r+1) T_{r+1}(f)(z) = S_r(y^r f)(z), \quad z \in \mathbb{C}\backslash(-\infty,0],\ r \in \mathbb{R}\backslash(-\mathbb{N}). \tag{2.13}$$

2.3. Modified Stieltjes transformation T_{r+1}. The modified Stieltjes transformation $T_{r+1}(f)$, $r \in \mathbb{R}\backslash(-\mathbb{N})$, is complex-valued function defined by

$$\Gamma(r+1)T_{r+1}(f)(s) = (r+1)_k \int_0^\infty \frac{F(t)}{(s+t)^{r+1+k}}dt, \quad r \in \mathbb{R}\backslash(-\mathbb{N}),\ s \in \mathbb{C}\backslash(-\infty,0],\ 0 < t < \infty, \tag{2.14}$$

where $(\alpha)_k = (\alpha)\cdot(\alpha+1)\cdot(\alpha+2)\cdots(\alpha+k-1)$.

2.4. Generalized modified Stieltjes transformation $\widetilde{T}_{r+1}$. The $\widetilde{T}_{r+1}$-transformation of a distribution $f \in S'_+(\mathbb{R})$ is complex-valued function $\widetilde{T}_{r+1}(f)$ defined by

$$\begin{aligned}\Gamma(r+1)&\widetilde{T}_{r+1}(f)(s)\\ &= \lim_{w\to\infty} \langle f(t), \eta(t)(s+t)^{-r-1}\exp(-wt)\rangle, \quad w \in \mathbb{R},\ s \in \Lambda \subset (\mathbb{C}\backslash(-\infty,0]),\ \eta \in A(s).\end{aligned} \tag{2.15}$$

Here Λ is the set of complex numbers for which this limit exists and $A(s)$ is the family of all smooth functions, defined on $\mathbb{R}$ for which there exists $\varepsilon = \varepsilon_{\eta,s} > o$ such that $0 \le \eta(t) \le 1$, $t \in \mathbb{R}$, $\eta(t) = 1$, if t belongs to the $\in$-neighbourhood of $\overline{\mathbb{R}}_+\eta(t) = 0$ or if it belongs to the complement of the 2ε-neighbourhood of $\overline{\mathbb{R}}_+$, where $\varepsilon > 0$ is arbitrary if $\operatorname{Im} s \neq 0$, and $0 < 2\varepsilon < \max \operatorname{Re} s$, if for some $\operatorname{Im} s = 0$ and $|D^p\eta(t)| \le C_p$, $t \in \mathbb{R}$. If $\eta(t) \in A(s)$, $s \in (\mathbb{C}\backslash\mathbb{R}_-)$, then $\eta(t)(s+t)^{-r-1}\exp(-w,t) \in S(\mathbb{R})$ for $w \in \mathbb{R}_+$, $r \in \mathbb{R}$.

3. Main results

3.1. Tauberian-type theorems for modified Stieltjes transformation. For the main results of this section, we need the following assertion from [5].

Theorem 3.1. *Suppose that for some $m > 0$ and*

$$x \longrightarrow \infty, \quad \int_0^\infty \frac{d\phi(\lambda)}{(\phi+x)^{m+1}} \sim \int_0^\infty \frac{d\varphi(\lambda)}{(\phi+x)^{m+1}} \tag{3.1}$$

and the following conditions are satisfied.

(1) *Functions ϕ and φ are defined for $x > 0$ and are nondecreasing.*

(2) $\lim_{x\to\infty}\phi(x) = \infty$.

(3) *For any $C > 1$, there are constants γ and $\mathbb{N}$, $0 < \gamma < m$, $\mathbb{N} > 0$, such that for any $x > y > \mathbb{N}$, $\phi(x)/\phi(y) \le C(x/y)$. Then for $\lambda \longrightarrow \infty$, $\phi(\lambda) \sim \varphi(\lambda)$.*

(This means $|\phi(\lambda)/\varphi(\lambda) - 1| < \varepsilon$ if $\lambda > \lambda_0(\varepsilon)$, $\lambda \in B$, $\operatorname{meas}((\lambda_0,\infty)\backslash B) = 0$.) Let us note that condition (3) is called as the Keldysh-type condition.

Now we are ready to prove the first Tauberian result.

Theorem 3.2. *Suppose that $s > 1$, $r + m - s > 0$, $f \in L'(r)$, and F (see (2.6)) is a nondecreasing function. Moreover, let*

$$\Gamma(r+1)(T_{r+1}f)(x) \sim \frac{\Gamma(s)}{\Gamma(r+1)}\cdot\frac{L(s)}{x^s}, \quad x \longrightarrow \infty, \tag{3.2}$$

where L is a slowly varying function at ∞ which is defined in some interval $[A,\infty)$, such that $x^{r=k-s}L(x)$ is a nondecreasing function. Then f has the quasiasymptotic behaviour at ∞ related to $k^{r-s}L(k)$ with the limit Cx^{r-s}, where $C \neq 0$.

Proof. Let us put

$$\phi(x) = \begin{cases} \dfrac{x^{r+m-s}L(x)}{\Gamma(r+m-s+1)}, & x > A, \\ 0, & x \leq A. \end{cases} \tag{3.3}$$

Then ϕ has the quasiasymptotic behaviour at ∞ related to $k^{r+m-s}L(k)$ with the limit

$$f_{r+m-s+1}. \tag{3.4}$$

Hence

$$\int_0^\infty \frac{d\phi(t)}{(x+t)^{r+m}} = (r+m)\int_0^\infty \frac{\phi(t)dt}{(x+t)^{r+m+1}} \sim \frac{(r+m)\Gamma(s)}{\Gamma(r+m+1)} \cdot \frac{L(x)}{x^s}, \quad x \longrightarrow \infty. \tag{3.5}$$

Now, we show that the conditions of Theorem 3.1 hold for ϕ and F. In fact, we have only to show that for some γ, $0 < \gamma < r+m-1$, and every $C > 1$, there exists $\mathbb{N} > 0$, such that

$$\frac{\phi(\lambda y)}{\phi(y)} < C\lambda^{\gamma}, \quad \text{for } \lambda > 1,\ y > \mathbb{N}. \tag{3.6}$$

Let us put $\gamma = r+m-s+\varepsilon$, where we choose $\varepsilon > 0$ such that $\gamma > 0$ and $\varepsilon < s-1$. After substituting ϕ in (3.6), we obtain $L(\lambda y) \leq C\lambda^{\varepsilon}L(y)$ and this inequality is true if $\lambda > 1$ and $y > \mathbb{N}$, where $\mathbb{N}$ depends on C.

From the assumption that $f \in L'(r)$ and from (3.5), we have

$$\begin{aligned} \Gamma(r+1)(T_{r+1}f)(x) &= (r+1)_m \int_0^\infty \frac{F(t)}{(x+t)^{r+m+1}}dt \\ &= (r+1)_{m-1}\int_0^\infty \frac{dF(t)}{(x+t)^{r+m}} \sim \frac{\Gamma(s)}{\Gamma(r+1)} \cdot \frac{L(x)}{x^s} \\ &\sim \int_0^\infty \frac{d\phi(t)}{(x+t)^{r+m}}, \quad x \longrightarrow \infty. \end{aligned} \tag{3.7}$$

This implies that

$$\Gamma(r+m+1)(T_{r+m+1}F)(x) \sim \Gamma(r+m+1)(T_{r+m+1}\phi)(x), \quad x \longrightarrow \infty, \tag{3.8}$$

and by Theorem 3.1, it follows that $F \sim \phi$, $x \to \infty$.

Thus we obtain that

$$F(x) \sim \frac{(x^{r+m-s}L(x))}{\Gamma(r+m-s+1)}, \quad x \longrightarrow \infty. \tag{3.9}$$

Since $r+m-s>0$, it follows that f has the quasiasymptotic behaviour at ∞ related to $k^{r+m-s}L(k)$ with the limit x^{r+m-s}.

Since $f=t^{-r}D^mF$, it easily follows that f has the quasiasymptotic behaviour at ∞ related to $k^{r-s}L(k)$ with the limit Cx^{r-s}, where C is a suitable nonzero constant. This completes the proof of the theorem. □

4. Tauberian-type results related to the quasiasymptotic behaviour

For the quasiasymptotic behaviour of all original f and for the ordinary asymptotic of the corresponding function $\Gamma(r+1)(T_{r+1}f)$, we need the following theorem and [5, Lemmas 1, 2, and 3].

Theorem 4.1. *Let $a_{n,m}$, $n,m\in\mathbb{N}$, be a matrix of complex numbers.*

(i) *If $a_{n,m}$ converges uniformly in $m\in\mathbb{N}$ to a_m as $n\to\infty$ and $\lim_{m\to\infty}a_m$ exists, then*

$$\lim_{n\to\infty}\lim_{m\to\infty}a_{n,m}=\lim_{m\to\infty}\lim_{n\to\infty}a_{n,m}=\lim_{\substack{n\to\infty\\ m\to\infty}}a_{n,m}. \tag{4.1}$$

(ii) *If $\lim_{n\to\infty}a_{n,m}$ exists for every $n\in\mathbb{N}$, $\lim_{m\to\infty}a_{n,m}$ exists for every $n\in\mathbb{N}$, $\lim_{n,m\to\infty}a_{n,m}$ exists, then $a_{n,m}$ converges uniformly in $n\in\mathbb{N}$ as $m\to\infty$.*

Lemma 4.2. *Let $r\in\mathbb{R}\backslash(-\mathbb{N})$, $k\in\mathbb{N}_0$ be given and let $\gamma\in\mathbb{C}$, then for every $n\in\mathbb{N}$,*

$$\begin{aligned}\sum_{i=1}^{n+1}\binom{n+1}{i}&(-1)^i(2n+r+k+3)\cdots(2n+r+k-i)(2n+r+k+\gamma+3-i)\\ &\cdots(r+k+\gamma+i+2)+(2n+r+k+\gamma+3)\cdots(r+k+\gamma+2)\\ &=(-1)^n\gamma(1-\gamma)\cdots(n-\gamma)(r+k+\gamma+2)_{(n+1)}.\end{aligned} \tag{4.2}$$

Lemma 4.3. *Suppose that $f\in S'_+$ and that f has the quasiasymptotic at ∞ related to $k^\nu L(k)$, where $\nu<r$. Then there exist $k\in\mathbb{N}_0$, $k+\nu>0$, and a continuous function F, $\operatorname{supp}F\subset[0,\infty)$, such that $f=t^{-r}D^kF$ and $F_1(x)=\int_0^xF(t)dt$, $x\in\mathbb{R}$, $(\angle_{n,r,k+1,x}\Gamma(r+k+2)(T_{r+k+2}F_1)(x)-F_1(x))/(x(1+x)^{\nu+k}L(x))$ converges uniformly to zero in $(0,\infty)$.*

Lemma 4.4. *Let $f\in L'(r)$ and $\Gamma(r+1)(T_{r+1}f)(x)\sim x^\nu L(x)$, $x\to\infty$, $\nu>-1$.*

Then $\overline{\Gamma(r+1)(T_{r+1}f)}_+(x)$ has the quasiasymptotic behaviour at ∞ related to $k^\nu L(k)$.

Proof. We have ($\phi\in S$),

$$\begin{aligned}\frac{1}{k^\nu L(k)}\left\langle\overline{\Gamma(r+1)(T_{r+1}f)}_+(kx),\,\phi(x)\right\rangle&=\frac{1}{k^{\nu+1}L(k)}\left\langle\overline{\Gamma(r+1)(T_{r+1}f)}_+(x),\,\phi\left(\frac{x}{k}\right)\right\rangle\\ &=\frac{1}{k^{\nu+1}L(k)}\int_0^1\Gamma(r+1)(T_{r+1}f)(x)\\ &\quad\times\left(\phi\left(\frac{x}{k}\right)-\phi(0)-\cdots-\left(\frac{x}{k}\right)^{l-1}\frac{\phi^{(l-1)}(0)}{(l-1)!}\right)\cdot dx\\ &\quad+\frac{1}{k^{\nu+1}L(k)}\int_1^\infty\Gamma(r+1)(T_{r+1}f)(x)\phi\left(\frac{x}{k}\right)dx.\end{aligned} \tag{4.3}$$

Since the first part on the right-hand side of (4.3) converges to zero as $k \to \infty$, we have to prove that $(1/k^{\nu}L(k))\int_{1/k}^{\infty}\Gamma(r+1)(T_{r+1}f)(kx)\phi(x)dx \to \int_0^{\infty}x^{\nu}L(x)\phi(x)dx$ as $x \to \infty$.

Let us recall $\Gamma(r+1)(T_{r+1}f)(x) \sim x^{\nu}L(x)$, $x \to \infty$.

This implies that for a given $\varepsilon > 0$, there exists $x_0 > 0$, such that

$$|\Gamma(r+1)(T_{r+1}f)(x) - x^{\nu}L(x)| \leq \varepsilon x^{\nu}L(x), \quad x \geq x_0 > 1. \tag{4.4}$$

We use the following decomposition:

$$\begin{aligned}\frac{1}{k^{\nu}L(k)}&\int_{1/k}^{\infty}\Gamma(r+1)(T_{r+1}f)(kx)\phi(x)dx\\ &=\frac{1}{k^{\nu}L(k)}\left[\begin{array}{l}\displaystyle\int_{1/k}^{x_0/k}\Gamma(r+1)(T_{r+1}f)(kx)\phi(x)dx\\ \displaystyle+\int_{x_0/k}^{\infty}\Gamma(r+1)(T_{r+1}f)(kx)\phi(x)dx\end{array}\right]S.\end{aligned}\tag{4.5}$$

The first member on the right-hand side of (4.5) tends to zero, when $k \to \infty$, because

$$\frac{1}{k^{\nu}L(k)}\int_{1/k}^{x_0/k}|\Gamma(r+1)(T_{r+1}f)(kx)\phi(x)|\,dx \leq \frac{M}{k^{\nu}L(k)}\max_{x\varepsilon\mathbb{R}}\{|\phi(x)|\}\frac{x_0-1}{k}, \tag{4.6}$$

where $M = \max\{|\Gamma(r+1)(T_{r+1}f)(x)| : 1 \leq x \leq x_0\}$. Also, one can prove easily that or a given $x_0 > 1$,

$$\frac{1}{L(k)}\int_0^{x_0/k}|x^{\nu}L(x)\phi(x)|\,dx \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.7}$$

Now by (4.4), (4.6), (4.7), we have

$$\begin{aligned}&\left|\frac{1}{k^{\nu}L(k)}\int_{1/k}^{\infty}\Gamma(r+1)(T_{r+1}f)(kx)\phi(x)dx - \frac{1}{k^{\nu}L(k)}\int_0^{\infty}(kx)^{\nu}L(kx)\phi(x)dx\right|\\ &\quad\leq \frac{1}{k^{\nu}L(k)}\int_{1/k}^{x_0/k}\left|\Gamma(r+1)(T_{r+1}f)(kx)\phi(x)\right|dx + \frac{1}{k^{\nu}L(k)}\int_0^{x_0/k}|(kx)^{\nu}L(kx)\phi(x)dx|\\ &\qquad+\frac{1}{k^{\nu}L(k)}\int_{x_0/k}^{\infty}\left|\Gamma(r+1)(T_{r+1}f)(kx)-(kx)^{\nu}L(kx)\right||\phi(x)|\,dx.\end{aligned}\tag{4.8}$$

Now we completed the proof of the lemma. □

Now we will assume that L satisfies the inequality $L(mx)/L(m) \leq C(1+x)$, $x > 0$, $m > 0$.

Now we are ready to prove the following.

Theorem 4.5. *If $f \in S'_+$ and f has the quasiasymptotic behaviour at ∞ related to $k^{\nu}L(k)$, $r-1<\nu<r$, $r \in \mathbb{R}\backslash(-\mathbb{N})$, then the double sequence*

$$\left\langle\frac{\overline{\Gamma(r+k+2)(T_{r+k+2}F_1)}_+(xm)}{m^{\nu+k+1}L(m)}, \angle_{n,r,k+1}\phi^{(k+1)}(x)\right\rangle, \quad m,n \in \mathbb{N},\ \phi \in s, \tag{4.9}$$

converges uniformly in $n \in \mathbb{N}$ *as* $m \to \infty$*, where* $k \in \mathbb{N}$*,* $k+\nu > 0$*, and* F_1 *are defined in Lemma 4.4.*

Proof. Let us put

$$a_{n,m} = \left\langle \frac{\widetilde{\angle}_{n,r,k,x}\, \overline{\Gamma(r+1)(T_{r+1}f)_+}(mx)}{m^{\nu}L(m)}, \phi(x) \right\rangle. \tag{4.10}$$

Since

$$a_{n,m} = (-1)^{k+1}(r+1)_{k+1} \left\langle \frac{\overline{\Gamma(r+k+2)(T_{r+k+2}F_1)_+}(mx)}{m^{\nu+k+1}L(m)}, \angle_{n,r,k+1,x}\, \phi^{(k+1)}(x) \right\rangle, \tag{4.11}$$

we have to prove that $a_{n,m}$ converges uniformly in $n \in \mathbb{N}$ as $m \to \infty$.

First, we will prove that the conditions of Theorem 4.1(i) are satisfied. Then, from Theorem 4.1(ii), the assertion of this theorem will follow

$$a_{n,m} \longrightarrow a_m = \left\langle \frac{f(mx)}{m^{\nu}L(m)}, \phi(x) \right\rangle, \quad n \longrightarrow \infty,\ m \in \mathbb{N}. \tag{4.12}$$

Since

$$\begin{aligned} a_{n,m} - a_m &= (-1)^{k+1}(r+1)_{k+1} \int_0^{\infty} \frac{\angle_{n,r,k+1,x}\Gamma(r+k+2)(T_{r+k+2}F_1)(mx)}{m^{\nu+k+1}L(m)} \phi^{(k+1)}(x)dx \\ &= (-1)^{k+1}(r+1)_{k+1} \int_0^{\infty} \frac{\angle_{n,r,k+1,x}\Gamma(r+k+2)(T_{r+k+2}F_1)(mx) - F_1(mx)(mx)}{(mx)(1+mx)^{\nu+k}} \\ &\qquad \times \frac{(1+mx)^{\nu+k}L(mx)}{m^{\nu+k+1}L(m)} \phi^{(k+1)}(x)dx, \end{aligned} \tag{4.13}$$

from Lemma 4.4 and the fact that

$$\frac{(mx)(1+mx)^{\nu+k}L(mx)}{m^{\nu+k+1}L(m)} \le 2C(1+x)\frac{mx(1+(mx)^{\nu+k})}{m^{\nu+k+1}} \le 2C(1+x)(x+x^{\nu+k+1}), \quad x \ge 0, \tag{4.14}$$

we obtain that $a_{n,m} - a_m \to 0$ uniformly in $n \in \mathbb{N}$ as $n \to \infty$.

We have $\Gamma(r+1)(T_{r+1}f)(x) \sim (\Gamma(r-\nu)/\Gamma(r+1))\, x^{\nu-r+2}L(x)$, $x \to \infty$. Since $-1 < \nu - r$, we have by Lemma 4.4 that $\overline{\Gamma(r+1)(T_{r+1}f)_+}$ has the quasiasymptotic at ∞ related to $k^{\nu-r}L(k)$ with the limit $(\Gamma(r-\nu)/\Gamma(r+1))x^{\nu-r}$.

By Leibniz formula, we have

$$
\begin{aligned}
&D^{n+1}x^{2n+r+k+3}D^{n+1}\Gamma(r+k+2)(T_{r+k+2}F_1)(x)\\
&\quad= D^{n+1}\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}\Big(\big(x^{2n+r+k+3}\big)^{(i)}\Gamma(r+k+2)\big(T_{r+k+2}F_1\big)(x)\Big)^{(n+1-i)}\\
&\quad= \sum_{i=1}^{n+1}(-1)^i\binom{n+1}{i}(2n+r+k+3)\cdots(2n+r+k+4-i)\\
&\qquad\times\Big(\big(x^{2n+r+k+3-i}\big)\Big(\Gamma(r+k+2)\big(T_{r+k+2}F_1\big)(x)\Big)^{(2n+2-i)}\\
&\qquad+\big(x^{2n+r+k+3}\big)\Gamma(r+k+3)\big(T_{r+k+3}F_1\big)(x)\Big)^{(2n+2)}.
\end{aligned}
\tag{4.15}
$$

Let $\gamma=\nu-r$. Then $x^{2n+r+k+3-i}\Gamma(r+k+2)(T_{r+k+2}F_1)(x)$ has the quasiasymptotic behaviour at ∞ related to $k^{2n+r+k+\gamma+3-i}L(k)$ with the limit $\Gamma(-\gamma)x^{2n+r+k+\gamma+3-i}/\Gamma(r+1)(r+1)_{k+1}$.

Lemma 4.2 and the properties of quasiasymptotic behaviour imply that $D^{n+1}x^{2n+r+k+3}D^{n+1}\Gamma(r+k+2)(T_{r+k+2}F_1)(x)$ has the quasiasymptotic behaviour at ∞ related to $k^{r+k+\gamma+1}L(k)$ with the limit

$$
(-1)^n\gamma(1-\gamma)\cdots(n-\gamma)(r+k+\gamma+2)_{n+1}\frac{\Gamma(-\gamma)}{\Gamma(r+1)(r+1)_{k+1}}x^{\gamma+k+r+1}. \tag{4.16}
$$

Thus, we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}(-1)^{(k+1)}(r+1)_{k+1}\left\langle\frac{\angle_{n,r,k+1,x}\Gamma(r+k+2)\big(T_{r+k+2}F_1\big)_+(mx)}{m^{\nu+k+1}L(m)},\phi(x)\right\rangle\\
&\quad= -\lim_{n\to\infty}\frac{(-1)^{k+1}\Gamma(r+k+2)}{(n+1)!\Gamma(n+r+k+2)}\frac{\Gamma(n-\gamma+1)}{\Gamma(-\gamma)}\frac{\Gamma(r+k+\gamma+n+3)}{\Gamma(r+k+\gamma+2)}\frac{\Gamma(-\gamma)}{\Gamma(r+1)}\\
&\qquad\times\langle x^{\nu+k+1},\phi^{(k+1)}(x)\rangle.
\end{aligned}
\tag{4.17}
$$

To prove that the last limit exists, we have to use the Stirling formula.

$\Gamma(s+1)\sim\sqrt{2\pi}e^{-s}s^{(s+1)/2}$, $s\to\infty$. Thus for the double sequence $a_{n,m}$, Theorem 4.1(i) holds and Theorem 4.1(ii) implies the assertion. □

Theorem 4.6. *Let $f\in L'(r)$ and let $\overline{\Gamma(r+1)(T_{r+1}f)_+}(x)$ have the quasiasymptotic at ∞ related to $k^\alpha L(k)$, $-1<\alpha<0$. If for any $\phi\in s$, the double sequence (4.9) converges uniformly in $n\in\mathbb{N}$ as $m\to\infty$, then f has the quasiasymptotic at ∞ related to $k^{\alpha+r}L(k)$.*

Proof. If $a_{n,m}$ is the double sequence defined in the proof of Theorem 4.5, we have by Theorem 4.1(i) the assertion. □

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