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Research Article

Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias Stabilities of an Additive Functional Equation in Several Variables

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It is well known that the concept of Hyers-Ulam-Rassias stability was originated by Th. M. Rassias (1978) and the concept of Ulam-Gavruta-Rassias stability was originated by J. M. Rassias (1982–1989) and by P. Găvruta (1999). In this paper, we give results concerning these two stabilities.

1. Introduction

In 1940, Ulam [13] proposed the Ulam stability problem of additive mappings. In the next year, Hyers [5] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies inequality $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that L is the unique additive mapping satisfying $\|f(x) - L(x)\| \le \varepsilon$. In 1978, Rassias [14] generalized the result to an approximation involving a sum of powers of norms. In 1982–1989, Rassias [8–11] treated the Ulam-Gavruta-Rassias stability on linear and nonlinear mappings and generalized Hyers result to the following theorem.

Theorem 1.1 (J. M. Rassias). *Let $f : E \to E'$ be a mapping, where E is a real-normed space and E' is a Banach space. Assume that there exist $\theta > 0$ such that*

$$\|f(x+y) - f(x) - f(y)\| \le \theta \|x\|^p \|y\|^q \tag{1.1}$$

for all $x, y \in E$, where $r = p + q \ne 1$. Then there exists a unique additive mapping $L : E \to E'$

such that

$$\|f(x)-L(x)\| \leq \frac{\theta}{|2-2^r|}\|x\|^r \tag{1.2}$$

for all $x \in E$.

However, the case $r = 1$ in the above inequality is singular. A counterexample has been given by Găvruta [2]. The above-mentioned stability involving a product of different powers of norms is called Ulam-Gavruta-Rassias stability by Bouikhalene and Elqorachi [1], Ravi and ArunKumar [12], and Nakmahachalasint [6]. In recent years, some other authors [3, 4, 7] have investigated the stability of additive mapping in various forms.

In this paper, we propose an n-dimensional additive functional equation and investigate its Hyers-Ulam-Rassias and Ulam-Gavruta-Rassias stabilities.

2. The functional equation and the solution

Theorem 2.1. *Let $n > 1$ be an integer and let X, Y be real vector spaces. A mapping $f : X \to Y$ satisfies the functional equation*

$$nf\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} f(x_i) + \sum_{1\leq i<j\leq n} f(x_i+x_j) \quad \forall x_1,x_2,\dots,x_n \in X \tag{2.1}$$

if and only if f satisfies the Cauchy functional equation

$$f(x+y) = f(x)+f(y) \quad \forall x,y \in X. \tag{2.2}$$

Proof. We first suppose that a mapping $f : X \to Y$ satisfies (2.2). By the additivity of the Cauchy functional equation, we have

$$\begin{aligned}\sum_{i=1}^{n} f(x_i) + \sum_{1\leq i<j\leq n} f(x_i+x_j) &= \sum_{i=1}^{n} f(x_i) + \sum_{1\leq i<j\leq n} \left(f(x_i)+f(x_j)\right)\\ &= n\sum_{i=1}^{n} f(x_i) = nf\left(\sum_{i=1}^{n} x_i\right)\end{aligned} \tag{2.3}$$

for all $x_1,x_2,\dots,x_n \in X$. Hence, f satisfies (2.1).

Now suppose that a mapping $f : X \to Y$ satisfies (2.1). Putting $x_1 = x_2 = \cdots = x_n = 0$ in (2.1), we have $nf(0) = nf(0) + \binom{n}{2} f(0)$, which leads to $f(0) = 0$. Putting $x_1 = x, x_2 = y$ and, if $n > 2$, $x_3 = x_4 = \cdots = x_n = 0$ in (2.1), we get

$$nf(x+y) = f(x)+f(y)+(n-2)f(x)+(n-2)f(y)+f(x+y) \quad \forall x,y \in X, \tag{2.4}$$

which simplifies to $f(x+y) = f(x)+f(y)$ as desired. □

3. Hyers-Ulam-Rassias stability

The following theorem treats the Hyers-Ulam-Rassias stability of (2.1).

Theorem 3.1. *Let $n > 1$ be an integer, let X be a real vector space, and let Y be a Banach space. Given real numbers $\delta, \theta \geq 0$ and $p \in (0,1) \cup (1,\infty)$ with $\delta = 0$ when $p > 1$. If a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| nf\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) - \sum_{1\leq i<j\leq n} f(x_i + x_j)\right\| \leq \delta + \theta \sum_{i=1}^{n} ||x_i||^p \tag{3.1}$$

for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique additive mapping $L : X \to Y$ that satisfies (2.1) and the inequality

$$||f(x) - L(x)|| \leq \frac{2\delta}{n} + \frac{2\theta}{(n-1)\,|\,2 - 2^p\,|}\,||x||^p \quad \forall x \in X. \tag{3.2}$$

The mapping L is given by

$$L(x) = \begin{cases} \lim_{m\to\infty} 2^{-m} f(2^m x) & \text{if } 0 < p < 1 \\ \lim_{m\to\infty} 2^{m} f(2^{-m} x) & \text{if } p > 1 \end{cases} \quad \forall x \in X. \tag{3.3}$$

Proof. Putting $x_1 = x_2 = \cdots = x_n = 0$ in (3.1), we have $\|nf(0) - nf(0) - \binom{n}{2} f(0)\| \leq \delta$. Thus, $\|f(0)\| \leq 2\delta/(n^2 - n)$. Setting $x_1 = x_2 = x$ and, if $n > 2$, $x_3 = x_4 = \cdots = x_n = 0$ in (3.1), we have

$$\left\| nf(2x) - 2f(x) - (n-2)f(0) - f(2x) - 2(n-2)f(x) - \binom{n-2}{2} f(0)\right\| \leq \delta + 2\theta\|x\|^p, \tag{3.4}$$

which simplifies to

$$(n-1)\left\| f(2x) - 2f(x) - \frac{n-2}{2} f(0)\right\| \leq \delta + 2\theta\|x\|^p. \tag{3.5}$$

Therefore,

$$||2f(x) - f(2x)|| \leq \frac{n-2}{2}||f(0)|| + \frac{\delta + 2\theta\|x\|^p}{n-1} \leq \frac{2\delta}{n} + \frac{2\theta}{n-1}\|x\|^p. \tag{3.6}$$

We first consider the case where $0 < p < 1$. Rewrite the above inequality (3.6) as

$$||f(x) - 2^{-1} f(2x)|| \leq \frac{\delta}{n} + \frac{\theta}{n-1}\|x\|^p. \tag{3.7}$$

For every positive integer m,

$$\begin{aligned}||f(x)-2^{-m}f(2^m x)|| &= \left\|\sum_{i=0}^{m-1}\left(2^{-i}f(2^i x)-2^{-(i+1)}f(2^{i+1}x)\right)\right\| \\ &\le \sum_{i=0}^{m-1}||2^{-i}f(2^i x)-2^{-(i+1)}f(2^{i+1}x)|| \\ &= \sum_{i=0}^{m-1}2^{-i}||f(2^i x)-2^{-1}f(2\cdot 2^i x)||.\end{aligned} \tag{3.8}$$

Substituting x with $x, 2x, 2^2x, \dots, 2^{m-1}x$ in (3.7), the above inequality becomes

$$||f(x)-2^{-m}f(2^m x)|| \le \frac{\delta}{n}\sum_{i=0}^{m-1}2^{-i}+\frac{\theta}{n-1}\|x\|^p\sum_{i=0}^{m-1}2^{i(p-1)}. \tag{3.9}$$

Consider the sequence $\{2^{-m}f(2^m x)\}$. For all positive integers $k<l$, we have

$$\begin{aligned}||2^{-k}f(2^k x)-2^{-l}f(2^l x)|| &= 2^{-k}||f(2^k x)-2^{-(l-k)}f(2^{l-k}\cdot 2^k x)|| \\ &\le 2^{-k}\left(\frac{\delta}{n}\sum_{i=0}^{l-k-1}2^{-i}+\frac{\theta}{n-1}||2^k x||^p\sum_{i=0}^{l-k-1}2^{i(p-1)}\right) \\ &\le \frac{2^{-k}\delta}{n}\sum_{i=0}^{\infty}2^{-i}+\frac{\theta}{n-1}2^{-k(1-p)}\|x\|^p\sum_{i=0}^{\infty}2^{i(p-1)}.\end{aligned} \tag{3.10}$$

The right-hand side of the above inequality approaches 0 as $k\to\infty$. Therefore, $L(x)=\lim_{m\to\infty}2^{-m}f(2^m x)$ is well defined. Taking the limit of (3.9) as $m\to\infty$, we have

$$||f(x)-L(x)|| \le \frac{\delta}{n}\sum_{i=0}^{\infty}2^{-i}+\frac{\theta}{n-1}\|x\|^p\sum_{i=0}^{\infty}2^{i(p-1)} = \frac{2\delta}{n}+\frac{2\theta}{(n-1)(2-2^p)}\|x\|^p \quad \forall x\in X. \tag{3.11}$$

To show that L satisfies (2.1), replace each x_i in (3.1) with $2^m x_i$. This results in

$$\left\|nf\left(\sum_{i=1}^{n}2^m x_i\right)-\sum_{i=1}^{n}f(2^m x_i)-\sum_{1\le i<j\le n}f(2^m x_i+2^m x_j)\right\| \le \left(\delta+\theta\sum_{i=1}^{n}||2^m x_i||^p\right). \tag{3.12}$$

Dividing the above inequality by 2^m and taking the limit as $m\to\infty$, we obtain

$$\left\|nL\left(\sum_{i=1}^{n}x_i\right)-\sum_{i=1}^{n}L(x_i)-\sum_{1\le i<j\le n}f(x_i+x_j)\right\| \le \lim_{m\to\infty}\left(\frac{\delta}{2^m}+\frac{\theta}{2^{m(1-p)}}\sum_{i=1}^{n}||x_i||^p\right)=0, \tag{3.13}$$

which verifies that L indeed satisfies (2.1).

To prove the uniqueness of L, suppose there is a mapping $L' : X \to Y$ such that L' satisfies (2.1) and (3.2). The additivity of L and L' is asserted by Theorem 2.1; hence,

$$\begin{aligned} \|L(x) - L'(x)\| &= 2^{-m}\|L(2^m x) - L'(2^m x)\| \\ &\le 2^{-m}\left(\|L(2^m x) - f(2^m x)\| + \|L'(2^m x) - f(2^m x)\|\right) \\ &\le 2^{-m}\cdot 2\left(\frac{2\delta}{n} + \frac{2\theta}{(n-1)(2-2^p)}\|2^m x\|^p\right) \underset{m\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.14}$$

Thus, $L(x) = L'(x)$ for all $x \in X$.

For the case $p > 1$, $\delta = 0$ and (3.7) must be replaced by

$$\|f(x) - 2f(2^{-1}x)\| \le \frac{2\theta}{n-1}\|2^{-1}x\|^p. \tag{3.15}$$

The rest of the proof can be done in the same fashion as that of the case $0 < p < 1$. □

4. Ulam-Gavruta-Rassias stability

The following theorem treats the Ulam-Gavruta-Rassias stability of (2.1).

THEOREM 4.1. *Let $n > 1$ be an integer, let X be a real vector space, and let Y be a Banach space. Given real numbers $\delta, \theta \ge 0$ and $p \in (0,1) \cup (1,\infty)$ with $\delta = 0$ when $p > 1$. If a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| nf\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n f(x_i) - \sum_{1\le i<j\le n} f(x_i + x_j)\right\| \le \delta + \theta \sum_{1\le i<j\le n} \|x_i\|^{p/2}\|x_j\|^{p/2} \tag{4.1}$$

for all $x_1, x_2, \dots, x_n \in X$, then there exists a unique additive mapping $L : X \to Y$ that satisfies (2.1) and the inequality

$$\|f(x) - L(x)\| \le \frac{2\delta}{n} + \frac{\theta}{(n-1)|2-2^p|}\|x\|^p \quad \forall x \in X. \tag{4.2}$$

The mapping L is given by (3.3).

Proof. We make the same substitution as in the proof of Theorem 3.1 and obtain instead of (3.5) the following inequality:

$$(n-1)\left\| f(2x) - 2f(x) - \frac{n-2}{2}f(0)\right\| \le \delta + \theta\|x\|^p \quad \forall x \in X. \tag{4.3}$$

The rest of the proof, apart from a multiplicative factor of 2 appears before θ, can be carried over from that of Theorem 3.1. □

It should be remarked that in the case where $n = 2$, functional equation (2.1) reduces to the Cauchy functional equation, and the Ulam-Gavruta-Rassias stability of this problem has been treated by J. M. Rassias, and the result has been restated in Theorem 1.1.

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