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Research Article

On a Class of Combinatorial Sums Involving Generalized Factorials

W.-S. Chou, L. C. Hsu, and P. J.-S. Shiue

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The object of this paper is to show that generalized Stirling numbers can be effectively used to evaluate a class of combinatorial sums involving generalized factorials.

1. Introduction

Let t and θ be real or complex numbers. We denote

$$(t \mid \theta)_p = \prod_{j=0}^{p-1}(t - j\theta) \quad (p \geq 1) \tag{1.1}$$

with $(t \mid \theta)_0 = 1$, and call it the *generalized falling factorial* of t with *increment* θ and *order* p. In particular, we write

$$(t \mid 1)_p = (t)_p, \qquad (t \mid 0)_p = t^p. \tag{1.2}$$

We are primarily concerned with an explicit closed formula of a class of combinatorial sums of the form

$$S_{j,p,\lambda,\theta}(n) = \sum_{k=j}^{n} \binom{k}{j} (k+\lambda \mid \theta)_p, \tag{1.3}$$

where both λ and θ may be real or complex numbers. Evidently, the class of sums includes some interesting particular ones, for example,

$$\begin{aligned} S_{0,p,\lambda,0}(n) &= \sum_{k=0}^{n}(k+\lambda)^p, \\ S_{j,p,0,0}(n) &= \sum_{k=j}^{n}\binom{k}{j}k^p, \\ \frac{S_{j,p,0,1}(n)}{p!} &= \sum_{k=j}^{n}\binom{k}{j}\binom{k}{p}. \end{aligned} \tag{1.4}$$

This paper is motivated from the work of Gould and Wetweerapong [1]. They have proved a pair of theorems involving two different closed formulas for the class of combinatorial sums

$$S_{j,p,0,0}(n) = \sum_{k=j}^{n}\binom{k}{j}k^p, \tag{1.5}$$

in which two special kinds of polynomials involved have been studied in much more detail. As is easily seen, Gould and Wetweerapong's formulas for $S_{j,p,0,0}(n)$ are of rank 2 and of rank 3, in as much as they consist of a double summation and of a triple summation (with elementary terms), respectively. As regards the general concept of "rank" for a summation formula, we refer the reader to Comtet's [2, Chapter 4].

The purpose of this paper is that a suitable utilization of generalized Stirling numbers may yield a general closed formula of (1.3) which is also of rank 2.

2. Preliminaries

It is well known that a kind of generalized Stirling number (GSN) with three parameters a, b, and c may be briefly defined by the basis transformation relation (cf. Hsu and Shiue [3])

$$(t \mid a)_p = \sum_{r=0}^{p} S(p,r;a,b,c)(t-c \mid b)_r, \tag{2.1}$$

where the GSNs, $S(p,r) = S(p,r;a,b,c)$, satisfy the recurrence relations

$$S(p+1,r) = S(p,r-1) + (rb - pa + c)S(p,r) \quad (r \geq 1) \tag{2.2}$$

with $S(0,0) = S(p,p) = 1$ and $S(1,0) = c$.

Now, replacing the variable t of (2.1) by $bt+c$, and applying Newton's interpolation formula to the polynomial $f(t) = (bt+c \mid a)_p$, we may find that the GSN as coefficients contained in (2.1) can be expressed using differences, viz.,

$$S(p,r;a,b,c) = \frac{1}{r!b^r}\left[\Delta^r (bt+c \mid a)_p\right]_{t=0}, \tag{2.3}$$

where Δ is the usual difference operator with increment 1, viz., $\Delta f(t) = f(t+1) - f(t)$, and $\Delta^r = \Delta\Delta^{r-1}$ $(r \geq 1)$ with $\Delta^0 = I$ $(I(f(x)) = f(x))$. Certainly an explicit expression of $S(p,r;a,b,c)$ can easily be deduced from (2.3). Of particular usefulness for our evaluation of sums are the numbers $S(p,r;\theta,1,j)$, $S(p,r;0,1,j)$, and $S(p,r;\theta,1,0)$, which are known as Howard's degenerate weighted Stirling numbers, Carlitz's weighted Stirling numbers, and degenerate Stirling numbers of the second kind, respectively (cf. [4, 5]).

In order to get a certain concise summation formula we also need an identity due to Knuth, namely,

$$\sum_{k=0}^{m} \binom{k}{r}\binom{k+s}{s} = \binom{m+1}{r}\binom{m+1+s}{s}\frac{m+1-r}{r+s+1}. \tag{2.4}$$

This identity may be found, for example, in Gould's formulary [6, formula (3.155)].

3. A general formula and its consequences

A main result to be proved is the following.

Theorem 3.1. *For any given integers $j \geq 0$ and $p \geq 0$, one has a summation formula as follows:*

$$\sum_{k=j}^{n} \binom{k}{j}(k+\lambda \mid \theta)_p = \binom{n+1}{j}\sum_{r=0}^{p}\frac{(n+1-j)_{r+1}}{r+1+j}S(p,r;\theta,1,\lambda+j), \tag{3.1}$$

where λ and θ are real or complex numbers.

Proof. Let us start with the relation (2.1) which is actually an algebraic identity involving variables a, b, c, and t. Making substitutions $a \to \theta$, $b \to 1$, $c \to \lambda + j$, and $t \to k+\lambda$, we find that (2.1) may be rewritten in the form

$$(k+\lambda \mid \theta)_p = \sum_{r=0}^{p} r!S(p,r;\theta,1,\lambda+j)\binom{k-j}{r}. \tag{3.2}$$

Using Knuth's identity (2.4) with substitutions $s \to j$ and $k \to k-j$, we have

$$\sum_{k=j}^{n}\binom{k-j}{r}\binom{k}{j} = \binom{n+1-j}{r}\binom{n+1}{j}\frac{n+1-j-r}{r+j+1} = \binom{n+1-j}{r+1}\binom{n+1}{j}\frac{r+1}{r+j+1}. \tag{3.3}$$

Thus, making use of (3.2) and (3.3), we find

$$\begin{aligned}
\sum_{k=j}^{n}\binom{k}{j}(k+\lambda\mid\theta)_p &= \sum_{r=0}^{p} r!S(p,r;\theta,1,\lambda+j)\sum_{k=j}^{n}\binom{k-j}{r}\binom{k}{j}\\
&= \binom{n+1}{j}\sum_{r=0}^{p}\binom{n+1-j}{r+1}\frac{r+1}{r+j+1}r!S(p,r;\theta,1,\lambda+j)\\
&= \binom{n+1}{j}\sum_{r=0}^{p}\frac{(n+1-j)_{r+1}}{r+1+j}S(p,r;\theta,1,\lambda+j).
\end{aligned}\tag{3.4}$$

This is what we desired. □

Note that the GSNs $S(p,r;\theta,1,\lambda+j)$ may be expressed in the forms

$$S(p,r;\theta,1,\lambda+j)=\frac{1}{r!}\left[\Delta^r(t+\lambda+j\mid\theta)_p\right]_{t=0},\tag{3.5}$$

$$S(p,r;\theta,1,\lambda+j)=\frac{1}{r!}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(i+\lambda+j\mid\theta)_p.\tag{3.6}$$

Note also that (3.5) is implied by (2.3) and that (3.6) just follows from the well-known expression for higher differences. It is clear that (3.6) is a formula of rank 1 since it consists of only a single summation involving elementary terms. Consequently, formula (3.1) is of rank 2.

Remark 3.2. In fact, that formula (3.1) is like a dual to [7, formula (35)], which states

$$\sum_{k=0}^{n}\binom{n}{k}(k+\lambda\mid\theta)_p=\sum_{r=0}^{p}\binom{n}{r}2^{n-r}r!S(p,r;\theta,1,\lambda),\tag{3.7}$$

where $S(p,r;\theta,1,\lambda)$s are given by (3.11) with $j=\lambda$. So, (3.7) is also a formula of rank 2. We can derive formula (3.7) using our method developed here, which is less complicated than that used in [7]. Indeed, from formula (3.2) and the identity

$$\binom{n}{k}\binom{k}{r}=\binom{n}{r}\binom{n-r}{k-r},\tag{3.8}$$

we have

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}(k+\lambda\mid\theta)_p &= \sum_{r=0}^{p}r!S(p,r;\theta,1,\lambda)\sum_{k=0}^{n}\binom{n}{k}\binom{k}{r}=\sum_{r=0}^{p}r!S(p,r;\theta,1,\lambda)\binom{n}{r}\sum_{k=r}^{n}\binom{n-r}{k-r}\\
&= \sum_{r=0}^{p}\binom{n}{r}r!S(p,r;\theta,1,\lambda)\sum_{k=0}^{n-r}\binom{n-r}{k}=\sum_{r=0}^{p}\binom{n}{r}2^{n-r}r!S(p,r;\theta,1,\lambda).
\end{aligned}\tag{3.9}$$

A number of corollaries giving some previously known and unknown formulas may be stated as consequences of (3.1) and (3.6) as follows.

Corollary 3.3. *There holds the formula*

$$\sum_{k=j}^{n}\binom{k}{j}(k\mid\theta)_p=\binom{n+1}{j}\sum_{r=0}^{p}\frac{(n+1-j)_{r+1}}{r+1+j}S(p,r;\theta,1,j),\tag{3.10}$$

where Howards's GSN $S(p,r;\theta,1,j)$ (see [5]) is written in the form

$$S(p,r;\theta,1,j)=\frac{1}{r!}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(i+j\mid\theta)_p.\tag{3.11}$$

In particular, if $\theta=0$, then (3.10) is reduced to

$$\sum_{k=j}^{n}\binom{k}{j}k^p=\binom{n+1}{j}\sum_{r=0}^{p}\frac{(n+1-j)_{r+1}}{r+1+j}S(p,r;0,1,j),\tag{3.12}$$

where Carlitz's GSN $S(p,r;0,1,j)$ (see [4]) is expressed in the form

$$S(p,r;0,1,j)=\frac{1}{r!}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(i+j)^p.\tag{3.13}$$

Note that (3.10) and (3.12) are different from the main results of Gould and Wetweerapong [1], nevertheless they are comparable with each other. By using residue method, Huang [8] provided an identity for (3.12) with $p=2$. Jones [9] also used a telescoping series to derive a different formula of (3.12), namely,

$$\sum_{k=j}^{n}\binom{k}{j}k^p=j^p\binom{n+1}{j+1}+\sum_{i=j+1}^{n}\{i^p-(i-1)^p\}\left\{\binom{n+1}{j+1}-\binom{i}{j+1}\right\}.\tag{3.14}$$

If $\theta=1$ in Corollary 3.3, then we have the following.

Corollary 3.4. *One has a pair of formulas of rank 1 as follows:*

$$\sum_{k=j}^{n}\binom{k}{j}\binom{k}{p}=\binom{n+1}{j+1}\sum_{r=0}^{p}\binom{n-j}{r}\binom{j}{p-r}\frac{j+1}{r+j+1},\tag{3.15}$$

$$\sum_{k=p}^{n}\binom{k}{p}^2=\binom{n+1}{p+1}\sum_{r=0}^{p}\binom{n-p}{r}\binom{p}{r}\frac{p+1}{r+p+1}.\tag{3.16}$$

Proof. Obviously, (3.16) is implied by (3.15) with $j=p$. Notice that formula (3.10) with $\theta=1$ implies that

$$\sum_{k=j}^{n}\binom{k}{j}\binom{k}{p}=\frac{1}{p!}\binom{n+1}{j}\sum_{r=0}^{p}\frac{(n+1-j)_{r+1}}{r+1+j}S(p,r;1,1,j).\tag{3.17}$$

Here, using (3.11) and the higher difference formula, that we easily find

$$S(p,r;1,1,j)=\frac{p!}{r!}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}\binom{i+j}{p}=\frac{p!}{r!}\left[\Delta^r\binom{t+j}{p}\right]_{t=0}=\frac{p!}{r!}\binom{j}{p-r}\quad(p\geq r). \tag{3.18}$$

Thus, by substitution of the above resultant expression into the right-hand side of (3.17), we will attain the desired expression (3.15) after simple computations. □

Corollary 3.5. *One has*

$$\sum_{k=0}^{n}(k\mid\theta)_p=\sum_{r=0}^{p}\binom{n+1}{r+1}r!S(p,r;\theta,1,0), \tag{3.19}$$

where Carlitz's degenerate Stirling numbers $S(p,r;\theta,1,0)$ are expressed in the form

$$S(p,r;\theta,1,0)=\frac{1}{r!}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(i\mid\theta)_p. \tag{3.20}$$

In particular, if $\theta=0$, one has the classical formula

$$\sum_{k=0}^{n}k^p=\sum_{r=0}^{p}\binom{n+1}{r+1}r!S(p,r), \tag{3.21}$$

where $S(p,r)=S(p,r;0,1,0)$ are the ordinary Stirling numbers of the second kind.

Corollary 3.6. *For any given real or complex number λ,*

$$\sum_{k=0}^{n}(k+\lambda)^p=\sum_{r=0}^{p}\frac{(n+1)_{r+1}}{r+1}S(p,r;0,1,\lambda), \tag{3.22}$$

where $S(p,r;0,1,\lambda)$ can be computed by (3.13) with $j=\lambda$.

Note that (3.22) implies (3.21) with $\lambda=0$.

Corollary 3.7. *One has the most simple identity*

$$\sum_{k=j}^{n}\binom{k}{j}=\binom{n+1}{j+1}. \tag{3.23}$$

Identity (3.23) is known as the most old formula found in the 14th century by ancient Chinese mathematician Zhu Shijie. Indeed, (3.23) appeared in the second mathematics book of Zhu which was published in 1303 AD. Certainly, (3.23) is a formulaof rank 0.

4. Final remarks

Remark 4.1. There is a closed formula for other types of combinatorial sums involving generalized factorials. For example, we have the following known results (cf. [7]):

$$\sum_{k=0}^{n}\binom{n}{k}^{2}(k+\lambda\mid\theta)_{p}=\sum_{r=0}^{p}\binom{2n-r}{n}(n)_{r}S(p,r;\theta,1,\lambda), \tag{4.1}$$

$$\sum_{k=0}^{[n/2]}\binom{n}{2k}(k+\lambda\mid\theta)_{p}=\sum_{r=0}^{p}2^{n-2r-1}\binom{n-r}{r}\frac{n}{n-r}r!S(p,r;\theta,1,\lambda), \tag{4.2}$$

where $S(p,r;\theta,1,\lambda)$s are given by (3.11) with $j=\lambda$, so that (4.1) and (4.2) are all formulas of rank 2. These two formulas can also be derived in our method easily.

For proving (4.1), we note that [6, formula (3.1)]

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}=\binom{x+y}{n}. \tag{4.3}$$

Then, using formula (3.2), we have

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}^{2}(k+\lambda\mid\theta)_{p}&=\sum_{r=0}^{p}r!S(p,r;\theta,1,\lambda)\sum_{k=0}^{n}\binom{n}{k}^{2}\binom{k}{r}\\&=\sum_{r=0}^{p}r!S(p,r;\theta,1,\lambda)\binom{n}{r}\sum_{k=0}^{n}\binom{n}{k}\binom{n-r}{k-r}\\&=\sum_{r=0}^{p}(n)_{r}S(p,r;\theta,1,\lambda)\sum_{k=0}^{n}\binom{n}{k}\binom{n-r}{n-k}.\end{aligned} \tag{4.4}$$

Taking $x=n$ and $y=n-r$ in (4.3), we have formula (4.1).

For proving (4.2), we have, from (3.2),

$$\sum_{k=0}^{[n/2]}\binom{n}{2k}(k+\lambda\mid\theta)_{p}=\sum_{r=0}^{p}r!S(p,r;\theta,1,\lambda)\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{k}{r}. \tag{4.5}$$

Since

$$\sum_{k=r}^{[n/2]}\binom{n}{2k}\binom{k}{r}=2^{n-2r-1}\binom{n-r}{r}\frac{n}{n-r} \tag{4.6}$$

(formula (3.120) in [6]), formula (4.2) follows immediately.

Remark 4.2 (Li Shanlan identity). The well-known classical identity

$$\sum_{k=0}^{n}\binom{n}{k}^{2}\binom{m+n+k}{2n}=\binom{m+n}{n}^{2} \tag{4.7}$$

due to Li Shanlan (1811–1882) appeared in Li's writings in 1860s and was given several proofs 50 years ago by a number of authors including Paul Turan, Loo-Keng Hua, et al. The related references are too numerous to be given. Here it may be worth mentioning that this identity is a particular consequence of (4.1). Recall that

$$\binom{m}{n}\binom{s}{m}=\binom{s}{n}\binom{s-n}{m-n}. \tag{4.8}$$

From (3.18), we have $j!S(2n,j;1,1,m+n)=(2n)!\binom{m+n}{2n-j}$. Thus, we see that from formula (4.1) with $\lambda=m+n$, $\theta=1$ and $p=2n$ implies that

$$\begin{aligned}
\text{LHS of (4.7)} &= \frac{1}{2n!}\sum_{k=0}^{n}\binom{n}{k}^2(k+m+n\mid 1)_{2n}=\frac{1}{2n!}\sum_{j=0}^{n}\binom{2n-j}{n}\binom{n}{j}j!S(2n,j;1,1,m+n)\\
&=\sum_{j=0}^{n}\binom{2n-j}{n}\binom{m+n}{2n-j}\binom{n}{j}=\sum_{j=0}^{n}\binom{m+n}{n}\binom{m}{n-j}\binom{n}{j}\quad\text{(by (4.8))}\\
&=\binom{m+n}{n}\binom{m+n}{n}\quad\text{(by (4.3))}\\
&=\text{RHS of (4.7).}
\end{aligned} \tag{4.9}$$

Remark 4.3 (simple asymptotics of combinatorial sums). It is clear that formula (3.1) is useful for practical computations whenever n is much bigger than p, say $n \gg p^2$. Observe that for n large, the asymptotic behavior of the combinatorial sums in (3.1) is mainly determined by those principal terms (i.e., the terms with $r=p$) within the closed formula. Also note that $S(p,p;\dots)=1$. Thus, we easily obtain a simple asymptotic relation for $n\to\infty$ as follows:

$$\sum_{k=j}^{n}\binom{k}{j}(k+\lambda\mid\theta)_p\sim\frac{n^{p+j+1}}{j!(p+j+1)}. \tag{4.10}$$

Certainly, the asymptotic estimate given by (4.10) could be refined by taking into account those terms with $r=p-1$, $r=p-2$, and so forth, within the closed formula. And accordingly, values of $S(p,p-1;\dots),S(p,p-2;\dots)$, and so forth are required to be evaluated.

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W.-S. Chou: Institute of Mathematics, Academia Sinica, Nankang, Taipei 11529, Taiwan
Email address: macws@math.sinica.edu.tw

L. C. Hsu: Institute of Mathematical Sciences, Dalian University of Technology, Dalian 116024, Liaoning, China
Email address: xulizhi63@hotmail.com

P. J.-S. Shiue: Department of Mathematical Sicences, University of Nevada, Las Vegas, NV 89154-4020, USA
Email address: shiue@unlv.nevada.edu