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Research Article

On a Class of Composition Operators on Bergman Space

Namita Das, R. P. Lal, and C. K. Mohapatra

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Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $A^2(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$ square integrable with respect to the measure $dA(z) = (1/\pi)dx\,dy$. Given $a \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define the function $C_a f$ by $C_a f(z) = f(\varphi_a(z))$, where $\varphi_a \in \mathrm{Aut}(\mathbb{D})$. The map C_a is a composition operator on $L^2(\mathbb{D}, dA)$ and $A^2(\mathbb{D})$ for all $a \in \mathbb{D}$. Let $\mathcal{L}(A^2(\mathbb{D}))$ be the space of all bounded linear operators from $A^2(\mathbb{D})$ into itself. In this article, we have shown that $C_a S C_a = S$ for all $a \in \mathbb{D}$ if and only if $\int_{\mathbb{D}} \widetilde{S}(\varphi_a(z))dA(a) = \widetilde{S}(z)$, where $S \in \mathcal{L}(A^2(\mathbb{D}))$ and $\widetilde{S}$ is the Berezin symbol of S.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let dA(z) be the area measure on $\mathbb{D}$ normalized so that the area of the disk $\mathbb{D}$ is 1. In rectangular and polar coordinates, $dA(z) = (1/\pi)dx\,dy = (1/\pi)r\,dr\,d\theta$. Let $L^2(\mathbb{D}, dA)$ be the Hilbert space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_2 = \left[\int_{\mathbb{D}} |f(z)|^2 dA(z)\right]^{1/2} < +\infty. \tag{1.1}$$

The inner product is defined as

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z) \tag{1.2}$$

for $f,g \in L^2(\mathbb{D},dA)$. The space $L^\infty(\mathbb{D},dA)$ will denote the Banach space of Lebesgue measurable functions f on $\mathbb{D}$ with

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(z)| : z \in \mathbb{D}\} < \infty. \tag{1.3}$$

The Bergman space $A^2(\mathbb{D})$ is defined to be the subspace of $L^2(\mathbb{D},dA)$ consisting of analytic functions. It is not so difficult to verify that (see [1]) $A^2(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D},dA)$. Since point evaluation at $z \in \mathbb{D}$ is a bounded linear functional on the Hilbert space $A^2(\mathbb{D})$, the Riesz representation theorem [2] implies that there exists a unique function K_z in $A^2(\mathbb{D})$ such that

$$f(w) = \int_{\mathbb{D}} f(z)\overline{\mathrm{K}_z(w)}\,dA(w) \tag{1.4}$$

for all f in $A^2(\mathbb{D})$. Let $K(z,w)$ be the function on $\mathbb{D}\times\mathbb{D}$ defined by $K(z,\omega) = \overline{\mathrm{K}_z(w)}$. The function $K(z,w)$ is called the Bergman kernel of $\mathbb{D}$ and it can be verified that (see [3])

$$K(z,\omega) = \frac{1}{(1-z\overline{w})^2}. \tag{1.5}$$

Let $k_a(z) = K(z,a)/\sqrt{K(a,a)} = (1-|a|^2)/(1-\overline{a}z)^2$. The function k_a is called the normalized reproducing kernel for $A^2(\mathbb{D})$. It is clear that $\|k_a\|_2 = 1$. Let $\mathrm{Aut}(\mathbb{D})$ be the Lie group of all automorphisms(biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$ an automorphism φ_a in $\mathrm{Aut}(\mathbb{D})$ such that

(i) $(\varphi_a \circ \varphi_a)(z) \equiv z$,

(ii) $\varphi_a(0) = a$, $\varphi_a(a) = 0$,

(iii) φ_a has a unique fixed point in $\mathbb{D}$.

In fact, $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ for all a and z in $\mathbb{D}$. An easy calculation shows that the derivative of φ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of φ_a at z is $\mathbf{J}_{\varphi_a(z)} = |k_a(z)|^2 = (1-|a|^2)^2/|1-\overline{a}z|^4$. Given $\lambda \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define a function $U_\lambda f$ on $\mathbb{D}$ by $U_\lambda f(z) = k_\lambda(z)f(\varphi_\lambda(z))$. Notice that U_λ is a bounded linear operator on $L^2(\mathbb{D},dA)$ and $A^2(\mathbb{D})$ for all $\lambda \in \mathbb{D}$. Further, it can be checked that $U_\lambda^2 = I$, the identity operator, $U_\lambda^* = U_\lambda, U_\lambda(A^2(\mathbb{D})) \subset (A^2(\mathbb{D}))$, and $U_\lambda((A^2(\mathbb{D}))^\perp) \subset (A^2(\mathbb{D}))^\perp$ for all $\lambda \in \mathbb{D}$. Thus $U_\lambda P = PU_\lambda$ for all $\lambda \in \mathbb{D}$, where P is the orthogonal projection from $L^2(\mathbb{D},dA)$ onto $A^2(\mathbb{D})$. Let $\phi : \mathbb{D} \to \mathbb{D}$ be analytic. Define the composition operator C_ϕ from $A^2(\mathbb{D})$ into itself as $C_\phi f = f \circ \phi$. Then C_ϕ is a bounded linear operator on $A^2(\mathbb{D})$ and $\|C_\phi\| \le (1+|\phi(0)|)/(1-|\phi(0)|)$ (see [3] for a proof). Given $a \in \mathbb{D}$ and f any measurable function on $\mathbb{D}$, we define the function $C_a f$ by $C_a f(z) = f(\varphi_a(z))$, where $\varphi_a \in \mathrm{Aut}(\mathbb{D})$. The map C_a is a composition operator on $A^2(\mathbb{D})$. Let $\mathcal{L}(A^2(\mathbb{D}))$ be the space of all bounded linear operator from $A^2(\mathbb{D})$ into itself. For $T \in \mathcal{L}(A^2(\mathbb{D}))$, define the map $\widetilde{T}$ on $\mathbb{D}$ as $\widetilde{T}(z) = \langle Tk_z, k_z\rangle$. The map $B : \mathcal{L}(A^2(\mathbb{D})) \to L^\infty(\mathbb{D})$ defined by $B(T) = \widetilde{T}$ is called the Berezin transform and $\widetilde{T}$ is called the Berezin symbol of T.

The Berezin transform associates operators on Hilbert spaces of holomorphic functions to smooth functions [4]. It is very effective in several contexts in the sense that $B(T)$ contains a lot of information about the operator T. Successful applications of the

Berezin transform are so far mainly in the study of Toeplitz and Hankel operators on the Bergman space. Regardless of the original motivation of Berezin for introducing it, the Berezin transform essentially provides a kind of "symbol" for certain natural operators on Hilbert spaces of analytic functions. Thus it is natural to ask the general question of how much information about the operator does its Berezin symbol carry. The problem is subtle and no general answer is known. In this work, we have shown that the Berezin symbol of a bounded linear operator S from the Bergman space into itself satisfies certain averaging condition if and only if the operator S satisfy the intertwining relation $C_aSC_a = S$ for all $a \in \mathbb{D}$. Recently, the spectra of composition operators have attracted much attention (see [5–7]) from operator theorists. To this purpose, it is important to know what are the essential commutants of the invertible operators $C_a, a \in \mathbb{D}$, and to characterize those $S \in \mathcal{L}(A^2(\mathbb{D}))$ such that $C_aS - SC_a = (C_aSC_a - S)C_a$ is compact for all $a \in \mathbb{D}$. In this work, we present a necessary and sufficient condition for $C_aSC_a - S = 0$ to happen for all $a \in \mathbb{D}$ in terms of the Berezin symbol of S. Related work in this area can be found in [5–8].

2. The unitary operator U_a and the Berezin transform

In this section, we will prove certain elementary properties of the unitary operator U_a and the Berezin transform.

Lemma 2.1. *For $z, \omega \in \mathbb{D}$, $U_z k_\omega = \alpha k_{\varphi_z(\omega)}$ for some complex constant α such that $|\alpha| = 1$.*

Proof. Suppose $z, \omega \in \mathbb{D}$. If $f \in A^2(\mathbb{D})$, then

$$\langle f, U_z K_\omega \rangle = \langle U_z f, K_\omega \rangle = (U_z f)(\omega) = -(f \circ \varphi_z)(\omega)\varphi_z'(\omega) = \langle f, (-\overline{\varphi_z'(\omega)})K_{\varphi_z(\omega)} \rangle. \tag{2.1}$$

Thus $U_z K_\omega = -\overline{\varphi_z'(\omega)} K_{\varphi_z(\omega)}$. Rewriting this in terms of the normalized reproducing kernels, we have

$$U_z k_\omega = \alpha k_{\varphi_z(\omega)} \tag{2.2}$$

for some complex constant α. Since U_z is unitary and $\|k_\omega\|_2 = \|k_{\varphi_z(\omega)}\|_2 = 1$, we obtain that $|\alpha| = 1$. □

Lemma 2.2. *For all $a \in \mathbb{D}$, $U_a k_a = 1$.*

Proof. If $a \in \mathbb{D}$, then first observe that $\varphi_a'(z) = -k_a(z)$. Since $(\varphi_a \circ \varphi_a)(z) = z$ for all $z \in \mathbb{D}$, taking derivatives with respect to z in both sides, we obtain

$$(U_a k_a)(z) = k_a(\varphi_a(z))k_a(z) = 1. \tag{2.3}$$

□

Notice that for all $a \in \mathbb{D}$, since $U_a k_a = 1$, hence $k_a \circ \varphi_a = 1/k_a$ and $k_a^{-1} \in H^\infty$, the space of bounded analytic functions on $\mathbb{D}$.

Lemma 2.3. *If $S, T \in \mathcal{L}(A^2(\mathbb{D}))$ and for all $z \in \mathbb{D}$, $\widetilde{S}(z) = \widetilde{T}(z)$, then $S = T$.*

Proof. If $\widetilde{S}(z) = \widetilde{T}(z)$ for all $z \in \mathbb{D}$, then

$$\langle (S-T)k_z, k_z \rangle = 0 \tag{2.4}$$

for all $z \in \mathbb{D}$. This implies

$$\langle (S-T)K_z, K_z \rangle = K(z,z)\langle (S-T)k_z, k_z \rangle = K(z,z) \cdot 0 = 0. \tag{2.5}$$

Let $L = S - T$ and define

$$F(x,y) = \langle LK_{\overline{x}}, K_y \rangle. \tag{2.6}$$

The function F is holomorphic in x and y and $F(x,y) = 0$ if $x = \overline{y}$. It can now be verified that such functions must vanish identically. Let $x = u + iv$, $y = u - iv$. Let $G(u,v) = F(x,y)$. The function G is holomorphic and vanishes if u and v are real. Hence $F(x,y) = G(u,v) \equiv 0$. Thus even $\langle LK_x, K_y \rangle = 0$ for any x, y. Since linear combinations of K_x, $x \in \mathbb{D}$, are dense in $A^2(\mathbb{D})$, it follows that $L = 0$. That is, $S = T$. □

3. Main result and its applications

In this section, we will prove that a bounded linear operator S from $A^2(\mathbb{D})$ into itself commutes with all the composition operators C_a, $a \in \mathbb{D}$, if and only if $\widetilde{S}$ satisfies certain averaging condition. We will also present some applications of this result.

Theorem 3.1. *A bounded linear operator $S \in \mathcal{L}(A^2(\mathbb{D}))$ commutes with all the composition operators $C_a, a \in \mathbb{D}$, if and only if*

$$\widetilde{S}(z) = \int_{\mathbb{D}} \widetilde{S}(\varphi_a(z))\,dA(a) \tag{3.1}$$

for all $z \in \mathbb{D}$.

Proof. Suppose $\widetilde{S}(z) = \int_{\mathbb{D}} \widetilde{S}(\varphi_a(z))dA(a)$ for all $z \in \mathbb{D}$.

Then by Lemma 2.1, there exists a constant α such that $|\alpha| = 1$ for all $z \in \mathbb{D}$,

$$\begin{aligned}\langle Sk_z, k_z \rangle &= \int_{\mathbb{D}} \langle Sk_{\varphi_a(z)}, k_{\varphi_a(z)} \rangle dA(a) = \int_{\mathbb{D}} \langle \alpha SU_a k_z, \alpha U_a k_z \rangle dA(a) \\ &= \int_{\mathbb{D}} \langle U_a S U_a k_z, k_z \rangle dA(a) = \left\langle \left(\int_{\mathbb{D}} U_a S U_a dA(a) \right) k_z, k_z \right\rangle = \langle \hat{S} k_z, k_z \rangle,\end{aligned} \tag{3.2}$$

where $\hat{S} = \int_{\mathbb{D}} U_a S U_a dA(a)$.

Thus by Lemma 2.3, $S = \hat{S}$. Hence for all $f, g \in A^2(\mathbb{D})$, $\langle Sf, g \rangle = \langle \hat{S} f, g \rangle$.

That is,

$$\int_{\mathbb{D}} \langle SU_a f, U_a g \rangle dA(a) = \int_{\mathbb{D}} Sf(z)\overline{g(z)}\,dA(z). \tag{3.3}$$

The boundedness of S and the antianalyticity of $K(z,a)$ in a imply that for each $z \in \mathbb{D}$, the function

$$S\left(\frac{f}{K(\cdot,a)}\right)(z)K(z,a) \tag{3.4}$$

is antianalytic in a. Therefore, by the mean value property of harmonic functions, we have

$$\int_{\mathbb{D}} S\left(\frac{f}{K(\cdot,a)}\right)(z)K(z,a)dA(a) = S\left(\frac{f}{K(\cdot,0)}\right)(z)K(z,0) = Sf(z). \tag{3.5}$$

Thus, from (3.5), it follows that

$$\langle Sf,g\rangle = \int_{\mathbb{D}} \overline{g(z)} \int_{\mathbb{D}} S\left(\frac{f}{K(\cdot,a)}\right)(z)K(z,a)dA(a)dA(z). \tag{3.6}$$

Using Fubini's theorem, we obtain

$$\langle Sf,g\rangle = \int_{\mathbb{D}} \int_{\mathbb{D}} S\left(\frac{f}{K(\cdot,a)}\right)(z)\overline{g(z)}K(z,a)dA(z)dA(a). \tag{3.7}$$

Now since $k_a(z) = K(z,a)/\sqrt{K(a,a)}$ and $(k_a \circ \varphi_a)(z)k_a(z) = 1$ for all $z,a \in \mathbb{D}$, the right-hand side of (3.7) is equal to

$$\begin{aligned}&\int_{\mathbb{D}} \int_{\mathbb{D}} S\left(\frac{f}{k_a}\right)(z)\overline{g(z)}k_a(z)dA(z)dA(a)\\ &\quad= \int_{\mathbb{D}} \int_{\mathbb{D}} S\left(\frac{f}{k_a}\right)(z)\overline{g(z)}\;\overline{k_a(\varphi_a(z))}\,|k_a(z)|^2 dA(z)dA(a).\end{aligned} \tag{3.8}$$

Finally, as $(\varphi_a \circ \varphi_a)(z) \equiv z$ and $\mathbf{J}_{\varphi_a(z)} = |k_a(z)|^2$, we obtain

$$\langle Sf,g\rangle = \int_{\mathbb{D}} \int_{\mathbb{D}} S\left(\frac{f}{k_a}\right)(\varphi_a(z))\overline{k_a(z)}\;\overline{g(\varphi_a(z))}dA(z)dA(a). \tag{3.9}$$

By hypothesis, $\langle Sf,g\rangle = \int_{\mathbb{D}} \langle SU_a f, U_a g\rangle dA(a)$ and by using Lemma 2.2,

$$\begin{aligned}\langle SU_a f, U_a g\rangle &= \left\langle S\left(\frac{f \circ \varphi_a}{k_a \circ \varphi_a}\right), (g \circ \varphi_a)k_a\right\rangle = \left\langle S\left(\frac{f}{k_a} \circ \varphi_a\right), (g \circ \varphi_a)k_a\right\rangle\\ &= \int_{\mathbb{D}} S\left(\frac{f}{k_a} \circ \varphi_a\right)(z)\overline{g(\varphi_a(z))}\;\overline{k_a(z)}dA(z).\end{aligned} \tag{3.10}$$

Thus we obtain for all $f,g \in A^2(\mathbb{D})$,

$$\int_{\mathbb{D}} S\left(\frac{f}{k_a} \circ \varphi_a\right)(z)\overline{g(\varphi_a(z))}\;\overline{k_a(z)}dA(z) = \int_{\mathbb{D}} S\left(\frac{f}{k_a}\right)(\varphi_a(z))\overline{k_a(z)}\;\overline{g(\varphi_a(z))}dA(z). \tag{3.11}$$

Hence for all $f,g \in A^2(\mathbb{D})$, $a \in \mathbb{D}$,

$$\left\langle S\left(\frac{f}{k_a} \circ \varphi_a\right), U_a g\right\rangle = \left\langle S\left(\frac{f}{k_a}\right) \circ \varphi_a, U_a g\right\rangle. \tag{3.12}$$

Since U_a is unitary, $U_a \in \mathcal{L}(A^2(\mathbb{D}))$, we get

$$S\left(\frac{f}{k_a} \circ \varphi_a\right) = S\left(\frac{f}{k_a}\right) \circ \varphi_a \tag{3.13}$$

for all $f \in A^2(\mathbb{D})$, $a \in \mathbb{D}$.

That is, for all $f \in A^2(\mathbb{D})$, $a \in \mathbb{D}$,

$$SC_a\left(\frac{f}{k_a}\right) = C_a S\left(\frac{f}{k_a}\right). \tag{3.14}$$

Since $k_a^{-1} \in H^\infty$, hence $SC_a = C_a S$ for all $a \in \mathbb{D}$. Thus $C_a S C_a = S$ for all $a \in \mathbb{D}$ as $C_a^2 = I$, the identity operator in $\mathcal{L}(A^2(\mathbb{D}))$.

Now we will prove the converse. Suppose $C_a S C_a = S$ for all $a \in \mathbb{D}$. Then $C_a S f = SC_a f$ for all $a \in \mathbb{D}$, $f \in A^2(\mathbb{D})$. That is, for all $f \in A^2(\mathbb{D})$, $a \in \mathbb{D}$,

$$(Sf) \circ \varphi_a = S(f \circ \varphi_a). \tag{3.15}$$

By Lemma 2.2, $(k_a \circ \varphi_a) k_a = 1$ for all $a \in \mathbb{D}$. Hence

$$SU_a f = S(k_a(f \circ \varphi_a)) = S\left(\frac{f \circ \varphi_a}{k_a \circ \varphi_a}\right) = S\left(\left(\frac{f}{k_a}\right) \circ \varphi_a\right) = \left(S\frac{f}{k_a}\right) \circ \varphi_a. \tag{3.16}$$

Thus for $f,g \in A^2(\mathbb{D})$, since $\overline{k_a(\varphi_a(z))}\,\overline{k_a(z)} = 1$, $\mathbf{J}_{\varphi_a(z)} = |k_a(z)|^2$, and $k_a(z) = K(z,a)/\sqrt{K(a,a)}$ for all $z,a \in \mathbb{D}$, we obtain

$$\begin{aligned}
\langle SU_a f, U_a g\rangle &= \int_{\mathbb{D}} \left(S\frac{f}{k_a}\right)(\varphi_a(z))\overline{(g \circ \varphi_a)(z)}\,\overline{k_a(z)}\,dA(z) \\
&= \int_{\mathbb{D}} S\left(\frac{f}{k_a}\right)(z)\overline{g(z)}\,\overline{(k_a \circ \varphi_a)(z)}\,|k_a(z)|^2 dA(z) \\
&= \int_{\mathbb{D}} S\left(\frac{f}{k_a}\right)(z)\overline{g(z)}k_a(z)\,dA(z) \\
&= \int_{\mathbb{D}} S\left(\frac{f}{K(\cdot,a)}\right)(z)\overline{g(z)}K(z,a)\,dA(z).
\end{aligned} \tag{3.17}$$

Hence by using Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{D}} \langle SU_a f, U_a g\rangle\, dA(a) &= \int_{\mathbb{D}}\int_{\mathbb{D}} S\left(\frac{f}{K(\cdot,a)}\right)(z)\overline{g(z)}K(z,a)\,dA(z)\,dA(a) \\
&= \int_{\mathbb{D}} \overline{g(z)}\,dA(z)\int_{\mathbb{D}} S\left(\frac{f}{K(\cdot,a)}\right)(z)K(z,a)\,dA(a).
\end{aligned} \tag{3.18}$$

We have already checked in the first part of the proof that for all $z \in \mathbb{D}$,

$$\int_{\mathbb{D}} S\left(\frac{f}{K(\cdot,a)}\right)(z)K(z,a)\,dA(a) = S\left(\frac{f}{K(\cdot,0)}\right)(z)K(z,0) = Sf(z). \tag{3.19}$$

Thus

$$\int_{\mathbb{D}} \langle SU_a f, U_a g\rangle\,dA(a) = \int_{\mathbb{D}} Sf(z)\overline{g(z)}\,dA(z) = \langle Sf,g\rangle. \tag{3.20}$$

When $f = g = k_z$, $z \in \mathbb{D}$, we obtain by Lemma 2.1 that

$$\langle Sk_z,k_z\rangle = \int_{\mathbb{D}} \langle SU_a k_z, U_a k_z\rangle\,dA(a) = \int_{\mathbb{D}} \langle Sk_{\varphi_a(z)},k_{\varphi_a(z)}\rangle\,dA(a) = \int_{\mathbb{D}} \widetilde{S}(\varphi_a(z))\,dA(a), \tag{3.21}$$

and this concludes the proof. □

Let P be the orthogonal projection from L^2 onto $A^2(\mathbb{D})$. For $\varphi \in L^\infty(\mathbb{D},dA)$, define the Toeplitz operator T_φ from $A^2(\mathbb{D})$ into itself as $T_\varphi f = P(\varphi f)$. For $\varphi \in L^\infty(\mathbb{D},dA)$, let

$$\begin{aligned} \widehat{\varphi}(z) &= \int_{\mathbb{D}} \varphi(\varphi_a(z))\,dA(a), \\ \widetilde{\varphi}(z) &= \int_{\mathbb{D}} \varphi(\varphi_z(\omega))\,dA(\omega). \end{aligned} \tag{3.22}$$

Notice that

$$\widetilde{\varphi}(z) = \langle \varphi k_z, k_z\rangle. \tag{3.23}$$

Corollary 3.2. *If $\varphi \in L^\infty(\mathbb{D},dA)$, then there exists a constant c of modulus 1 such that*

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \varphi(\varphi_{\varphi_a(z)}(\omega))\,dA(\omega)\,dA(a) = \int_{\mathbb{D}}\int_{\mathbb{D}} \varphi(c\varphi_{\varphi_z(a)}(\omega))\,dA(a)\,dA(\omega). \tag{3.24}$$

Proof. From (3.23), it follows that

$$\begin{aligned} \int_{\mathbb{D}} \widetilde{T}_\varphi(\varphi_a(z))\,dA(a) &= \int_{\mathbb{D}} \langle T_\varphi k_{\varphi_a(z)},k_{\varphi_a(z)}\rangle\,dA(a) = \int_{\mathbb{D}} \langle \varphi k_{\varphi_a(z)},k_{\varphi_a(z)}\rangle\,dA(a) \\ &= \int_{\mathbb{D}} \widetilde{\varphi}(\varphi_a(z))\,dA(a) = \int_{\mathbb{D}}\int_{\mathbb{D}} \varphi(\varphi_{\varphi_a(z)}(\omega))\,dA(\omega)\,dA(a). \end{aligned} \tag{3.25}$$

Given $f, g \in A^2(\mathbb{D})$, by Lemma 2.2 and Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} \langle U_a T_\varphi U_a f, g\rangle dA(a) \\
&\quad = \int_{\mathbb{D}} dA(a) \int_{\mathbb{D}} \varphi(z)(f \circ \varphi_a)(z) k_a(z) \overline{(g \circ \varphi_a)(z)}\, \overline{k_a(z)} dA(a) \\
&\quad = \int_{\mathbb{D}} dA(a) \int_{\mathbb{D}} \varphi(\varphi_a(\omega)) f(\omega)\overline{g(\omega)} \left| (k_a \circ \varphi_a)(\omega) \right|^2 \left| k_a(\omega) \right|^2 dA(\omega) \\
&\quad = \int_{\mathbb{D}} dA(a) \int_{\mathbb{D}} \varphi(\varphi_a(\omega)) f(\omega)\overline{g(\omega)} dA(\omega) \\
&\quad = \int_{\mathbb{D}} f(\omega)\overline{g(\omega)} dA(\omega) \int_{\mathbb{D}} \varphi(\varphi_a(\omega)) dA(a) \\
&\quad = \int_{\mathbb{D}} \widehat{\varphi}(\omega) f(\omega)\overline{g(\omega)} dA(\omega).
\end{aligned} \tag{3.26}$$

Thus

$$\begin{aligned}
\int_{\mathbb{D}} \widetilde{T}_\varphi(\varphi_a(z)) dA(a) &= \int_{\mathbb{D}} \langle U_a T_\varphi U_a k_z, k_z\rangle dA(a) = \int_{\mathbb{D}} \widehat{\varphi}(\omega) \left| k_z(\omega) \right|^2 dA(\omega) \\
&= \int_{\mathbb{D}} \widehat{\varphi}(\varphi_z(\omega)) dA(\omega) = \int_{\mathbb{D}} \int_{\mathbb{D}} (\varphi \circ \varphi_a \circ \varphi_z)(\omega) dA(a) dA(\omega).
\end{aligned} \tag{3.27}$$

Thus by Theorem 3.1, we obtain

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \varphi(\varphi_{\varphi_a(z)}(\omega)) dA(\omega) dA(a) = \int_{\mathbb{D}} \int_{\mathbb{D}} \varphi(\varphi_a \circ \varphi_z)(\omega) dA(a) dA(\omega). \tag{3.28}$$

Let

$$\mathbf{U} = \varphi_a \circ \varphi_z \circ \varphi_{\varphi_z(a)}. \tag{3.29}$$

Then $\mathbf{U} \in \operatorname{Aut}(\mathbb{D})$ and $\mathbf{U}(0) = \varphi_a \circ \varphi_z(\varphi_z(a)) = \varphi_a(a) = 0$ and $\mathbf{U}\varphi_{\varphi_z(a)} = \varphi_a \circ \varphi_z$.

We know that (see [9]) if $\varphi \in \operatorname{Aut}(\mathbb{D})$, then

$$\varphi(z) = e^{i\theta} \frac{z - b}{1 - \bar{b} z} \tag{3.30}$$

for some θ real and $b \in \mathbb{D}$. Furthermore, $\varphi(0) = 0$ if and only if $\varphi(z) = e^{i\theta} z$. Thus $\mathbf{U}z = e^{i\theta} z$. Hence

$$\varphi_a \circ \varphi_z = \mathbf{U}\varphi_{\varphi_z(a)} = e^{i\theta} \varphi_{\varphi_z(a)} = c\varphi_{\varphi_z(a)}, \tag{3.31}$$

where $c = e^{i\theta}$, $\theta \in \mathbb{R}$. Thus it follows that

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \varphi(\varphi_{\varphi_a(z)}(\omega)) dA(\omega) dA(a) = \int_{\mathbb{D}} \int_{\mathbb{D}} \varphi(c\varphi_{\varphi_z(a)}(\omega)) dA(a) dA(\omega). \tag{3.32}$$

□

Notice that we can define U_a on $L^2(\mathbb{D},dA)$ also. Suppose $\varphi \in L^\infty(\mathbb{D},dA)$, $f,g \in L^2(\mathbb{D},dA)$. Then by using Fubini's theorem and making a change of variable, we obtain

$$\begin{aligned}\int_{\mathbb{D}} \langle \varphi U_a f, U_a g\rangle dA(a) &= \int_{\mathbb{D}} dA(a) \int_{\mathbb{D}} \varphi(z)(f\circ\varphi_a)(z)k_a(z)\overline{(g\circ\varphi_a)(z)}\,\overline{k_a(z)}dA(z)\\ &= \int_{\mathbb{D}} dA(a)\int_{\mathbb{D}} \varphi(\varphi_a(\omega))f(\omega)\overline{g(\omega)}dA(\omega)\\ &= \int_{\mathbb{D}} f(\omega)\overline{g(\omega)}dA(\omega)\int_{\mathbb{D}}\varphi(\varphi_a(\omega))dA(a)\\ &= \int_{\mathbb{D}} \widehat{\varphi}(\omega)f(\omega)\overline{g(\omega)}dA(\omega) = \langle \widehat{\varphi} f, g\rangle.\end{aligned} \tag{3.33}$$

Define $J : L^2 \to L^2$ as $Jf(z) = f(\overline{z})$. The map J is a unitary operator and $J^* = J$. Let $\overline{A^2(\mathbb{D})} = \{\overline{f} : f \in A^2(\mathbb{D})\}$. Define $h_\varphi : A^2(\mathbb{D}) \to \overline{A^2(\mathbb{D})}$ such that $h_\varphi f = \overline{P}(\varphi f)$, where $\overline{P}$ is the orthogonal projection from L^2 onto $\overline{A^2(\mathbb{D})}$. The operator h_φ is called the little Hankel operator on $A^2(\mathbb{D})$. The following holds.

Corollary 3.3. *If $\varphi \in L^\infty(\mathbb{D})$, $f \in A^2(\mathbb{D})$, $g \in \overline{A^2(\mathbb{D})}$, then*

$$\int_{\mathbb{D}} \langle U_a h_\varphi U_a f, g\rangle dA(a) = \langle h_{\widehat{\varphi}} f, g\rangle. \tag{3.34}$$

Proof. From the above discussion it follows that for $f \in A^2(\mathbb{D})$, $g \in \overline{A^2(\mathbb{D})}$,

$$\int_{\mathbb{D}} \langle \varphi U_a f, U_a g\rangle dA(a) = \langle \widehat{\varphi} f, g\rangle. \tag{3.35}$$

That is,

$$\int_{\mathbb{D}} \langle \varphi U_a P f, U_a \overline{P} g\rangle dA(a) = \langle \widehat{\varphi} P f, \overline{P} g\rangle. \tag{3.36}$$

Notice that $\overline{P} = JPJ$. Therefore,

$$\int_{\mathbb{D}} \langle \varphi U_a P f, U_a JPJ g\rangle dA(a) = \langle \widehat{\varphi} P f, JPJ g\rangle. \tag{3.37}$$

Since $U_a P = P U_a$, we obtain

$$\int_{\mathbb{D}} \langle U_a JPJ\varphi P U_a f, g\rangle dA(a) = \int_{\mathbb{D}} \langle \varphi U_a P f, JPJ U_a g\rangle dA(a) = \langle JPJ\widehat{\varphi} P f, g\rangle. \tag{3.38}$$

Thus

$$\int_{\mathbb{D}} \langle U_a h_\varphi U_a f, g\rangle dA(a) = \langle h_{\widehat{\varphi}} f, g\rangle. \tag{3.39}$$

□

Corollary 3.4. *If $\varphi \in L^\infty(\mathbb{D}, dA)$, then there exists a constant C such that*

$$\frac{1}{3C}\|h_{\hat{\varphi}}\| \le \operatorname{Sup}_{z\in\mathbb{D}} \left| \int_{\mathbb{D}} \langle U_a h_\varphi U_a k_z, \overline{k_z}\rangle \, dA(a) \right| \le \frac{C}{3}\|h_{\hat{\varphi}}\|. \tag{3.40}$$

Proof. From a result in [3], it follows that there exists a constant C such that

$$\frac{1}{3C}\|h_{\hat{\varphi}}\| \le \operatorname{Sup}_{z\in\mathbb{D}} \left| \langle h_{\hat{\varphi}} k_z, \overline{k_z}\rangle \right| \le \frac{C}{3}\|h_{\hat{\varphi}}\|. \tag{3.41}$$

By Corollary 3.3,

$$\operatorname{Sup}_{z\in\mathbb{D}} \left| \langle h_{\hat{\varphi}} k_z, \overline{k_z}\rangle \right| = \operatorname{Sup}_{z\in\mathbb{D}} \left| \int_{\mathbb{D}} \langle U_a h_\varphi U_a k_z, \overline{k_z}\rangle \, dA(a) \right|, \tag{3.42}$$

and the result follows. □

Let $h^\infty(\mathbb{D})$ be the space of bounded harmonic functions on $\mathbb{D}$. Then $h^\infty(\mathbb{D}) \subset L^\infty(\mathbb{D})$. It is well known (see [9]) that every harmonic function on $\mathbb{D}$ is the sum of an analytic function and the conjugate of another analytic function. Hence if $f \in h^\infty(\mathbb{D})$, then $f(z) = \sum_{n=0}^\infty a_n z^n + \sum_{n=0}^\infty b_n \overline{z}^n$. It is not so difficult to verify (see [3]) that in this case $\hat{f}(z) = a_0 - (a_1/2)\, z - (b_1/2)\, \overline{z}$. Thus if $f \in h^\infty(\mathbb{D})$, then $\hat{f} = f$ if and only if f is a constant.

Corollary 3.5. *If $f \in h^\infty(\mathbb{D})$, then $C_a T_f = T_f C_a$ for all $a \in \mathbb{D}$ if and only if $\hat{f} = f$. That is, if and only if f is a constant.*

Proof. If $f \in h^\infty(\mathbb{D})$, then T_f is bounded linear operator on $A^2(\mathbb{D})$. Further, for $z \in \mathbb{D}$,

$$\widetilde{T}_f(z) = \langle T_f k_z, k_z\rangle = \langle f k_z, k_z\rangle = \widetilde{f}(z) = f(z). \tag{3.43}$$

This is so because by the invariant mean value property [3], $f \in h^\infty(\mathbb{D})$ implies $\widetilde{f} = f$. Hence

$$\widetilde{T}_f(\varphi_a(z)) = f(\varphi_a(z)) \tag{3.44}$$

for all $a, z \in \mathbb{D}$. By Theorem 3.1, $C_a T_f = T_f C_a$ for all $a \in \mathbb{D}$ if and only if for all $z \in \mathbb{D}$,

$$\int_{\mathbb{D}} \widetilde{T}_f(\varphi_a(z))\, dA(a) = \widetilde{T}_f(z). \tag{3.45}$$

That is, if and only if for all $z \in \mathbb{D}$,

$$\int_{\mathbb{D}} f(\varphi_a(z))\, dA(a) = f(z). \tag{3.46}$$

Hence $C_a T_f = T_f C_a$ for all $a \in \mathbb{D}$ if and only if $\hat{f} = f$. That is, if and only if f is a constant. □

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Namita Das: P. G. Department of Mathematics, Utkal University, Vani Vihar, Bhubaneswar, Orissa 751004, India
Email address: namitadas440@yahoo.co.in

R. P. Lal: Institute of Mathematics and Applications, 2nd Floor, Surya Kiran Building, Sahid Nagar, Bhubaneswar, Orissa 751007, India
Email address: rlal77@yahoo.com

C. K. Mohapatra: Institute of Mathematics and Applications, 2nd Floor, Surya Kiran Building, Sahid Nagar, Bhubaneswar, Orissa 751007, India
Email address: compact100@yahoo.com