FILTERS OF R_0-ALGEBRAS

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The notion of fated filters in R_0-algebras is introduced. Characterizations of (fated) filters are given. A filter generated by a set is established. By introducing the notion of finite $\odot$-property, we show that if F is a nonempty subset of an R_0-algebra L that has the finite $\odot$-property, then there exists a maximal filter of L containing F.

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1. Introduction

In order to research the logical system whose propositional value is given in a lattice from the semantic viewpoint, Xu [7] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [8] introduced the notion of implicative filters in a lattice implication algebra, and investigated some of their properties. Turunen [5] introduced the notion of Boolean deductive system, or equivalently, Boolean filter in BL-algebras which rise as Lindenbaum algebras from many valued logic introduced by Hájek [2]. Boolean filters are important because the quotient algebras induced by Boolean filters are Boolean algebras, and a BL-algebra is bipartite if and only if it has proper Boolean filter. In [6], Wang introduced the notion of R_0-algebras in order to provide an algebraic proof of the completeness theorem of a formal deductive system. We note that R_0-algebras are different from BL-algebras because the identity $x \wedge y = x \odot (x \to y)$ holds in BL-algebras, but does not hold in R_0-algebras. R_0-algebras are also different from lattice implication algebras because the identity $(x \to y) \to y = (y \to x) \to x$ holds in lattice implication algebras, but does not hold in R_0-algebras. Although they are different in essence, they have some similarities, that is, they all have the implication operator $\to$. Therefore, it is meaningful to generalize some aspects of lattice implication algebras and BL-algebras to R_0-algebras. In [1], Esteva and Godo introduced the MTL-algebra; the MTL-algebra is an extension of a BL-algebra, which is obtained by eliminating the condition $x \wedge y = x \odot (x \to y)$ in a BL-algebra. In fact, the MTL-algebra is an algebra induced by a left continuous t-norm and its corresponding residuum, but the BL-algebra is an algebra induced by a continuous t-norm and its corresponding residuum. It is proved an that R_0-algebra is a particular MTL-algebra and

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its t-norm $\odot$ is a nilpotent minimum t-norm [1], which is obtained by taking a negation operator as $1-x$. Hence, the theory of R_0-algebras becomes one of the guides to the development of the theory of MTL-algebras. Lianzhen and Kaitai [3] extended the notions of implicative filters and Boolean filters to R_0-algebras, considered the fuzzification of such notions, and gave characterizations of fuzzy implicative filters. They also proved that fuzzy implicative filters and fuzzy Boolean filters coincide in R_0-algebras.

In this paper, we introduce the notion of fated filters, and give characterizations of fated filters. We study how to generate a filter by a set. By introducing the notion of finite $\odot$-property, we show that if F is a nonempty subset of an R_0-algebra L that has the finite $\odot$-property, then there exists a maximal filter of L containing F.

2. Preliminaries

Definition 2.1 [6]. Let L be a bounded distributive lattice with order-reversing involution $\neg$ and a binary operation $\rightarrow$. Then $(L, \wedge, \vee, \neg, \rightarrow)$ is called an R_0-*algebra* if it satisfies the following axioms:

(R1) $x \rightarrow y = \neg y \rightarrow \neg x$,
(R2) $1 \rightarrow x = x$,
(R3) $(y \rightarrow z) \wedge ((x \rightarrow y) \rightarrow (x \rightarrow z)) = y \rightarrow z$,
(R4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(R5) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$,
(R6) $(x \rightarrow y) \vee ((x \rightarrow y) \rightarrow (\neg x \vee y)) = 1$.

Let L be an R_0-algebra. For any $x, y \in L$, we define $x \odot y = \neg(x \rightarrow \neg y)$ and $x \oplus y = \neg x \rightarrow y$. It is proved that $\odot$ and $\oplus$ are commutative, associative, and $x \oplus y = \neg(\neg x \odot \neg y)$, and $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice.

Example 2.2 [3]. Let $L = [0,1]$. For any $x, y \in L$, we define $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, $\neg x = 1 - x$, and

$$x \longrightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ \neg x \vee y & \text{if } x > y. \end{cases} \tag{2.1}$$

Then $(L, \wedge, \vee, \neg, \rightarrow)$ is an R_0-algebra which is neither a BL-algebra nor a lattice implication algebra.

An R_0-algebra has the following useful properties.

Proposition 2.3 [4]. *For any elements x, y, and z of an R_0-algebra L, there exist the following properties:*

(a1) $x \leq y$ *if and only if* $x \rightarrow y = 1$,
(a2) $x \leq y \rightarrow x$,
(a3) $\neg x = x \rightarrow 0$,
(a4) $(x \rightarrow y) \vee (y \rightarrow x) = 1$,
(a5) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$,
(a6) $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y$,
(a7) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$,

(a8) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(a9) $x \odot \neg x = 0$ *and* $x \oplus \neg x = 1$,
(a10) $x \odot y \leq x \wedge y$ *and* $x \odot (x \to y) \leq x \wedge y$,
(a11) $(x \odot y) \to z = x \to (y \to z)$,
(a12) $x \leq y \to (x \odot y)$,
(a13) $x \odot y \leq z$ *if and only if* $x \leq y \to z$,
(a14) $x \leq y$ *implies* $x \odot z \leq y \odot z$,
(a15) $x \to y \leq (y \to z) \to (x \to z)$,
(a16) $(x \to y) \odot (y \to z) \leq x \to z$.

3. Fated filters

Definition 3.1 [4]. A nonempty subset F of L is called a *filter* of L if it satisfies
(i) $1 \in F$,
(ii) (for all $x \in F$) (for all $y \in L$) ($x \to y \in F \Rightarrow y \in F$).

Definition 3.2 [3]. A nonempty subset F of L is called an *implicative filter* of L if it satisfies
(i) $1 \in F$,
(ii) (for all $x, y, z \in L$) ($x \to (y \to z) \in F, x \to y \in F \Rightarrow x \to z \in F$).

Note that every implicative filter is a filter. The following is a characterization of filters.

Lemma 3.3 [4]. *Let F be a nonempty subset of L. Then F is a filter of L if and only if it satisfies*
(i) (*for all* $x \in F$) (*for all* $y \in L$) ($x \leq y \Rightarrow y \in F$),
(ii) (*for all* $x, y \in F$) ($x \odot y \in F$).

Proposition 3.4. *Let F be a nonempty subset of L. Then F is a filter of L if and only if it satisfies*

$$(\forall x, y \in L) \quad (\forall a \in F)((x \longrightarrow a) \longrightarrow y \in F \Longrightarrow x \longrightarrow y \in F). \tag{3.1}$$

Proof. Suppose that F is a filter of L. Let $x, y \in L$ and $a \in F$ be such that $(x \to a) \to y \in F$. Since $a \leq x \to a$ by (a2), we have $x \to a \in F$ by Lemma 3.3(i) and so $y \in F$ by Definition 3.1(ii). Using (a2) and Lemma 3.3, we conclude that $x \to y \in F$. Conversely, let F be a nonempty subset of L for which (3.1) is valid. Then there exists $a \in F$. Since $(0 \to a) \to a = 1 \to a = a \in F$, it follows from (3.1) that $1 = 0 \to a \in F$. Let $x \in F$ and $y \in L$ be such that $x \to y \in F$. Then $(1 \to x) \to y = x \to y \in F$, and thus $y = 1 \to y \in F$ by (3.1). Therefore, F is a filter of L. □

Proposition 3.5. *Every filter F of L satisfies the following implication:*

$$(\forall x, y, z \in L) \quad ((x \longrightarrow y) \longrightarrow z \in F \Longrightarrow x \longrightarrow (y \longrightarrow z) \in F). \tag{3.2}$$

Proof. Let $x, y, z \in L$ be such that $(x \to y) \to z \in F$. Then

$$1 = y \longrightarrow (x \longrightarrow y) \leq ((x \longrightarrow y) \longrightarrow z) \longrightarrow (y \longrightarrow z), \tag{3.3}$$

and so $((x \to y) \to z) \to (y \to z) \in F$ by Lemma 3.3(i). Using Definition 3.1(ii), we have $y \to z \in F$. Since $y \to z \le x \to (y \to z)$ by (a2), it follows from Lemma 3.3(i) that $x \to (y \to z) \in F$. This completes the proof. □

The following example shows that the converse of Proposition 3.5 may not be true in general.

Example 3.6. Let $L = \{0, a, b, c, d, 1\}$ be a set with the order $0 \le a \le b \le c \le d \le 1$, and the following Cayley tables:

x	¬ x
0	1
a	d
b	c
c	b
d	a
1	0

→	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	1	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \neg, \to)$ is an R_0-algebra (see [3]), where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Then the set $D := \{a, b, c\}$ satisfies the condition (3.2), but D is not a filter of L.

Let F be a nonempty subset of L. Then the least filter containing F is called the *filter generated* by F and denoted by $\langle F \rangle$.

The next statement gives a description of elements of $\langle F \rangle$.

THEOREM 3.7. *If F is a nonempty subset of L, then*

$$\langle F \rangle = \left\{ x \in L \;\middle|\; \begin{array}{l} a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_n \longrightarrow x) \cdots)) = 1 \\ \text{for some } a_1, a_2, \ldots, a_n \in F \end{array} \right\}. \tag{3.4}$$

Proof. Denote

$$G := \left\{ x \in L \;\middle|\; \begin{array}{l} a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_n \longrightarrow x) \cdots)) = 1 \\ \text{for some } a_1, a_2, \ldots, a_n \in F \end{array} \right\}. \tag{3.5}$$

Obviously, $G \ne \emptyset$ and $F \subseteq G$. Let $x, y \in L$ and $z \in G$ be such that $(x \to z) \to y \in G$. Then there exist $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in F$ such that

$$a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_n \longrightarrow ((x \longrightarrow z) \longrightarrow y)) \cdots)) = 1, \tag{3.6}$$

$$b_1 \longrightarrow (b_2 \longrightarrow (\cdots \longrightarrow (b_m \longrightarrow z) \cdots)) = 1. \tag{3.7}$$

Using (R4) repeatedly, (3.7) implies that

$$\begin{aligned} & b_1 \longrightarrow (b_2 \longrightarrow (\cdots \longrightarrow (b_m \longrightarrow (x \longrightarrow z)) \cdots)) \\ & = x \longrightarrow (b_1 \longrightarrow (b_2 \longrightarrow (\cdots \longrightarrow (b_m \longrightarrow z) \cdots))) \\ & = x \longrightarrow 1 = 1. \end{aligned} \tag{3.8}$$

Since $x \to z \le ((x \to z) \to y) \to y$, we have

$$b_1 \longrightarrow (b_2 \longrightarrow (\cdots \longrightarrow (b_m \longrightarrow (((x \longrightarrow z) \longrightarrow y) \longrightarrow y))\cdots)) = 1, \tag{3.9}$$

that is, $((x \to z) \to y) \to (b_1 \to (b_2 \to (\cdots \to (b_m \to y)\cdots))) = 1$. It follows from (3.6) and (a6) that

$$\begin{aligned} 1 &= a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_n \longrightarrow ((x \longrightarrow z) \longrightarrow y))\cdots)) \\ &\le a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_n \longrightarrow (b_1 \longrightarrow (b_2 \longrightarrow (\cdots \longrightarrow (b_m \longrightarrow y)\cdots))))\cdots)), \end{aligned} \tag{3.10}$$

so that

$$a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_n \longrightarrow (b_1 \longrightarrow (b_2 \longrightarrow (\cdots \longrightarrow (b_m \longrightarrow y)\cdots))))\cdots)) = 1. \tag{3.11}$$

Consequently,

$$\begin{aligned} 1 &= x \longrightarrow 1 \\ &= x \longrightarrow (a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_n \longrightarrow (b_1 \longrightarrow (b_2 \longrightarrow (\cdots \longrightarrow (b_m \longrightarrow y)\cdots))))\cdots))) \\ &= a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_n \longrightarrow (b_1 \longrightarrow (b_2 \longrightarrow (\cdots \longrightarrow (b_m \longrightarrow (x \longrightarrow y))\cdots))))\cdots)), \end{aligned} \tag{3.12}$$

which shows that $x \to y \in G$. Using Proposition 3.4, we know that G is a filter of L. Now let H be a filter of L containing F and let $x \in G$. Then

$$a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_n \longrightarrow x)\cdots)) = 1 \tag{3.13}$$

for some $a_1, a_2, \ldots, a_n \in F \subseteq H$. Since H is a filter of L, it follows that $x \in H$ so that $G \subseteq H$. Therefore, $G = \langle F \rangle$, completing the proof. □

Proposition 3.8. *For any $x, a_1, a_2, \ldots, a_n \in L$,*

$$(a_1 \odot a_2 \odot \cdots \odot a_n) \longrightarrow x = 1 \Longleftrightarrow a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_n \longrightarrow x)\cdots)) = 1. \tag{3.14}$$

Proof. We will prove (3.14) by induction on n. If $n = 1$, then it is clear. If $n = 2$, then we have

$$\begin{aligned} (a_1 \odot a_2) \longrightarrow x = 1 &\Longleftrightarrow \neg(a_1 \longrightarrow \neg a_2) \longrightarrow x = 1 \\ &\Longleftrightarrow \neg x \longrightarrow (a_1 \longrightarrow \neg a_2) = 1 \Longleftrightarrow a_1 \longrightarrow (\neg x \longrightarrow \neg a_2) = 1 \\ &\Longleftrightarrow a_1 \longrightarrow (a_2 \longrightarrow x) = 1. \end{aligned} \tag{3.15}$$

Assume that (3.14) holds for $n = k$, that is,

$$(a_1 \odot a_2 \odot \cdots \odot a_k) \longrightarrow x = 1 \Longleftrightarrow a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_k \longrightarrow x)\cdots)) = 1. \tag{3.16}$$

Then

$$\begin{aligned}(a_1 \odot a_2 \odot \cdots \odot a_k \odot a_{k+1}) \longrightarrow x = 1 \\ &\Longleftrightarrow \neg((a_1 \odot a_2 \odot \cdots \odot a_k) \longrightarrow \neg a_{k+1}) \longrightarrow x = 1 \\ &\Longleftrightarrow \neg x \longrightarrow ((a_1 \odot a_2 \odot \cdots \odot a_k) \longrightarrow \neg a_{k+1}) = 1 \\ &\Longleftrightarrow (a_1 \odot a_2 \odot \cdots \odot a_k) \longrightarrow (\neg x \longrightarrow \neg a_{k+1}) = 1 \\ &\Longleftrightarrow (a_1 \odot a_2 \odot \cdots \odot a_k) \longrightarrow (a_{k+1} \longrightarrow x) = 1 \\ &\Longleftrightarrow a_1 \longrightarrow (a_2 \longrightarrow (\cdots \longrightarrow (a_k \longrightarrow (a_{k+1} \longrightarrow x))\cdots)) = 1,\end{aligned} \tag{3.17}$$

which shows that (3.14) holds for $n = k+1$. This completes the proof. □

Combining Theorem 3.7 and Proposition 3.8, the following corollary is straightforward.

Corollary 3.9 [4]. *If F is a nonempty subset of L, then*

$$\langle F \rangle = \left\{ x \in L \;\middle|\; \begin{array}{l} (a_1 \odot a_2 \odot \cdots \odot a_n) \longrightarrow x = 1 \\ \text{for some } a_1, a_2, \ldots, a_n \in F \end{array} \right\}. \tag{3.18}$$

A subset F of L is said to have the *finite $\odot$-property* if $a_1 \odot a_2 \odot \cdots \odot a_n \neq 0$ for any finite members $a_1, a_2, \ldots, a_n$ of F.

Example 3.10. Let $L = \{0, a, b, c, d, 1\}$ be an R_0-algebra in Example 3.6. The set $\{c, d\}$ satisfies the finite $\odot$-property, but the set $\{a, b, c\}$ does not satisfy the finite $\odot$-property since $a \odot b = \neg(a \to \neg b) = \neg(a \to c) = \neg 1 = 0$.

The following corollary is an immediate consequence of Corollary 3.9.

Corollary 3.11. *If F is a nonempty subset of L, then $\langle F \rangle$ is a proper filter of L if and only if F has the finite $\odot$-property.*

Theorem 3.12. *Let F be a filter of L that satisfies*

$$(\forall x \in L) \quad (x \in F \Longleftrightarrow \neg x \in L \setminus F). \tag{3.19}$$

Then F is a maximal filter of L.

Proof. Let F be a filter that satisfies the condition (3.19). Since $1 \in F$, we get $0 = \neg 1 \in L \setminus F$ and so F is proper. If G is a filter of L and $F \subsetneq G$, then there exists $a \in G \setminus F \subseteq L \setminus F$. Thus $\neg a \in F \subsetneq G$. By (a9) and Lemma 3.3, we have $0 = a \odot \neg a \in G$ and thus $G = L$. Therefore, F is a maximal filter of L. □

Proposition 3.13. *Let F be a proper filter of L. Then F satisfies condition (3.19) if and only if it satisfies*

$$(\forall x, y \in L) \quad (x \oplus y \in F \Longrightarrow x \in F \text{ or } y \in F). \tag{3.20}$$

Proof. Assume that F satisfies the condition (3.20). Let $x \in L$ be such that $\neg x \in L \setminus F$. Then $x \oplus \neg x = \neg x \to \neg x = 1 \in F$, and so $x \in F$ by (3.20). Suppose that $x \in F$. If $\neg x \in F$, then $0 = x \odot \neg x \in F$ since F is closed under $\odot$. This is a contradiction because F is proper. Hence, $\neg x \in L \setminus F$. Conversely, suppose that F satisfies (3.19). Let $x, y \in L$ be such that $x \oplus y \in F$. If $x \notin F$, then $\neg x \in F$ by (3.19). Since F is a filter, it follows from $\neg x \to y = x \oplus y \in F$ that $y \in F$. Hence, (3.20) is valid. □

Corollary 3.14. *Every proper filter F of L satisfying the condition (3.20) is maximal.*

The following example shows that there exists a maximal filter F which does not satisfy the condition (3.19).

Example 3.15. Let $L = \{0, a, b, c, 1\}$ be a set with the order $0 \le a \le b \le c \le 1$, and the following Cayley tables:

x	¬ x
0	1
a	c
b	b
c	a
1	0

⟶	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

Then $(L, \wedge, \vee, \neg, \to)$ is an R_0-algebra (see [3]), where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. This is neither a BL-algebra nor a lattice implication algebra because $c \wedge a \neq c \odot (c \to a)$ and $(b \to c) \to c \neq (c \to b) \to b$, respectively. It is not difficult to verify that the set $F = \{1, c\}$ is a maximal filter of L and $\neg b \in L \setminus F$ but $b \notin F$.

Theorem 3.16. *Let F be a nonempty subset of L. If F has the finite $\odot$-property, then there exists a maximal filter of L containing F.*

Proof. Let $\Omega := \{G \mid G$ is a proper filter of L containing $F\}$. Then $\Omega \neq \emptyset$ since $\langle F \rangle \in \Omega$. Suppose that $G_1 \subseteq G_2 \subseteq \ldots$ is a chain of elements of Ω and let $H = \bigcup_i G_i$. Then (i) $F \subseteq H$, (ii) $0 \notin H$ (because $0 \notin G_i$ for all i), (iii) $1 \in H$, and (iv) if $x \in H$ and $x \to y \in H$, then there exists i such that $x \in G_i$ and $x \to y \in G_i$. It follows from Definition 3.1(ii) that $y \in G_i \subseteq H$. This shows that H is a proper filter of L containing F, and so $H \in \Omega$. Using Zorn's lemma, Ω has a maximal element. This completes the proof. □

Example 3.17. Let $L = \{0, a, b, c, d, 1\}$ be an R_0-algebra in Example 3.6. Then $\{c, d\}$ satisfies the finite $\odot$-property, and we know that $F = \{1, c, d\}$ is a maximal proper filter of L containing $\{c, d\}$.

Definition 3.18. A nonempty subset F of L is called a *fated filter* of L if it satisfies

(i) $1 \in F$,

(ii) (for all $x, y \in L$) (for all $a \in F$) $(a \to ((x \to y) \to x) \in F \Rightarrow x \in F)$.

Example 3.19. Let $L = \{0, a, b, c, d, 1\}$ be an R_0-algebra in Example 3.6. It is not difficult to verify that the set $F := \{1, c, d\}$ is a fated filter of L.

Theorem 3.20. *Every fated filter is a filter.*

Proof. Let F be a fated filter of L and let $x,y\in L$ be such that $x\in F$ and $x\to y\in F$. Replacing a and x by x and y, respectively, in Definition 3.18(ii), we have

$$x\longrightarrow((y\longrightarrow y)\longrightarrow y)=x\to(1\longrightarrow y)=x\longrightarrow y\in F, \tag{3.21}$$

and so $y\in F$ by Definition 3.18(ii). Hence, F is a filter of L. □

The following example shows that the converse of Theorem 3.20 is not true in general.

Example 3.21. Let $L=\{0,a,b,c,1\}$ be a R_0-algebra in Example 3.15. Then $F=\{1,c\}$ is a filter of L but not a fated filter of L since $b\notin F$ and $c\to((b\to a)\to b)=1\in F$.

Theorem 3.22. *A filter F of L is fated if and only if it satisfies*

$$(\forall x,y\in L)\quad((x\longrightarrow y)\longrightarrow x\in F\Longrightarrow x\in F). \tag{3.22}$$

Proof. Assume that F is a fated filter of L and let $x,y\in L$ be such that $(x\to y)\to x\in F$. Since $1\to((x\to y)\to x)=(x\to y)\to x\in F$ and $1\in F$, it follows from Definition 3.18(ii) that $x\in F$. Thus (3.22) is valid. Conversely, let F be a filter of L that satisfies the condition (3.22). Let $a\in F$ and $x,y\in L$ be such that $a\to((x\to y)\to x)\in F$. Then $(x\to y)\to x\in F$ by Definition 3.1(ii), and so $x\in F$ by (3.22). Therefore, F is a fated filter of L. □

Theorem 3.23. *Let F be a filter of L. Then F is a fated filter if and only if it is an implicative filter.*

Proof. Let F be a fated filter of L and assume that $x\to(y\to z)\in F$ and $x\to y\in F$ for all $x,y,z\in L$. Using (R4) and (a15), we have

$$x\longrightarrow(y\longrightarrow z)=y\longrightarrow(x\longrightarrow z)\leq(x\longrightarrow y)\longrightarrow(x\longrightarrow(x\longrightarrow z)), \tag{3.23}$$

and so $(x\to y)\to(x\to(x\to z))\in F$ by Lemma 3.3(i). Since $x\to y\in F$, it follows from Definition 3.1(ii) that $x\to(x\to z)\in F$. Since

$$((x\longrightarrow z)\longrightarrow z)\longrightarrow(x\longrightarrow z)=x\longrightarrow(((x\longrightarrow z)\longrightarrow z)\longrightarrow z)=x\longrightarrow(x\longrightarrow z)\in F, \tag{3.24}$$

it follows from Theorem 3.22 that $x\to z\in F$. Therefore, F is an implicative filter of L. Conversely, suppose that F is an implicative filter of L. Let $x,y\in L$ be such that $(x\to y)\to x\in F$. Since $\neg x\leq x\to y$ and $\neg$ is involution, we have

$$(x\longrightarrow y)\longrightarrow x\leq\neg x\longrightarrow x=\neg x\longrightarrow(\neg x\longrightarrow 0) \tag{3.25}$$

by (a5). Since F is a filter, we get $\neg x\to(\neg x\to 0)\in F$ by Lemma 3.3(i). Since F is an implicative filter and $\neg x\to\neg x=1\in F$, it follows from Definition 3.2(ii) that $x=\neg x\to 0\in F$. Hence, by Theorem 3.22, we conclude that F is a fated filter of L. □

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