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Research Article

Relationship of Algebraic Theories to Powerset Theories and Fuzzy Topological Theories for Lattice-Valued Mathematics

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Recommended by Robert Lowen

This paper deals with a broad question—to what extent is topology algebraic—using two specific questions: (1) what are the algebraic conditions on the underlying membership lattices which insure that categories for topology and fuzzy topology are indeed topological categories; and (2) what are the algebraic conditions which insure that algebraic theories in the sense of Manes are a foundation for the powerset theories generating topological categories for topology and fuzzy topology? This paper answers the first question by generalizing the Höhle-Šostak foundations for fixed-basis lattice-valued topology and the Rodabaugh foundations for variable-basis lattice-valued topology using semi-quantales; and it answers the second question by giving necessary and sufficient conditions under which certain theories—the very ones generating powerset theories generating (fuzzy) topological theories in the sense of this paper—are algebraic theories, and these conditions use unital quantales. The algebraic conditions answering the second question are much stronger than those answering the first question. The syntactic benefits of having an algebraic theory as a foundation for the powerset theory underlying a (fuzzy) topological theory are explored; the relationship between these two specific questions is discussed; the role of pseudo-adjoints is identified in variable-basis powerset theories which are algebraically generated; the relationships between topological theories in the sense of Adámek-Herrlich-Strecker and topological theories in the sense of this paper are fully resolved; lower-image operators introduced for fixed-basis mathematics are completely described in terms of standard image operators; certain algebraic theories are given which determine powerset theories determining a new class of variable-basis categories for topology and fuzzy topology using new preimage operators; and the theories of this paper are undergirded throughout by several extensive inventories of examples.

1. Introduction and preliminaries

1.1. Motivation. It is a fascinating question to what extent one area of mathematics is needed in another area of mathematics, a question fascinating in part because of the twin facts of mathematics' almost overwhelming diversity and mathematics' holistic coherence. In particular, it is of interest to formulate and understand the extent to which algebra is needed for topology or the extent to which topology is algebraic.

The first approach to the question of the extent to which topology needs algebra or is algebraic uses fuzzy sets—lattice-valued subsets—and may be formulated as follows: what are the minimal lattice-theoretic conditions on the base lattices, in which these subsets take their values, which guarantee that the resulting categories for doing lattice-valued topology are in fact topological over their respective ground categories? This question was studied for both fixed-basis and (the more general) variable-basis topologies in [1, 2] and is generalized in this paper using the notion of "topological powerset theory". The current answer is as follows: if the underlying lattices $(L, \leq, \otimes)$ are *semi-quantales* (s-quantales)—meaning that $(L, \leq)$ is a complete lattice and $\otimes : L \times L \to L$ is simply a binary operation on L, with morphisms preserving arbitrary $\bigvee$ and binary $\otimes$ (and the unit e if we restrict ourselves to *unital s-quantales* (us-quantales))—then all the concrete categories for topology considered or constructed in [1, 2] and this paper, and these include many well-known categories, are topological over their ground categories with respect to the underlying forgetful functors. Meta-mathematically restated, the conditions of semi-quantales guarantee that one is doing topology when working in these categories. Analogously, the answer in the case of "fuzzy topological theories" being topological is similar and uses (u)s-quantales which are ordered.

According to this first approach, the amount of algebraic information needed for doing topology is small, namely a binary operation $\otimes$ (with a unit in the us-quantale case)—no associativity, no commutivity, no zero, and so forth, is assumed of this binary operation. (We regard arbitrary joins and arbitrary meets to be primarily order-theoretic and limit-theoretic rather than algebraic since they involve infinitary operations and since **PreSet**, **PoSet**, **CLat** are nonalgebraic constructs—indeed **PreSet** is a topological construct and **PoSet** is a monotopological construct.)

A second approach to the question (of the extent to which topology needs algebra or is algebraic) also uses fuzzy sets to formulate a crucial bridge between algebra and topology, namely *powersets and powerset operators and powerset theories*. This second approach may be stated "in reverse" as follows: as shown in Section 3.4, to obtain a "topological theory" or topological category **A** (over a ground category with respect to a specified functor) means that **A** has complete fibres and that the preimage operator lifting ground morphisms to continuous **A** morphisms is a contravariant functor; the syntax of this preimage operator is part of an underlying "topological" powerset theory which sometimes has an image operator generating the preimage operator via the adjoint functor theorem; and this underlying powerset theory, or at least the image operator—viewed as the lifting of a ground morphism to a morphism between powersets over ground objects—could be possibly generated from an algebraic theory (or monad) in clone form [3]. The second approach to this question may be summed up thusly: to what extent or under what conditions do algebraic theories (in clone form) account for powerset theories constructing (fuzzy) topological theories occurring in traditional and lattice-valued mathematics? We note that relationships between powerset theories and algebraic theories are considered briefly in [3] via a few examples, and these relationships are stated in greater detail (without proofs) in [4, 5].

It is the joint-purpose of this paper to extend from [1, 2] the first approach using s-quantales and us-quantales while thoroughly resolving all aspects of the second approach. More specifically:

(1) This paper reformulates in Section 1.5 current axioms of topology and fuzzy topology by expressing the notion of the whole space being open by the tensor unit of a us-quantale—this stems from a suggestion of Professor U. Höhle.

(2) This paper extends the first approach in Sections 3.2–3.4 by axiomatizing powerset theories, specifying which powerset theories are topological, axiomatizing topological and fuzzy topological theories induced by topological powerset theories, and showing that under very weak lattice-theoretic conditions, all such topological and fuzzy topological theories are topological categories over their ground categories with respect to forgetful functors—the fuzzy case needs the tensor to be isotone. It is an intriguing fact that topological theories are not inherently fixed-basis or variable-basis, while fuzzy topological theories are inherently variable-basis; see [1, 2] for the notions of "fixed-basis" and "variable-basis" as well as Sections 1.5 and 3.5 for convenient examples of fixed-basis and variable-basis powerset theories and categories for topology.

(3) This paper shows in Section 3.6 that the categories of spaces of topological theories *à la* [6] functorially embed into certain topological theories *à la* Definition 3.7 and in this sense are generalized by Definition 3.7.

(4) This paper axiomatizes in Section 4.2 what it means for an algebraic theory in a ground category to generate a powerset theory in that category.

(5) This paper gives necessary and sufficient conditions in Section 5.1 for theories in **Set** of standard construction (defined following Example 3.15) to be algebraic, theories which in Section 5.2 generate the fixed-basis topological powerset theories which induce fixed-basis topological categories for topology and fuzzy topology fuzzy topology (Sections 1.5, 3.5). This completely resolves the second approach for algebraic theories of standard construction in the fixed-basis case.

(6) This paper gives necessary and sufficient conditions in Section 6.1 for *left-adjoint* theories in ground categories of the form $\mathbf{Set} \times \mathbf{C}$ to be algebraic theories of standard construction, theories which in Section 6.2 generate topological powerset theories in $\mathbf{Set} \times \mathbf{C}$, which in turn generate well-known variable-basis categories for topology and fuzzy topology (Sections 1.5, 3.5). The term "left-adjoint" refers to the fact that the construction of left-adjoint theories uses pseudo-left-adjoints (Definition 1.12) of certain isotone maps. The topological powerset theories which these algebraic theories generate are also designated as *left-adjoint* as the topological categories produced from these powerset theories are. This completely resolves the second approach for algebraic theories of standard construction type in the variable-basis, left-adjoint case.

(7) This paper gives necessary and sufficient conditions in Section 6.1 for *right-adjoint* theories in ground categories of the form $\mathbf{Set} \times \mathbf{C}$ to be algebraic theories of standard construction, theories which in Section 6.3 generate new topological powerset theories

in **Set** $\times$ **C**, which in turn generate new categories for topology and fuzzy topology. These right-adjoint topological categories have the same objects as their left-adjoint siblings, but have fundamentally different morphisms (i.e., a fundamentally different continuity) based on new preimage operators based on the notion of a pseudo-right-adjoint. This completely resolves the second approach for algebraic theories of standard construction in this new variable-basis, right-adjoint case.

(8) To accomplish the goals of (1)–(7), it is necessary to develop subsidiary ideas of potential importance in their own right: *pseudo-adjoints* of mappings between complete lattices, critical to the characterizations of Section 6 and the new kind of topology developed in Section 6.3, are introduced in Definition 1.12 and studied in Remark 1.13 and Proposition 1.15, particularly their relationship to complete distributivity; the *lower-image operator* of ground morphisms, a concept long known in the folklore and crucial in some of the examples inventoried in Section 1.3, is explicitly defined in Definition 1.19 and its relationship to the standard image operator in the fixed-basis case is completely resolved in Remark 1.20 and Proposition 1.21; and several extensive inventories of examples are given, particularly in Sections 1.2, 1.3, 3.5, to undergird and justify this paper's several intertwining developments.

With regard to the two approaches described earlier, the algebraic and order-theoretic price of the second approach is much higher than that of the first approach. The price for the first approach is that L be an s-quantale, and there is no cost regarding morphisms. But the object cost for the second approach is that L additionally be a unital quantale (an associative s-quantale with unit in which $\otimes$ distributes from both sides across arbitrary joins)—such a condition is both necessary and sufficient for the topological L-powerset theory to be generated from an algebraic theory of standard construction; and in the variable-basis cases, there is also a significant morphism cost (see Sections 1.3 and 6.1) which is both necessary and sufficient. On the other hand, the second approach coherently generates topological powerset theories which induce topological categories, while the first approach makes use of topological powerset theories which have been individually created for each overlying topological category.

The proper role of the second approach appears to be the following: it generates the proper syntax for powerset theories, particularly of the image and preimage operators, *syntax which makes sense even when the conditions needed for an algebraic theory are not remotely satisfied*. For example, under quite restrictive conditions (which are necessary and sufficient), the left-adjoint topological powerset theories are generated from left-adjoint algebraic theories of standard construction, and the resulting preimage operator has indeed the very syntax—known twenty-five years ago—of the standard preimage operator needed to produce topological categories in variable-basis topology. The conditions needed to generate this syntax algebraically are far more restrictive than the conditions needed for this syntax to function in a topological powerset theory giving us topological categories. In fact, all that is needed for this preimage operator to work properly are the conditions of s-quantales and s-quantalic morphisms! In example after example, both fixed-basis and variable-basis, the second approach generates under very restrictive conditions the exact machinery already successfully used for many years by the first approach under much weaker conditions. Therefore, while the charge is true that topology done in the first approach is partly *ad hoc*, remarkably it is even more true in the left-adjoint case that this charge turns out to be irrelevant.

With the above paragraphs in mind, we may now summarize the main contributions of this paper: necessary and sufficient conditions—relatively restrictive—are given under which topological powerset theories, already in use in lattice-valued mathematics under much less restrictive conditions, can be given an algebraic foundation using algebraic theories in clone form [3] of standard construction. These algebraic theories *confirm the syntax of what was already in place*. But other algebraic theories *generate new topological powerset theories* and *new kinds of topology and continuity* whose potential opens new avenues of topological research and whose syntax functions successfully under conditions much weaker than needed for those algebraic theories—see Section 6.3.

This paper's limitations include the following, which may be regarded as open questions:

(1) Although the relationship between topological theories in the sense of [6] and those in the sense of this paper is fully resolved, this paper gives no relationship between algebraic theories and topological theories in the sense of [6].

(2) Powerset theories given an algebraic foundation in this paper do not include those which address powersets of fuzzy subsets and subspaces based on fuzzy subsets in lattice-valued topology. Such structures are addressed at length by the theories of [7–13]. An important question—in what way or to what degree can the powerset theories and approaches of [7–13] be given an algebraic foundation in clone form—is open and awaits future work.

(3) This paper axiomatizes the algebraic and categorical foundations for (some) topology from the standpoint of single powersets and open sets. Alternative algebraic and categorical foundations for topology from the standpoint of double powersets and "point" and "density" have been studied in [14]. It is an important question how to link these two foundational approaches.

1.2. Lattice-theoretic preliminaries and examples. This section gives types of lattices needed throughout this paper, particularly for algebraic, powerset, and (fuzzy) topological theories which are fixed-basis. An important difference compared to [4] is the generalization from complete quasi-monoidal lattices to semi-quantales and from strictly two-sided quantales to unital quantales. The categories associated with these types of lattices are needed for variable-basis algebraic, powerset, and (fuzzy) topological theories.

Throughout the following definition, the composition and identities are taken from **Set**.

Definition 1.1 (lattice structures and associated categories). (1) A *semi-quantale* $(L, \leq, \otimes)$, abbreviated as *s-quantale*, is a complete lattice $(L, \leq)$ equipped with a binary operation $\otimes : L \times L \to L$, with no additional assumptions, called a *tensor product*. **SQuant** comprises all semi-quantales together with mappings preserving $\otimes$ and arbitrary $\bigvee$.

(2) An *ordered semi-quantale* $(L, \leq, \otimes)$, abbreviated as *os-quantale*, is an s-quantale in which $\otimes$ is isotone in both variables. **OSQuant** is the full subcategory of **SQuant** of all os-quantales. The class of all os-quantales is precisely that (strict) subclass of po-groupoids (see [15, page 319]) in each of which there is closure under arbitrary $\bigvee$.

(3) A *unital semi-quantale* $(L, \leq, \otimes)$, abbreviated as *us-quantale*, is an s-quantale in which $\otimes$ has an identity element $e \in L$ called the *unit* [16]—units are necessarily unique. **USQuant** comprises all us-quantales together with all mappings preserving arbitrary $\bigvee$, $\otimes$, and e.

(4) A *complete quasi-monoidal lattice* $(L, \leq, \otimes)$, abbreviated as *CQML*, is an os-quantale having $\top$ idempotent. **CQML** comprises all cqmls together with mappings preserving arbitrary $\bigvee$, $\otimes$, and $\top$ [1, 2]. Note that **CQML** is a subcategory of **OSQuant**. If it is additionally stipulated that $\otimes = \wedge$ (binary), then such cqmls are *semiframes* and **SFrm** is the full subcategory of **CQML** of all semiframes.

(5) A *DeMorgan s-quantale* is an s-quantale which is equipped with an order-reversing involution. **DmSQuant** is the category of DeMorgan s-quantales and s-quantalic morphisms also preserving $'$. If it is additionally stipulated that $\otimes = \wedge$ (binary), then such DeMorgan s-quantales are *complete DeMorgan algebras* and **Dmrg** is the full subcategory of **DmSQuant** of all complete DeMorgan algebras.

(6) A *quantale* $(L, \leq, \otimes)$ is an s-quantale with $\otimes$ associative and distributing across arbitrary $\bigvee$ from both sides (implying that $\perp$ is a two-sided *zero*) [1, 16, 17]. **Quant** is the full subcategory of **OSQuant** of all quantales. If it is additionally stipulated that $\otimes = \wedge$ (binary), then such quantales are *frames* and **Frm** is the subcategory of **Quant** of all frames and quantalic morphisms preserving finite $\wedge$. That **Frm** is not full in **Quant** is demonstrated by Example 1.17(6)(a)(ii). It should be noted that Girard quantales are DeMorgan quantales [17] using (5) above.

(7) A *strictly two-sided semi-quantale* $(L, \leq, \otimes)$ is a us-quantale with $e = \top$ [1], abbreviated as *st-s-quantale*. **ST-SQuant** is the full subcategory of **USQuant** of all st-s-quantales. Note that **ST-SQuant** is also a full subcategory of **CQML** and that the notion of an st-s-quantale is the same as a *complete monoidal lattice*, abbreviated as *cml*.

(8) There are various combinations of the above terms. Of particular importance to subsequent sections are **unital ordered semi-quantales**—abbreviated as *uos-quantales* and forming the full subcategory **UOSQuant** of **USQuant** of all uos-quantales, **unital quantales** [1, 16, 17]—abbreviated as *u-quantales* and forming the full subcategory **UQuant** of **UOSQuant** of all u-quantales, and **strictly two-sided quantales**—abbreviated as *st-quantales* and forming the full subcategory **STQuant** of **UQuant** of all st-quantales. **Frm** is a full subcategory of **STQuant**. All of these combinations yield subclasses of po-monoids [15].

(9) The dual of any category formed using the above terms is additionally called *localic* and the term *localic* is appended to the attributes for the structures in question. Examples in subsequent sections include $\mathbf{SQuant}^{\mathrm{op}} \equiv$ **LoSQuant**, $\mathbf{USQuant}^{\mathrm{op}} \equiv$ **LoUSQuant**, $\mathbf{OSQuant}^{\mathrm{op}} \equiv$ **LoOSQuant**, $\mathbf{UOSQuant}^{\mathrm{op}} \equiv$ **LoUOSQuant**, $\mathbf{CQML}^{\mathrm{op}} \equiv$ **LoQML** (in this case, the objects are *localic quasi-monoidal lattices*), $\mathbf{Quant}^{\mathrm{op}} \equiv$ **LoQuant**, $\mathbf{UQuant}^{\mathrm{op}} \equiv$ **LoUQuant**. In the cases of **SFrm** and **Frm**, the dual categories are already known in the literature as **SLoc** and **Loc**, respectively.

(10) **CSLat**$(\bigvee)$ is the category of complete join-semilattices, together with all mappings preserving arbitrary $\bigvee$; and **CSLat**$(\bigwedge)$ is the category of complete meet-semi-lattices, together with all mappings preserving arbitrary $\bigwedge$. These two categories share the same objects.

(11) **CLat** is the category of all complete lattices and mappings preserving arbitrary $\bigvee$ and arbitrary $\bigwedge$; and **CBool** is the full subcategory of **CLat** of all complete Boolean algebras.

PROPOSITION 1.2. *Let $(L,\leq,\otimes)$ be an s-quantale. Then the following hold.*

(1) $(L,\leq^{\mathrm{op}},\otimes)$ *is an s-quantale. Further,* $(L,\leq,\otimes)$ *is an os-quantale [us-quantale, uos-quantale] if and only if* $(L,\leq^{\mathrm{op}},\otimes)$ *is an os-quantale [us-quantale, uos-quantale, resp.].*

(2) $(L,\leq,\otimes)$ *is a quantale [u-quantale, st-quantale] implies* $(L,\leq^{\mathrm{op}},\otimes)$ *is an associative os-quantale [uos-quantale, uos-quantale, resp.], and these implications do not reverse.*

Remark 1.3. **SQuant** provides the general lattice-theoretic framework within which we can give necessary and sufficient conditions for certain powerset theories to be generated from standard construction algebraic theories (Example 3.15) and sufficient conditions for certain powerset theories to generate (fuzzy) topological theories, in the sense of Section 3.3, which are topological categories; it is also the unifying lattice-theoretic framework for the following examples. The use of the appellative *semi* for s-quantales is analogous to its use for semiframes and semilocales in [2] and subsequent papers.

Examples 1.4. A wealth of examples justifying the above lattice-theoretic notions can be seen in [14, 1, 18, 19, 16, 20, 2, 21, 22] and references therein. We sample the following examples to illustrate the above notions. The author is grateful to Professor Höhle for bringing example (11) to his attention.

(1) Each semiframe, and hence each complete lattice, is a commutative st-s-quantale ($\otimes = \wedge$ (binary)). Such examples need not be quantales nor distributive.

(2) Both the five-point asymmetric diamond and the five-symmetric diamond are nondistributive DeMorgan algebras, and hence are DeMorgan st-s-quantales which are not quantales. Each nondistributive lattice has a subposet which is order-isomorphic to one of these diamonds.

(3) The complete DeMorgan algebra $\mathcal{K}(\mathbf{H}^n)$ comprising all closed linear subspaces of Hilbert space with $n \geq 2$, equipped with the inclusion order and orthocomplementation and with $\otimes = \cap$ (binary), is a nondistributive DeMorgan st-s-quantale which is not a quantale.

(4) Each frame is a commutative st-quantale.

(5) Each $(L,\leq,T)$ with $(L,\leq)$ a complete lattice (or semiframe) and $\otimes = T$ a t-norm on L is a commutative st-s-quantale. Such examples need not be quantales; see [19] for t-norms and co-t-norms.

(6) Each $(L,\leq,S)$ with $(L,\leq)$ a complete lattice (or semiframe) and $\otimes = S$ a co-t-norm on L is a commutative us-quantale (with unit $\perp$). Such examples need not be quantales.

(7) Each $([0,1],\leq,T)$ with $\leq$ the usual ordering on $[0,1]$ and $\otimes = T$ a left-continuous t-norm on $[0,1]$ is a commutative st-quantale. If T is not left-continuous, then $([0,1],\leq,T)$ is still a commutative st-s-quantale.

(8) $(\mathrm{Max}A,\leq,\&)$—where A is a unital C^*-algebra with unit 1, $\mathrm{Max}A$ is the set of all closed linear subspaces of A, the join is the closure of the algebraic sum, and $\otimes = \&$ is

the closure of the algebraic product—is a u-quantale which is not an st-quantale (since $A \neq \langle 1 \rangle$); see [16].

(9) $\mathcal{N}(G)$, the family of all normal subgroups of a group G equipped with the inclusion order and $\otimes$ given by $H \otimes K = HK$, is a commutative u-quantale which is not a quantale nor an st-s-quantale if $|G| \geq 2$ ($G \otimes \{e\} \neq \{e\}$, and the unit is $\{e\}$ but $\top = G$). It is well known in these examples that $H \otimes K$ reduces to $H \vee K$.

(10) $\mathcal{S}(S)$, the family of all subsemigroups of an abelian semigroup S without identity, equipped with the inclusion order and $\otimes$ given by $H \otimes K = HK$, is a nonunital os-quantale; and for many S (e.g., $(\mathbb{N}, +), ((0,1), \cdot)$, etc.), there exist H, K such that $H \otimes K$ is not $H \vee K$ (e.g., there exist H, K such that $H \not\subset HK$ and $K \not\subset HK$).

(11) $(S(L), \leq, \circ)$—where L is a complete lattice,

$$S(L) = \left\{ f : L \longrightarrow L \mid f \text{ preserves arbitrary } \bigvee \right\}, \tag{1.1}$$

$\leq$ is the pointwise order, and $\otimes = \circ$ is composition of functions—is a u-quantale which for $|L| > 2$ is not an st-quantale (since id_L is not the universal upper bound mapping

$$\mathbf{1}(b) = \begin{cases} \top, & b > \perp, \\ \perp, & b = \perp) \end{cases} \tag{1.2}$$

nor commutative.

The construction in Example 1.4(11) of a class of examples merits a brief categorical analysis of its degree of “naturality”, an analysis needing the following ideas and proposition.

Definition 1.5 (partial-left adjunction). Let $\mathscr{C}$, $\mathscr{D}$ be categories and $G : \mathscr{C} \leftarrow \mathscr{D}$ a functor. Then an object function $F : |\mathscr{C}| \to |\mathscr{D}|$ is a *partial-left adjoint* of G with F, G forming a *partial-left adjunction*, written as

$$F \dashv_l G, \tag{1.3}$$

if the *partial-lifting diagram* holds: for each $A \in |\mathscr{C}|$, there exists a *unit* $\eta_A : A \to G(F(A))$ in $\mathscr{C}$, for each $B \in |\mathscr{D}|$, for each $f \in \mathscr{C}(A, G(B))$, there exists $\overline{f} \in \mathscr{D}(F(A), B)$,

$$f = G(\overline{f}) \circ \eta_A. \tag{1.4}$$

Note that $\overline{f}$ is not assumed to be unique given f. It is well known that if $\overline{f}$ is uniquely determined from f, then each $g \in \mathscr{C}(A, B)$ uniquely determines $\overline{\eta_B \circ g} \in \mathscr{D}(F(A), F(B))$ such that choosing $F(g) = \overline{\eta_B \circ g}$ both makes F a functor and left-adjoint of G. Hence every adjunction yields a partial-left adjunction, but not conversely.

Construction 1.6. For each $L \in |\mathbf{CSLat}(\bigvee)|$, define $\eta_L : L \to S(L)$ by $\eta_L(a)(b) = a$ if $|L| = 1$ and by

$$\eta_L(a)(b) = \begin{cases} a, & b > \perp, \\ \perp, & b = \perp, \end{cases} \tag{1.5}$$

if $|L| \geq 2$. Further, define

$$
\begin{aligned}
S &: \left|\mathbf{CSLat}\left(\bigvee\right)\right| \longrightarrow |\mathbf{UQuant}| \quad \text{by} \quad (L,\leq) \longmapsto (S(L),\leq,\circ),\\
V &: \mathbf{CSLat}\left(\bigvee\right) \longleftarrow \mathbf{UQuant} \quad \text{by} \quad V(L)=L,\ V(f)=f.
\end{aligned}
\tag{1.6}
$$

Proposition 1.7. *The following hold concerning Example 1.4(11) and Construction 1.6:*

(1) *For each $L \in |\mathbf{CSLat}(\bigvee)|$, $\eta_L : L \to S(L)$ is well defined, injective, preserves arbitrary $\bigvee$ and arbitrary $\bigwedge$, and so is isotone; and hence η_L order embeds L into $S(L)$ in the sense that $(\eta_L)^{\rightarrow}(L)$ is a complete sublattice of $S(L)$ which is order-isomorphic to L via η_L.*

(2) *Redefining S to have codomain $|\mathbf{CSLat}(\bigvee)|$ and V to have domain $\mathbf{CSLat}(\bigvee)$, then $S \dashv_l V$ with unit η and the bar operator given by*

$$\overline{f}(g) = f(g(\top_L)) \tag{1.7}$$

for $f : L \to M$ in $\mathbf{CSLat}(\bigvee)$ and $g \in S(L)$.

(3) *The partial-left adjunction $S \dashv_l V$ of (2)—with η and the indicated bar operator—does not hold for each forgetful functor $V : \mathbf{CSLat}(\bigvee) \leftarrow \mathfrak{D}$ for which $\mathfrak{D}$ is a subcategory of $\mathbf{SQuant}$ containing $\mathbb{I}$ equipped with $\otimes$ any t-norm.*

(4) *The forgetful functor $V : \mathbf{CSLat}(\bigvee) \leftarrow \mathbf{UQuant}$ does not have a left-adjoint using S on objects, η as the unit, and the bar operator of (2).*

Proof. Ad(1) Well-definedness and the preservation of $\bigvee$ both partly depend on $\bigvee$ distributing across the pieces of a piecewise defined mapping; and the preservation of $\bigwedge$ partly depends on $\bigwedge$ distributing across the pieces of a piecewise defined function. Injectivity is immediate if $|L| = 1$ and straightforward if $|L| \geq 2$.

Ad(2) To check the partial-lifting diagram, we note that $\overline{f}$ preserves arbitrary $\bigvee$, a consequence of f preserving arbitrary $\bigvee$; so $\overline{f}$ is in $\mathbf{CSLat}(\bigvee)$. Clearly

$$\overline{f}(\eta_B(a)) = f(\eta_B(a)(\top)) = f(a), \tag{1.8}$$

so that $\overline{f} \circ \eta_B = f$.

Ad(3) Assume to the contrary that $S \dashv_l V$ holds with η and the bar operator of (2) and with forgetful functor $V : \mathbf{CSLat}(\bigvee) \leftarrow \mathfrak{D}$, where $\mathfrak{D}$ is a subcategory of $\mathbf{SQuant}$ containing $\mathbb{I}$ equipped with $\otimes$ any t-norm. Then for any $f : L \to M$ in $\mathbf{CSLat}(\bigvee)$, we have $\overline{f} \in \mathfrak{D}(S(L), M)$. Choose $L = M = \mathbb{I}$ with $\otimes = T$ a t-norm, $f = \mathrm{id}_L$, and $g_1, g_2 \in S(L)$ by $g_1(a) = a/2$, $g_2(a) = a^2/2$. Now $\overline{f} \in \mathfrak{D}(S(L), M)$ implies that $\overline{f}$ preserves tensor products, taking $\circ$ on $S(L)$ to T on M. The commutivity of T yields

$$
\begin{aligned}
\frac{1}{4} &= g_1(g_2(1)) = \overline{f}(g_1 \circ g_2) = T(\overline{f}(g_1), \overline{f}(g_2))\\
&= T(\overline{f}(g_2), \overline{f}(g_1)) = f(g_2 \circ g_1) = g_2(g_1(1)) = \frac{1}{8},
\end{aligned}
\tag{1.9}
$$

a contradiction.

Ad(4) This is a corollary of (3). □

1.3. Category-theoretic preliminaries, pseudo-adjoints, and examples. Variable-basis theories are more subtle from a category-theoretic point of view than fixed-basis theories. This section introduces the critical notion of pseudo-adjoints and uses this notion to outline categorical preliminaries needed for this paper's variable-basis theories and verifies through an extensive inventory of classes of examples that the morphism classes of certain categories, defined in terms of pseudo-adjoints and used in the variable-basis characterizations of algebraic theories in this paper, are nontrivial. Throughout this paper, the notation $\mathscr{C} \subset \mathscr{D}$ for $\mathscr{D}$ a category means that $\mathscr{C}$ is a category which is a subcategory of $\mathscr{D}$. Unless stated otherwise, $\mathbf{C} \subset \mathbf{LoSQuant} \equiv \mathbf{SQuant}^{\mathrm{op}}$ in this section.

Definition 1.8 (variable-basis ground categories). The product category $\mathbf{Set} \times \mathbf{C}$ comprises the following data:

(1) *Objects*: (X, L), with $X \in |\mathbf{Set}|$ and $L \in |\mathbf{C}|$.

(2) *Morphisms*: $(f, \phi) : (X, L) \to (Y, M)$, with $f : X \to Y$ in $\mathbf{Set}$ and $\phi : L \to M$ in $\mathbf{C}$, that is,

$$\phi^{\mathrm{op}} : L \longleftarrow M \tag{1.10}$$

is a concrete morphism in $\mathbf{C}^{\mathrm{op}} \subset \mathbf{SQuant}$.

(3) *Composition, identities*: these are taken component-wise from $\mathbf{Set}$ and $\mathbf{C}$.

Categories of the form $\mathbf{Set} \times \mathbf{C}$ play a fundamental role in topology as the ground categories of topological variable-basis categories such as $\mathbf{C}$-$\mathbf{Top}$ and $\mathbf{C}$-$\mathbf{FTop}$ in [23–26, 20, 2, 21, 27].

Definition 1.9. Let $f : L \to M$, $g : L \leftarrow M$ be isotone maps between presets. Then $f \dashv g$ provided that

$$\begin{aligned} &\forall a \in L, \quad a \le g(f(a)), \\ &\forall b \in M, \quad f(g(b)) \le b, \end{aligned} \tag{1.11}$$

or equivalently,

$$\forall a \in L, \quad b \in M, \quad a \le g(b) \Longleftrightarrow f(a) \le b. \tag{1.12}$$

If $f \dashv g$, then one writes $g = f^{\vdash}$ and $f = g^{\dashv}$ and calls $f^{\vdash}$ the *right-adjoint* of f and $g^{\dashv}$ the *left-adjoint* of g.

Theorem 1.10 (adjoint functor theorem (AFT) [18]). *Let $f : L \to M$ $[g : L \leftarrow M]$ be a mapping between posets such that L $[M]$ has arbitrary $\bigvee$ $[\bigwedge]$ and f $[g]$ preserves arbitrary $\bigvee$ $[\bigwedge$, resp.$]$. Then f $[g]$ is isotone, there exists a unique $f^{\vdash} : L \leftarrow M$ $[g^{\dashv} : L \to M]$, and $f^{\vdash}$ $[g^{\dashv}]$ preserves all $\bigwedge$ $[\bigvee]$ existing in M $[L]$, where $f^{\vdash} : L \leftarrow M$ $[g^{\dashv} : L \to M]$ is given by*

$$\begin{aligned} f^{\vdash}(\beta) &= \bigvee \{\alpha \in L : f(\alpha) \le \beta\} \\ [g^{\dashv}(\alpha) &= \bigwedge \{\beta \in M : \alpha \le g(\beta)\}]. \end{aligned} \tag{1.13}$$

Notation 1.11 [20, 21]. Given a mapping $f : X \to Y$ and any set Z, we may lift f to $\langle f \rangle : X^Z \to Y^Z$ by

$$\langle f \rangle (a) = f \circ a. \tag{1.14}$$

Note that if L, M are presets and $f : L \to M$, $g : L \leftarrow M$ are isotone maps such that

$$f \dashv g, \tag{1.15}$$

then for each set X, L^X, M^X are presets and $\langle f \rangle : L^X \to M^X$ and $\langle g \rangle : L^X \leftarrow M^X$ are isotone maps such that

$$\langle f \rangle \dashv \langle g \rangle. \tag{1.16}$$

Restated,

$$\langle f \rangle^{\vdash} = \langle f^{\vdash} \rangle, \qquad \langle g \rangle^{\dashv} = \langle g^{\dashv} \rangle. \tag{1.17}$$

Definition 1.12 (pseudo-adjoints and liftings). Let $f : L \leftarrow M$ be any function between two complete lattices. Define the *pseudo-left-adjoint* $f^{\dashv} : L \to M$ in **Set** and the *pseudo-right-adjoint* $f^{\vdash} : L \to M$ in **Set** by

$$f^{\dashv}(a) = \bigwedge_{a \le f(b)} b, \qquad f^{\vdash}(a) = \bigvee_{f(b) \le a} b. \tag{1.18}$$

In particular, for $\phi : L \to M$ in **LoSQuant**, put $\phi^{\dashv} \equiv (\phi^{\mathrm{op}})^{\dashv} : L \to M$ in **Set** and $\phi^{\vdash} \equiv (\phi^{\mathrm{op}})^{\vdash} : L \to M$ in **Set**, so that

$$\phi^{\dashv}(a) = \bigwedge_{a \le \phi^{\mathrm{op}}(b)} b, \qquad \phi^{\vdash}(a) = \bigvee_{\phi^{\mathrm{op}}(b) \le a} b. \tag{1.19}$$

There is also reason to use $\phi^{\dashv\dashv} \equiv (\phi^{\dashv})^{\dashv} : L \leftarrow M$ and $\phi^{\vdash\vdash} \equiv (\phi^{\vdash})^{\vdash} : L \leftarrow M$. Given a set X, then

$$\langle \phi^{\vdash} \rangle : L^X \longrightarrow M^X, \qquad \langle \phi^{\dashv} \rangle : L^X \longrightarrow M^X \tag{1.20}$$

are given by Notation 1.11 to be

$$\langle \phi^{\vdash} \rangle (a) = \phi^{\vdash} \circ a, \qquad \langle \phi^{\dashv} \rangle (a) = \phi^{\dashv} \circ a. \tag{1.21}$$

The same notation is used for adjoints (Definition 1.9) and pseudo-adjoints (Definition 1.12), with context identifying the distinction as needed. Of course, many pseudo-adjoints really are adjoints, one of several issues addressed in Remark 1.13 and Proposition 1.15. Pseudo-adjoints play a critical role in the characterizations of Section 6 and the new kind of topology developed in Section 6.3.

Remark 1.13. (1) If $f : L \leftarrow M$ is any function between two complete lattices, then both $f^{\dashv}$ and $f^{\vdash}$ are isotone, $f^{\dashv}$ preserves $\bot$, and $f^{\vdash}$ preserves $\top$.

(2) Let $\phi : L \to M$ in **LoSQuant**. Note that $\phi^{op} : L \leftarrow M$ in **SQuant**, so that ϕ^{op} preserves arbitrary $\bigvee$; and by AFT, $\phi^{\vdash}$ preserves arbitrary $\bigwedge$ and

$$\phi^{op} \dashv \phi^{\vdash}, \tag{1.22}$$

in which case the pseudo-right-adjoint is the right-adjoint. Also by (1), $\phi^{\dashv}$ is isotone and preserves $\perp$; and if ϕ^{op} preserves arbitrary $\bigwedge$, then by AFT, $\phi^{\dashv}$ preserves arbitrary $\bigvee$ and

$$\phi^{\dashv} \dashv \phi^{op}, \tag{1.23}$$

in which case the pseudo-left-adjoint is the left-adjoint.

(3) It is easy to verify, given $\phi : L \to M$ and $\psi : M \to N$ in **LoSQuant** and using the identity

$$(g \circ f)^{\vdash} = f^{\vdash} \circ g^{\vdash} \tag{1.24}$$

for right-adjoints of arbitrary $\bigvee$ preserving, concrete maps, that

$$(\psi \circ \phi)^{\vdash} = \psi^{\vdash} \circ \phi^{\vdash}. \tag{1.25}$$

(4) The condition

$$(\psi \circ \phi)^{\dashv} = \psi^{\dashv} \circ \phi^{\dashv} \tag{1.26}$$

need not always hold for $\phi : L \to M$ and $\psi : M \to N$ in **LoSQuant**, is rather delicate, and is dealt with in Proposition 1.15.

Definition 1.14 (data for adjunction categories). Let $\mathbf{C}(\vdash)[\mathbf{C}(\dashv), \mathbf{C}(\vdash\vdash), \mathbf{C}^*]$ denote the following data:

(AC1) *Objects*: objects in $\mathbf{C}$.

(AC2) *Morphisms*: given two objects L, M, $\phi : L \to M$ is a morphism in $\mathbf{C}(\vdash)$ $[\mathbf{C}(\dashv)$, $\mathbf{C}(\vdash\vdash), \mathbf{C}^*]$ if the following hold:

(a) $\phi : L \to M$ is a morphism in $\mathbf{C}$;

(b) $\phi^{\vdash} : L \to M$ $[\phi^{\dashv} : L \to M$, $\phi^{\vdash\vdash} : L \leftarrow M$, there exists $\phi^* : L \to M$ which$]$ is a morphism in $\mathbf{C}^{op}$.

(AC3) *Composition, identities*: composition and identities of $\mathbf{C}$.

Morphisms in $\mathbf{C}(\vdash)$ $[\mathbf{C}(\dashv)]$ have the unusual property that their duals have right-adjoints [pseudo-left-adjoints] which preserve arbitrary $\bigvee$ *and* binary $\otimes$. Examples are given for such morphisms in Example 1.17; and it should be noted that these conditions are unavoidable in Section 6 for variable-basis powerset theories to be generated by an algebraic theory of standard construction. It should also be noted that Definition 1.14 simply defines data for $\mathbf{C}(\vdash)$, $\mathbf{C}(\dashv)$, $\mathbf{C}(\vdash\vdash)$, $\mathbf{C}^*$ without any claim of whether any is a category. The next proposition addresses these issues, gives sufficient conditions for $\mathbf{C}(\dashv)$ to be a category, and applies in its proof examples suggested by Professor Höhle for which the author is grateful.

Proposition 1.15. *The following hold:*

(1) $\mathbf{C}(\vdash)$, $\mathbf{C}(\vdash\vdash)$, $\mathbf{C}^* \subset$ **LoSQuant**.

(2) *Let* $\mathbf{C}_{\dashv}(\dashv)$ *be that data with* $|\mathbf{C}_{\dashv}(\dashv)| = \mathbf{C}$ *and morphisms of the form* $\phi^{\dashv}$ *for* ϕ *in* $\mathbf{C}(\dashv)$, *with composition and identities from* $\mathbf{C}^{\mathrm{op}}$. *The following hold:*

(a) $\mathbf{C}_{\dashv}(\dashv) \subset$ **SQuant** *if and only if for each* $\phi : L \to M$, $\psi : M \to N$ *in* $\mathbf{C}(\dashv)$,

$$(\psi \circ \phi)^{\dashv} = \psi^{\dashv} \circ \phi^{\dashv}. \tag{1.27}$$

(b) *If* $\mathbf{C}_{\dashv}(\dashv) \subset$ **SQuant**, *then* $\mathbf{C}(\dashv) \subset$ **LoSQuant**.

(c) *If the morphisms of* $\mathbf{C}(\dashv)$ *have the additional property that their duals preserve arbitrary* $\bigwedge$, *then* $\mathbf{C}_{\dashv}(\dashv) \subset$ **SQuant**.

(d) *If each object in* $\mathbf{C}$ *is completely distributive as a lattice and has* $\otimes = \vee$ *(binary), then* $\mathbf{C}(\dashv) \subset$ **LoSQuant**.

(e) *None of the implications in* (b), (c) *reverses.*

The proof of Proposition 1.15 uses the following lemma, which stems from a remark of Professor Höhle.

Lemma 1.16. *Let L be a complete lattice, let M be a completely distributive lattice, and let $f : L \leftarrow M$ be a map which preserves arbitrary nonempty $\bigvee$ [$\bigwedge$]. Then the pseudo-left-adjoint [pseudo-right-adjoint] $f^{\dashv}$ [$f^{\vdash}$] $: L \to M$ preserves arbitrary $\bigvee$ [$\bigwedge$].*

Proof. We prove the join case and leave the meet case for the reader. By Remark 1.13(1), $f^{\dashv}$ preserves $\perp$ (the empty join); so we need only to check the preservation of nonempty joins. Let $\{a_\gamma : \gamma \in \Gamma\} \subset L$ with $\Gamma \neq \varnothing$. By Remark 1.13(1), $f^{\dashv}$ is isotone, so immediately we have

$$f^{\dashv}\left(\bigvee_{\gamma\in\Gamma} a_\gamma\right) \geq \bigvee_{\gamma\in\Gamma} f^{\dashv}(a_\gamma). \tag{1.28}$$

For the reverse inequality, for each $\gamma \in \Gamma$, set

$$B_\gamma = \{b \in M : a_\gamma \leq f(b)\}. \tag{1.29}$$

Invoking the definition of $f^{\dashv}$ and complete distributivity in M yields

$$\bigvee_{\gamma\in\Gamma} f^{\dashv}(a_\gamma) = \bigvee_{\gamma\in\Gamma}\left(\bigwedge_{b\in B_g} b\right) = \bigwedge_{x\in\Pi_{\gamma\in\Gamma}B_\gamma}\left(\bigvee_{\gamma\in\Gamma} x(\gamma)\right). \tag{1.30}$$

Now for each $x \in \Pi_{\gamma\in\Gamma}B_\gamma$ and each $\gamma \in \Gamma$, we have, using the preservation of arbitrary nonempty $\bigvee$ by f, that

$$a_\gamma \leq f(x(\gamma)), \quad \bigvee_{\gamma\in\Gamma} a_\gamma \leq \bigvee_{\gamma\in\Gamma} f(x(\gamma)) = f\left(\bigvee_{\gamma\in\Gamma} x(\gamma)\right), \tag{1.31}$$

and hence that $\bigvee_{\gamma\in\Gamma} x(\gamma) \in B \equiv \{b \in M : \bigvee_{\gamma\in\Gamma} a_\gamma \leq f(b)\}$. It follows that

$$\begin{gathered}\left\{\bigvee_{\gamma\in\Gamma} x(\gamma) : x \in \Pi_{\gamma\in\Gamma} B_\gamma\right\} \subset B, \\ \bigwedge_{x\in\Pi_{\gamma\in\Gamma}B_\gamma}\left(\bigvee_{\gamma\in\Gamma} x(\gamma)\right) \geq \bigwedge B = f^{\dashv}\left(\bigvee_{\gamma\in\Gamma} a_\gamma\right).\end{gathered} \tag{1.32}$$

Hence

$$\bigvee_{\gamma\in\Gamma} f^{\dashv}(a_\gamma) \geq f^{\dashv}\left(\bigvee_{\gamma\in\Gamma} a_\gamma\right), \tag{1.33}$$

which completes the proof. □

Proof of Proposition 1.15. (1) and (2)(a)–(c) are immediate. Now (2)(d) is a consequence of Lemma 1.16 as follows: $\phi : L \to M$, $\psi : M \to N$ are in $\mathbf{C}(\dashv)$, then ϕ, ψ are in $\mathbf{C}$, so that $\psi \circ \phi \in \mathbf{C}$; and since

$$(\psi \circ \phi)^{\dashv} \equiv \left((\psi \circ \phi)^{\mathrm{op}}\right)^{\dashv} = \left(\phi^{\mathrm{op}} \circ \psi^{\mathrm{op}}\right)^{\dashv}, \tag{1.34}$$

then Lemma 1.16 says that $(\psi \circ \phi)^{\dashv} \in \mathbf{C}^{\mathrm{op}}$ (with $\otimes = \vee$), so that $\psi \circ \phi \in \mathbf{C}(\dashv)$.

To confirm (2)(e), let the following maps be defined from $\mathbb{I}$ to $\mathbb{I}$:

$$\begin{gathered}\phi^{\mathrm{op}}(x) = \begin{cases} \dfrac{3}{4}x + \dfrac{1}{4}, & x \in \left(\dfrac{1}{2}, 1\right], \\ x, & x \in \left[0, \dfrac{1}{2}\right], \end{cases} \\ \psi^{\mathrm{op}}(x) = \begin{cases} 1, & x \in \left(\dfrac{3}{4}, 1\right], \\ \dfrac{1}{2}, & x \in \left(0, \dfrac{3}{4}\right], \\ 0, & x = 0. \end{cases}\end{gathered} \tag{1.35}$$

Then the following hold:

(i) $(\psi \circ \phi)^{\dashv} \neq \psi^{\dashv} \circ \phi^{\dashv}$;

(ii) neither ϕ^{op} nor ψ^{op} preserves arbitrary $\bigwedge$.

To see that (2)(b) does not reverse, choose $\mathbf{C}$ in this way: $|\mathbf{C}| = \{\mathbb{I}\}$ with $\otimes = \vee$ (binary), $\mathrm{Morph}(\mathbf{C}) = \langle\{\mathrm{id}_{\mathbb{I}}, \phi, \psi\}\rangle$, meaning all **LoSQuant** morphisms obtained through composition from these three morphisms. Then $\mathbf{C}(\dashv) \subset$ **LoSQuant** by (2)(d), yet by (i) above, $\mathbf{C}_{\dashv}(\dashv)$ is not a subcategory of **SQuant** (and indeed is not a category); so (2)(b) does not reverse. Now to see that (2)(c) does not reverse, choose $\mathbf{C}$ in this way: $|\mathbf{C}| = \{\mathbb{I}\}$ with $\otimes = \vee$ (binary), $\mathrm{Morph}(\mathbf{C}) = \{\mathrm{id}_{\mathbb{I}}, \psi\}$. In this case, it is easy to show that

$$\psi \circ \psi = \psi, \qquad \psi^{\dashv} \circ \psi^{\dashv} = \psi^{\dashv}. \tag{1.36}$$

This observation shows that $\mathbf{C}_{\dashv}(\dashv) \subset \mathbf{SQuant}$, and hence by (2)(b) we have $\mathbf{C}(\dashv) \subset \mathbf{LoSQuant}$. But by (ii) above, the antecedent of (2)(c) does not hold; so (2)(c) does not reverse. □

Example 1.17. It is fundamental to Section 6 being meaningful (nonempty, nonredundant) that for many $\mathbf{C} \subset \mathbf{LoSQuant}$, $\mathbf{C}(\vdash)$ and $\mathbf{C}(\dashv)$ (and hence $\mathbf{C}^*$) and $\mathbf{C}(\vdash\vdash)$ are nontrivial with respect to morphisms; namely, there are nonidentity—even nonisomorphism—morphisms in each of these categories. In particular, the examples below include many classes of nontrivial ϕ in $\mathbf{LoUQuant}(L,M)$ such that $\phi^{\vdash}$ and/or $\phi^{\dashv}$ and/or $\phi^{\vdash\vdash}$ are in **UQuant**, the latter being (more than) sufficient for the new powerset theories of Section 6.3 to yield topological and fuzzy topological theories which are meaningful. Note that ((1)–(5), (10)) give classes of examples of nonisomorphic ϕ, ((5)–(10)) give examples in which $\otimes$ need not be order-theoretic (binary $\wedge$ or binary $\vee$) on L and M, and (10) gives an algorithm for generating from ((1)–(9)) many classes of examples of *nonisomorphism* ϕ in $\mathbf{C}(\vdash)$ and/or $\mathbf{C}(\dashv)$ and/or $\mathbf{C}(\vdash\vdash)$ with *non-order-theoretic* $\otimes$. Also note that in all examples, Proposition 1.15(2)(c)—and hence Proposition 1.15(2)(b)—applies.

(1) Let $L = \{\bot, a, \top\}$ and $M = \{\bot, \top\}$, let $\phi^{\mathrm{op}} : L \leftarrow M$ be given by

$$\phi^{\mathrm{op}}(\bot) = \bot, \qquad \phi^{\mathrm{op}}(\top) = \top. \tag{1.37}$$

Then each of ϕ^{op}, $\phi^{\vdash}$, $\phi^{\dashv}$ preserves each of $\bigvee$, $\bigwedge$, $\top$.

(a) Let L, M be equipped with $\otimes = \wedge$ (binary). Then both $\phi^{\vdash}, \phi^{\dashv} : L \to M$ are in **UQuant** and so $\phi \in \mathbf{LoUQuant}(\dashv) \cap \mathbf{LoUQuant}(\vdash)$.

(b) Let L, M be equipped with $\otimes = \vee$ (binary). Then both $\phi^{\vdash}, \phi^{\dashv} : L \to M$ are in **UOSQuant** and so $\phi \in \mathbf{LoUOSQuant}(\dashv) \cap \mathbf{LoUOSQuant}(\vdash)$.

(2) This example essentially shifts the previous example one pseudo-adjoint to the left. To be precise, let $L = \{\bot, \top\}$ and $M = \{\bot, a, \top\}$, let $\psi^{\mathrm{op}} : L \leftarrow M$ be given by

$$\psi^{\mathrm{op}}(\bot) = \bot, \qquad \psi^{\mathrm{op}}(a) = \top, \qquad \psi^{\mathrm{op}}(\top) = \top. \tag{1.38}$$

Note that ψ^{op} is the $\phi^{\dashv}$ of the previous example. Then it follows that each of ψ^{op}, $\psi^{\vdash}$, $\psi^{\vdash\vdash}$ preserves each of $\bigvee$, $\bigwedge$, $\top$.

(a) Let L, M be equipped with $\otimes = \wedge$ (binary). Then both $\psi^{\vdash} : L \to M$, $\psi^{\vdash\vdash} : L \leftarrow M$ are in **UQuant**, and so $\psi \in \mathbf{LoUQuant}(\vdash) \cap \mathbf{LoUQuant}(\vdash\vdash)$.

(b) Let L, M be equipped with $\otimes = \vee$ (binary). Then both $\psi^{\vdash} : L \to M$, $\psi^{\vdash\vdash} : L \leftarrow M$ are in **UOSQuant**, and so $\psi \in \mathbf{LoUOSQuant}(\vdash) \cap \mathbf{LoUOSQuant}(\vdash\vdash)$.

(3) Let $M = \{\bot, \alpha, \top\}$ and let $L = \{\bot, a, b, c, \top\}$ be the product topology of the Šierpinski topology with itself with the coordinate-wise ordering—$\bot$ is meet-irreducible (prime) and $\{a, b, c, \top\}$ is the four-point diamond with

$$\bot \le a \le b \le \top, \qquad \bot \le a \le c \le \top, \tag{1.39}$$

and b, c unrelated. Now let $\phi^{\mathrm{op}} : L \leftarrow M$ be given by

$$\phi^{\mathrm{op}}(\bot) = \bot, \qquad \phi^{\mathrm{op}}(\top) = \top, \qquad \phi^{\mathrm{op}}(\alpha) = c. \tag{1.40}$$

Then each of $\phi^{\mathrm{op}} : L \leftarrow M$, $\phi^{\dashv} : L \to M$ preserves each of $\bigvee$, $\bigwedge$, $\top$.

(a) Let L, M be equipped with $\otimes = \wedge$ (binary). Then $\phi^{\dashv} \in \mathbf{UQuant}$, and so $\phi \in \mathbf{LoUQuant}(\dashv)$.

(b) Let L, M be equipped with $\otimes = \vee$ (binary). Then $\phi^{\dashv} \in \mathbf{UOSQuant}$, and so $\phi \in \mathbf{LoUOSQuant}(\dashv)$.

Now shift our mappings one pseudo-adjoint to the left as in (2), namely, put

$$\psi^{\mathrm{op}} = \phi^{\dashv} : M \longleftarrow L. \tag{1.41}$$

Then each of $\psi^{\mathrm{op}} : L \leftarrow M$, $\psi^{\vdash} : L \to M$ preserves each of $\bigvee$, $\bigwedge$, $\top$.

(c) Let M, L be equipped with $\otimes = \wedge$ (binary). Then $\psi^{\vdash} \in \mathbf{UQuant}$, and so $\psi \in \mathbf{LoUQuant}(\vdash)$.

(d) Let M, L be equipped with $\otimes = \vee$ (binary). Then $\psi^{\vdash} \in \mathbf{UOSQuant}$, and so $\psi \in \mathbf{LoUOSQuant}(\vdash)$.

(4) Let M be as in the previous example, let L be L^{op} of the previous example, and let ϕ, ψ be formally defined as in the previous example. Then each of $\phi^{\mathrm{op}} : L \leftarrow M$, $\phi^{\vdash} : L \to M$, $\psi^{\mathrm{op}} : L \leftarrow M$, $\psi^{\dashv} : L \to M$ preserves each of $\bigvee$, $\bigwedge$, $\top$.

(a) Let L, M be equipped with $\otimes = \wedge$ (binary). Then $\phi^{\vdash} \in \mathbf{UQuant}$, and so $\phi \in \mathbf{LoUQuant}(\vdash)$.

(b) Let L, M be equipped with $\otimes = \vee$ (binary). Then $\phi^{\vdash} \in \mathbf{UOSQuant}$, and so $\phi \in \mathbf{LoUOSQuant}(\vdash)$.

(c) Let M, L be equipped with $\otimes = \wedge$ (binary). Then $\psi^{\dashv} \in \mathbf{UQuant}$, and so $\psi \in \mathbf{LoUQuant}(\dashv)$.

(d) Let M, L be equipped with $\otimes = \vee$ (binary). Then $\psi^{\dashv} \in \mathbf{UOSQuant}$, and so $\psi \in \mathbf{LoUOSQuant}(\dashv)$.

(5) Let $L = M = \mathbb{I} \equiv [0,1]$ and let $\phi^{\mathrm{op}} : L \leftarrow M$ be as follows: $[0,1/4]$ scales to $[0,1/2]$ with $0 \mapsto 0$ and $1/4 \mapsto 1/2$; $[1/4,3/4]$ maps to $\{1/2\}$; and $[3/4,1]$ scales to $[1/2,1]$ with $3/4 \mapsto 1/2$ and $1 \mapsto 1$. Since ϕ^{op} is isotone and continuous and preserves 0 and 1, it preserves arbitrary $\bigvee$ and arbitrary $\bigwedge$ (cf. [22, Lemma 3.6.2]). It should be noted that $\phi^{\dashv} : L \to M$ preserves arbitrary $\bigvee$ and finite $\wedge$, but not arbitrary $\bigwedge$, that $\phi^{\vdash} : L \to M$ preserves arbitrary $\bigwedge$ and finite $\vee$, but not arbitrary $\bigvee$, and that

$$\phi^{\dashv} \dashv \phi^{\mathrm{op}} \dashv \phi^{\vdash}, \qquad \phi^{\dashv\dashv} = \phi^{\mathrm{op}} = \phi^{\vdash\vdash}, \qquad \phi^{\dashv\dashv} \not\dashv \phi^{\dashv}, \qquad \phi^{\vdash} \not\dashv \phi^{\vdash\vdash}. \tag{1.42}$$

Further, if we shift to the left as above in (2) and choose $\psi^{\mathrm{op}} : M \leftarrow L$ by

$$\psi^{\mathrm{op}} = \phi^{\dashv}, \tag{1.43}$$

then we also have

$$\psi^{\mathrm{op}} \dashv \psi^{\vdash} \dashv \psi^{\vdash\vdash}. \tag{1.44}$$

We therefore have the following examples, in some of which occurs $\otimes = T_{\mathbf{D}}$ (the *drastic product t*-norm [19]).

(a) Let L, M be equipped with $\otimes = \wedge$ (binary). Then $\phi^{\dashv} \in \mathbf{UQuant}$, and so $\phi \in \mathbf{LoUQuant}(\dashv)$.

(b) Let L, M be equipped with $\otimes = \vee$ (binary). Then $\phi^{\dashv} \in \mathbf{UOSQuant}$, and so $\phi \in \mathbf{LoUOSQuant}(\dashv)$.

(c) Let L, M be equipped with $\otimes = T_{\mathbf{D}}$. Then $\phi^{\dashv} \in$ **UOSQuant**, and so $\phi \in$ **LoUOSQuant**$(\dashv)$.

(d) Let L, M be equipped with $\otimes = \wedge$ (binary). Then $\psi^{\vdash} \in$ **UQuant**, and so $\psi \in$ **LoUQuant**$(\vdash)$.

(e) Let L, M be equipped with $\otimes = \vee$ (binary). Then $\psi^{\vdash} \in$ **UOSQuant**, and so $\psi \in$ **LoUOSQuant**$(\vdash)$.

(f) Let L, M be equipped with $\otimes = T_{\mathbf{D}}$. Then $\psi^{\vdash} \in$ **UOSQuant**, and so $\psi \in$ **LoUOSQuant**$(\dashv)$.

(g) Let L, M be equipped with $\otimes = \wedge$ (binary). Then $\phi^{\vdash\vdash} \in$ **UQuant**, and so $\phi \in$ **LoUQuant**$(\vdash\vdash)$.

(h) Let L, M be equipped with $\otimes = \vee$ (binary). Then $\phi^{\vdash\vdash} \in$ **UOSQuant**, and so $\phi \in$ **LoUOSQuant**$(\vdash\vdash)$.

(i) Let L, M be equipped with $\otimes = T_{\mathbf{D}}$. Then $\phi^{\vdash\vdash} \in$ **UOSQuant**, and so $\phi \in$ **LoUOSQuant**$(\vdash\vdash)$.

The maps ϕ^{op}, $\phi^{\vdash}$, $\phi^{\vdash\vdash}$ show that the converse of AFT (Theorem 1.10) need not hold and that $\mathbf{C}(\vdash\vdash)$ need not be a subcategory of $\mathbf{C}(\vdash)$ (it is also the case that $\mathbf{C}(\vdash)$ need not be a subcategory of $\mathbf{C}(\vdash\vdash)$); and in particular, we *cannot* in these examples shift two positions to the left and rechoose $\psi^{\mathrm{op}} = \phi^{\dashv\dashv}$—reinforcing Remark 1.13 and Proposition 1.15 and the fact that the converse of AFT (Theorem 1.10) need not hold.

(6) Let X, Y be sets, let $f : X \to Y$ be a function, and let N be a complete lattice. For powerset operator notation, see Definition 1.19.

(a) Put

$$L = N^X, \qquad M = N^Y \tag{1.45}$$

and define $\phi^{\mathrm{op}} : L \leftarrow M$ by

$$\phi^{\mathrm{op}} = f_N^{\leftarrow}. \tag{1.46}$$

Then $\phi^{\dashv} \equiv f_N^{\rightarrow} : L \to M$.

(i) Let L, M be equipped with $\otimes = \vee$ (binary). Then $\phi^{\dashv} \in$ **UOSQuant**, and so $\phi \in$ **LoUOSQuant**$(\dashv)$ (note $f_N^{\rightarrow}$ preserves arbitrary joins and hence the unit $\perp$ of $\otimes$).

(ii) Let L,M be equipped with $\otimes = \wedge$ (binary) and f be injective and N be a frame. Then $\phi^{\dashv} \in$ **Quant**, and so $\phi \in$ **LoQuant**$(\dashv)$. If it is further assumed that f is *not* surjective, then $\phi^{\dashv} \in$ **Quant** $-$ **Frm** and $\phi \in$ **LoQuant**$(\dashv)$ $-$ **Loc**$(\dashv)$ (cf. 1.2.1(6) above).

(b) Put

$$L = N^Y, \qquad M = N^X \tag{1.47}$$

and define $\phi^{\mathrm{op}} : L \leftarrow M$ by

$$\phi^{\mathrm{op}} = f_N^{\rightarrow}. \tag{1.48}$$

Then $\phi^{\vdash} \equiv f_N^{\leftarrow} : L \to M$.

(i) Let L, M be equipped with $\otimes = \wedge$ (binary) and let f be injective. Then $\phi \in \mathbf{LoOSQuant}$, $\phi^{\vdash} \in \mathbf{UQuant}$, and so $\phi \in \mathbf{LoOSQuant}(\vdash)$.

(ii) In (i) above, restrict ϕ^{op}, using $\mathcal{A} \subset M$, $\mathcal{B} \subset L$ *à la* Proposition 1.21(6), by

$$\phi^{\mathrm{op}} = \left(f_N^{\rightarrow}\right)_{|\mathcal{A}} = \left(f_{N\rightarrow}\right)_{|\mathcal{A}} : \mathcal{B} \leftarrow \mathcal{A}, \tag{1.49}$$

let f be surjective, and let $(N, \leq, \otimes)$ be a unital quantale with $\otimes$ arbitrary. Then $\mathcal{A}$, $\mathcal{B}$ are unital quantales, $\phi \in \mathbf{LoUQuant}$, and

$$\begin{aligned} \phi^{\dashv} &= \left(f_N^{\leftarrow}\right)_{|\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{A}, \\ \phi^{\vdash} &= \left(f_N^{\leftarrow}\right)_{|\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{A}, \\ \phi^{\vdash\vdash} &= \left(f_{N\rightarrow}\right)_{|\mathcal{A}} : \mathcal{B} \longleftarrow \mathcal{A} \end{aligned} \tag{1.50}$$

are in $\mathbf{UQuant}$ (see Remark 1.20, Proposition 1.21). So

$$\phi \in \mathbf{LoUQuant}(\dashv) \cap \mathbf{LoUQuant}(\vdash) \cap \mathbf{LoUQuant}(\vdash\vdash). \tag{1.51}$$

(c) Every function $f : X \rightarrow Y$ gives rise to the examples in (a)((i), (ii))–(b)((i), (ii)) using the epi-mono decomposition of f (cf. Remark 1.20).

(7) Let $L = M = \mathbb{I} \equiv [0,1]$, $n \in \mathbb{N}$, and $\phi_n^{\mathrm{op}} : L \leftarrow M$ by $\phi_n^{\mathrm{op}}(x) = x^n$; and let L, M be equipped with the t-norm $\otimes = T_{\mathbf{P}} = \cdot$ (multiplication). Then ϕ_n^{op} preserves arbitrary $\bigvee$, arbitrary $\bigwedge$, and $\otimes$. So we have that both $\phi_n^{\dashv} : L \rightarrow M$ and $\phi_n^{\vdash} : L \rightarrow M$ exist and $\phi_n^{\dashv} \dashv \phi_n^{\mathrm{op}} \dashv \phi_n^{\vdash}$. Further, since both ϕ_n^{op} and $(\phi_n^{\mathrm{op}})^{-1}$ are isotone bijections, $\phi_n^{\dashv} = (\phi_n^{\mathrm{op}})^{-1} = \phi_n^{\vdash}$. Now $(\phi_n^{\mathrm{op}})^{-1}(x) = x^{1/n}$ and $(\phi_n^{\mathrm{op}})^{-1}$ preserves arbitrary $\bigvee$, arbitrary $\bigwedge$ and $\otimes$. Hence each of $\phi_n^{\dashv} : L \rightarrow M$ and $\phi_n^{\vdash} : L \rightarrow M$ is in $\mathbf{UQuant}$. Further, $\phi_n^{\dashv\dashv} = \phi_n^{\mathrm{op}}$ and $\phi_n^{\vdash\vdash} = \phi_n^{\mathrm{op}}$ are also in $\mathbf{UQuant}$. So $\phi \in \mathbf{LoUQuant}(\dashv) \cap \mathbf{LoUQuant}(\vdash) \cap \mathbf{LoUQuant}(\vdash\vdash)$.

(8) To extend the examples of (7) above to classes of examples, let $\phi^{\mathrm{op}} : L \leftarrow M$ be any order-isomorphism and assume that $(L, \leq, \otimes)$ is a unital quantale. Put $T : M \times M \rightarrow M$ by

$$T = \left(\phi^{\mathrm{op}}\right)^{-1} \circ \otimes \circ \left(\phi^{\mathrm{op}} \times \phi^{\mathrm{op}}\right). \tag{1.52}$$

Then $(M, \leq, T)$ is a unital quantale and $\phi \in \mathbf{LoUQuant}(L, M)$. Further,

$$\phi^{\dashv} = \left(\phi^{\mathrm{op}}\right)^{-1} = \phi^{\vdash}, \qquad \phi^{\dashv\dashv} = \phi^{\mathrm{op}} = \phi^{\vdash\vdash}, \tag{1.53}$$

all these morphisms are in $\mathbf{UQuant}$, and so

$$\phi \in \mathbf{LoUQuant}(\dashv) \cap \mathbf{LoUQuant}(\vdash) \cap \mathbf{LoUQuant}(\vdash\vdash). \tag{1.54}$$

To indicate the richness of this class of examples, let ϕ^{op} be any of the ϕ_n^{op}'s of (7) with $L = M = \mathbb{I}$, and choose $\otimes$ to be any left-continuous t-norm on $\mathbb{I}$, of which there are uncountably many [28, 19].

(9) To indicate how the classes of examples of (8) can be further extended, apply the setup of (7) to $L \equiv \mathbb{I}^3$ equipped with

$$\otimes = T_{\wedge} \times T_{\mathbf{P}} \times T_{Ł}, \tag{1.55}$$

where $T_\wedge$, $T_{\mathbf{P}}$, $T_{Ł}$ are the continuous min, product, and Łukasiewicz t-norms on $\mathbb{I}$, and finally choose

$$\psi^{\text{op}} = \phi_n^{\text{op}} \times \phi_n^{\text{op}} \times \phi_n^{\text{op}} : L \equiv \mathbb{I}^3 \longleftarrow M \equiv \mathbb{I}^3, \tag{1.56}$$

where ϕ_n^{op} is any of the mappings of (7). Then *à la* the construction of (8), put

$$T = \left(\psi^{\text{op}}\right)^{-1} \circ \otimes \circ \left(\psi^{\text{op}} \times \psi^{\text{op}}\right) : M \times M \longrightarrow M. \tag{1.57}$$

Then $\psi \in \mathbf{LoUQuant}(L, M)$ and indeed

$$\psi \in \mathbf{LoUQuant}(\dashv) \cap \mathbf{LoUQuant}(\vdash) \cap \mathbf{LoUQuant}(\vdash\vdash). \tag{1.58}$$

(10) All of the examples in (1)–(9) can be used to generate example classes having whichever combination of desired properties simply by using products of semi-quantales, where the underlying sets are Cartesian product of sets and the partial orders and limits and colimits and tensors are taken pointwise, and by noting that adjunctions occur pointwise as well. We illustrate this with two constructions.

(a) Suppose we wish an example class of nonisomorphism morphisms in

$$\mathbf{LoUQuant}(\dashv) \cap \mathbf{LoUQuant}(\vdash) \tag{1.59}$$

with $\otimes$'s not order-theoretic. We simply "cross" (1) and (7) as follows. Put

$$\begin{gathered} L = \{\perp, a, \top\} \times \mathbb{I}, \qquad M = \{\perp, \top\} \times \mathbb{I}, \\ \otimes = \wedge \times T_{\mathbf{P}} \quad \text{on each of } L,\ M, \qquad \psi^{\text{op}} = \phi^{\text{op}} \times \phi_n^{\text{op}}, \end{gathered} \tag{1.60}$$

where ϕ^{op} is taken from (1) and ϕ_n^{op} is taken from (7) for some $n \in \mathbb{N}$. Then ψ is not an isomorphism, $\otimes$ is not order-theoretic, and $\psi \in \mathbf{LoUQuant}(\dashv) \cap \mathbf{LoUQuant}(\vdash)$.

(b) Supppose we wish an example class of nonisomorphism morphisms in

$$\mathbf{LoUQuant}(\vdash) \cap \mathbf{LoUQuant}(\vdash\vdash) \tag{1.61}$$

with $\otimes$'s not order-theoretic. We simply "cross" (2) and (7) as follows. Put

$$\begin{gathered} L = \{\perp, \top\} \times \mathbb{I}, \qquad M = \{\perp, a, \top\} \times \mathbb{I}, \\ \otimes = \wedge \times T_{\mathbf{P}} \quad \text{on each of } L,\ M, \qquad \psi^{\text{op}} = \phi^{\text{op}} \times \phi_n^{\text{op}}, \end{gathered} \tag{1.62}$$

where ϕ^{op} is taken from (2) and ϕ_n^{op} is again taken from (7) for some $n \in \mathbb{N}$. Then ψ is not an isomorphism, $\otimes$ is not order-theoretic, and

$$\psi \in \mathbf{LoUQuant}(\vdash\ ) \cap \mathbf{LoUQuant}(\vdash\vdash). \tag{1.63}$$

1.4. Powerset operator preliminaries. We catalogue the powerset operators for traditional and lattice-valued mathematics used in this paper.

Definition 1.18 (powersets). Let $X \in |\mathbf{Set}|$ and $L \in |\mathbf{SQuant}|$. Then L^X is the *L-powerset of* X. The constant member of L^X having value α is denoted $\underline{\alpha}$. All order-theoretic operations (e.g., $\bigvee, \bigwedge$) and algebraic operations (e.g., $\otimes$) on L lift point-wise to L^X and are denoted by the same symbols. In the case $L \in |\mathbf{USQuant}|$, the unit e lifts to the constant map $\underline{e}$, which is the unit of $\otimes$ as lifted to L^X.

Definition 1.19 (powerset operators). (1) Let $X, Y \in |\mathbf{Set}|$, let $f : X \to Y$ be in $\mathbf{Set}$, and define the standard (traditional) image and preimage operators $f^{\to} : \wp(X) \to \wp(Y)$, $f^{\leftarrow} : \wp(X) \leftarrow \wp(Y)$ by

$$f^{\to}(A) = \{f(x) \in Y : x \in A\}, \qquad f^{\leftarrow}(B) = \{x \in X : f(x) \in B\}. \tag{1.64}$$

(2) Fix $L \in |\mathbf{SQuant}|$, let $X, Y \in |\mathbf{Set}|$, let $f : X \to Y$ be in $\mathbf{Set}$, and define the standard image and preimage operators $f_L^{\to} : L^X \to L^Y$, $f_L^{\leftarrow} : L^X \leftarrow L^Y$ [29] by

$$f_L^{\to}(a)(y) = \bigvee \{a(x) : x \in f^{\leftarrow}(\{y\})\}, \qquad f_L^{\leftarrow}(b) = b \circ f. \tag{1.65}$$

If L is understood, then it may be dropped provided that the context distinguishes these operators from the traditional operators of (1). It is also needed in this paper (e.g., Example 1.17(6)) to consider the *lower-image operator* $f_{L\to} : L^X \to L^Y$ given by

$$f_{L\to} = \left(f_L^{\leftarrow}\right)^{\vdash} \tag{1.66}$$

using Definition 1.12. Note $f_L^{\to}(a)(y) = \bigwedge\{a(x) : x \in f^{\leftarrow}(\{y\})\}$.

(3) Let $(X,L), (Y,M) \in |\mathbf{Set} \times \mathbf{LoSQuant}|$ and let $(f,\phi) : (X,L) \to (Y,M)$ be in $\mathbf{Set} \times \mathbf{LoSQuant}$. Define $(f,\phi)^{\to} : L^X \to M^Y$, $(f,\phi)^{\leftarrow} : L^X \leftarrow M^Y$, (cf. [30–33, 23–26, 20, 2, 21, 34, 27]), by

$$(f,\phi)^{\to}(a) = \bigwedge \{b \in M^Y : f_L^{\to}(a) \le \langle \phi^{\mathrm{op}} \rangle (b)\}, \qquad (f,\phi)^{\leftarrow} = \langle \phi^{\mathrm{op}} \rangle \circ f_M^{\leftarrow}, \tag{1.67}$$

where $\langle \phi^{\mathrm{op}} \rangle$ uses Notation 1.11.

The above powerset operators get repackaged in various ways in Sections 2.2 and 3.5.

Remark 1.20. This remark resolves the relationship between the upper- (or standard) image operator and the lower-image operator. We consider the general lattice-valued case first and then various special cases including the traditional case.

(1) Let $f : X \to Y$ be any function. Then the following relationship holds between the standard and lower-image operators:

$$\forall a \in L^X, \quad f_{L\to}(a) = \chi_{Y - f^{\to}(X)} \vee f_L^{\to}\left(q_L^{\leftarrow}(\hat{a})\right), \tag{1.68}$$

where

$$\begin{gathered}\forall x \in X, \quad [x] = \{z \in X : f(x) = f(z)\}, \\ q : X \longrightarrow X/f \equiv \{[x] : x \in X\} \subset \wp(X) \quad \text{by} \quad q(x) = [x], \\ \hat{a} : \wp(X) \longrightarrow L : \hat{a}(B) = \bigwedge_{x \in B} a(x).\end{gathered} \tag{1.69}$$

(2) The identity (1.68) for f injective reduces to

$$\forall a \in L^X, \quad f_{L\rightarrow}(a) = \chi_{Y - f^{\rightarrow}(X)} \vee f_L^{\rightarrow}(a). \tag{1.70}$$

(3) The identity (1.68) for f surjective reduces to

$$\forall a \in L^X, \quad f_{L\rightarrow}(a) = f_L^{\rightarrow}\left(q_L^{\leftarrow}(\hat{a})\right). \tag{1.71}$$

(4) The identity (1.68) for the traditional case (with $L = 2$) reduces to

$$\forall A \in \wp(X), \quad f_{\rightarrow}(A) = \left(Y - f^{\rightarrow}(X)\right) \cup f^{\rightarrow}\left(q^{\leftarrow}\left(\wp(A)\right)\right). \tag{1.72}$$

Proof. We prove only (1.68) in (1). Let $a \in L^X$. It is helpful to consider the set $D = \{d \in L^Y : f_L^{\leftarrow}(d) \le a\}$ and write $f_{L\rightarrow}(a) = \bigvee D$ (using AFT). We observe that for $x \in X$,

$$\chi_{Y - f^{\rightarrow}(X)}\left(f(x)\right) = \perp \le a(x); \tag{1.73}$$

so that $f_L^{\leftarrow}(\chi_{Y - f^{\rightarrow}(X)}) \le a$, $\chi_{Y - f^{\rightarrow}(X)} \in D$, and

$$f_{L\rightarrow}(a) = \chi_{Y - f^{\rightarrow}(X)} \quad \text{on } Y - f^{\rightarrow}(X). \tag{1.74}$$

Now let $y \in f^{\rightarrow}(X)$; then there exist $x \in X$, $y = f(x)$. Since $q(x) \neq \varnothing$, we first observe that

$$\begin{aligned} f_L^{\leftarrow}\left(f_L^{\rightarrow}\left(q_L^{\leftarrow}(\hat{a})\right)\right)(x) &= f_L^{\rightarrow}\left(q_L^{\leftarrow}(\hat{a})\right)\left(f(x)\right) = \bigvee_{w \in q(x)} \hat{a}\left(q(w)\right) \\ &= \bigvee_{w \in q(x)} \left[\bigwedge_{z \in q(w)} a(z)\right] = \bigvee_{w \in q(x)} \left[\bigwedge_{z \in q(x)} a(z)\right] \\ &= \bigwedge_{z \in q(x)} a(z) \le a(x) \end{aligned} \tag{1.75}$$

so that $f_L^{\leftarrow}(f_L^{\rightarrow}(q_L^{\leftarrow}(\hat{a}))) \le a$ on X, which means that $f_L^{\rightarrow}(q_L^{\leftarrow}(\hat{a})) \in D$. We further observe for fixed $d_0 \in D$, again for $y = f(x)$, that for each $z \in q(x)$, we have

$$d_0(y) = d_0\left(f(z)\right) = f_L^{\leftarrow}\left(d_0\right)(z) \le a(z). \tag{1.76}$$

From this it follows, using the sixth term and then the third term from (1.75), that

$$d_0(y) \le \bigwedge_{z \in q(x)} a(z) = f_L^{\rightarrow}\left(q_L^{\leftarrow}(\hat{a})\right)\left(f(x)\right) = f_L^{\rightarrow}\left(q_L^{\leftarrow}(\hat{a})\right)(y). \tag{1.77}$$

Hence

$$f_{L\rightarrow}(a) = \bigvee D = f_L^{\rightarrow}(q_L^{\leftarrow}(\hat{a})) \quad \text{on } f^{\rightarrow}(X). \tag{1.78}$$

It is now easy to check that

$$f_{L\rightarrow}(a) = \chi_{Y-f^{\rightarrow}(X)} \vee f_L^{\rightarrow}(q_L^{\leftarrow}(\hat{a})) \quad \text{on } Y, \tag{1.79}$$

concluding the proof. □

Proposition 1.21. *Under the conditions of Definitions 1.19(2), 1.19(3) and with the notation of Remark 1.20, the following hold:*

(1) $f_L^{\leftarrow}$ $[(f,\phi)^{\leftarrow}]$ *preserves arbitrary* $\bigvee$, *arbitrary* $\bigwedge$, $\otimes$, *and all constant maps.*
(2) $f_L^{\rightarrow}$ $[(f,\phi)^{\rightarrow}]$ *preserves arbitrary* $\bigvee$ *[if* ϕ^{op} *preserves arbitrary* $\bigwedge$*].*
(3) $f_L^{\leftarrow}$ $[(f,\phi)^{\leftarrow}]$ *preserves the unit if* $L \in |\mathbf{USQuant}|$ $[L, M \in |\mathbf{USQuant}|]$.
(4) $f_{L\rightarrow}$ *preserves arbitrary* $\bigwedge$.
(5) *If f is injective, then* $f_{L\rightarrow}$ *preserves arbitrary nonempty* $\bigvee$.
(6) *If f is surjective, then* $(f_{L\rightarrow})_{|\mathcal{A}} = (f_L^{\rightarrow})_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ *preserves arbitrary* $\bigvee$ *and arbitrary* $\bigwedge$ *and* $\otimes$, *where*

$$\mathcal{A} = \{a \in L^X : \forall x \in X,\ \forall w \in [x],\ a(w) = a(x)\}, \qquad \mathcal{B} = (f_L^{\rightarrow})^{\rightarrow}(\mathcal{A}) = (f_{L\rightarrow})^{\rightarrow}(\mathcal{A}). \tag{1.80}$$

Proof. Key properties of the powerset operators of Definition 1.19, as well as those of other powerset operators, can be found in [8, 9, 30–32, 10, 11, 13, 25, 26, 20, 2, 21, 34]. We comment only on (4), (5), (6) regarding $f_{L\rightarrow}$. First, $f_{L\rightarrow}$ preserves arbitrary $\bigwedge$ by AFT since it is the right-adjoint of $f_L^{\leftarrow}$ and the latter is a map preserving arbitrary joins. Second, the claim that $f_{L\rightarrow}$ also preserves nonempty $\bigvee$ when f is injective follows from $f_L^{\rightarrow}$ preserving arbitrary $\bigvee$ and from the identity of Remark 1.20(2). Finally, the claims of (6) when f is surjective follow from the identity of Remark 1.20(3), the fact that $(f_{L\rightarrow})_{|\mathcal{A}} = (f_L^{\rightarrow})_{|\mathcal{A}}$, and the fact that $\mathcal{A}$ $[\mathcal{B}]$ is a complete sublattice of L^X $[L^Y]$ which is closed under $\otimes$ as lifted to L^X $[L^Y]$. □

1.5. Topological preliminaries. We catalogue the notions of topology used in this paper. Such notions generalize those of [1, 2] with respect to the underlying membership lattices, essentially generalizing **CQML** to **SQuant** and/or **USQuant**.

Definition 1.22 (cf. [35, 33, 36, 1, 2]). Fixing $L \in |\mathbf{SQuant}|$, the category L-**QTop** has ground category **Set** and comprises the following data:

(1) *Objects*: (X,τ), where $X \in |\mathbf{Set}|$ and $\tau \subset L^X$ is closed under $\otimes$ and arbitrary $\bigvee$. The structure τ is a(n) (L-)*q(uasi)-topology* and the object (X,τ) is a(n) (L-)*q(uasi)-topological space*.
(2) *Morphisms*: $f : (X,\tau) \to (Y,\sigma)$, where $f : X \to Y$ is from **Set** and $\tau \supset (f_L^{\leftarrow})^{\rightarrow}(\sigma)$.
(3) *Composition, identities*: from **Set**.

If $L \in |\mathbf{USQuant}|$, the category L-**Top** is defined as L-**QTop** with the additional condition on objects (X,τ) that τ also contains the constant subset $\underline{e}$, where e is the unit of

$\otimes$ in L; and the structure τ is this case is a(n) (L-)*topology* and the object (X,τ) is a(n) (L-)*topological space*.

Definition 1.23 (cf. [37, 1, 38–50]). Fixing $L \in |\mathbf{SQuant}|$, the category L-**FQTop** has ground category **Set** and comprises the following data:

(1) *Objects*: $(X,\mathcal{T})$, where $X \in |\mathbf{Set}|$ and $\mathcal{T} : L^X \to L$ satisfies the following conditions.

(a) For each indexing set J, for each $\{u_j : j \in J\} \subset L^X$,

$$\bigwedge_{j\in J} \mathcal{T}(u_j) \le \mathcal{T}\left(\bigvee_{j\in J} u_j\right). \tag{1.81}$$

(b) For each indexing set J with $|J| = 2$, for each $\{u_j : j \in J\} \subset L^X$,

$$\bigotimes_{j\in J} \mathcal{T}(u_j) \le \mathcal{T}\left(\bigotimes_{j\in J} u_j\right). \tag{1.82}$$

The structure $\mathcal{T}$ is a(n) (L-)*q(uasi)-fuzzy topology* and the object $(X,\mathcal{T})$ is a(n) (L-)*q(uasi)-fuzzy topological space*.

(2) *Morphisms*: $f : (X,\mathcal{T}) \to (Y,\mathcal{S})$, where $f : X \to Y$ is from **Set** and $\mathcal{T} \circ f_L^{\leftarrow} \ge \mathcal{S}$ on L^Y.

(3) *Composition, identities*: from **Set**.

If $L \in |\mathbf{USQuant}|$, the category L- **FTop** is defined as L-**QFTop** with objects $(X,\mathcal{T})$ in (1) additionally satisfying that

(c) $\mathcal{T}(\underline{e}) = e$ (where e is the identity of $\otimes$ on L and $\underline{e}$ is the corresponding constant L-subset).

For an object $(X,\mathcal{T})$ in L-**FTop**, the structure $\mathcal{T}$ in this case is a(n) (L-)*fuzzy topology* and the object (X,τ) is a(n) (L-)*fuzzy topological space*.

Definition 1.24 [30–33, 23–26, 20, 2, 27] (cf. [51, 52]). Fixing $\mathscr{C} \subset \mathbf{SQuant}$ and $\mathbf{C} = \mathscr{C}^{\mathrm{op}}$, the category **C-QTop** has ground category $\mathbf{Set} \times \mathbf{C}$ and comprises the following data:

(1) *Objects*: (X,L,τ), where $(X,L) \in |\mathbf{Set} \times \mathbf{C}|$ and $(X,\tau) \in |L\text{-}\mathbf{QTop}|$. The object (X, L,τ) is a *q(uasi)-topological space* and τ is a *q-topology* on (X,L).

(2) *Morphisms*: $(f,\phi) : (X,L,\tau) \to (Y,M,\sigma)$, where $(f,\phi) : (X,L) \to (Y,M)$ is from $\mathbf{Set} \times \mathbf{C}$ and $\tau \supset ((f,\phi)^{\leftarrow})^{\rightarrow}(\sigma)$.

(3) *Composition, identities*: from $\mathbf{Set} \times \mathbf{C}$.

If $\mathscr{C} \subset \mathbf{USQuant}$, the category **C-Top** is defined as **C-QTop** with the additional condition that objects (X,L,τ) satisfy the condition that $(X,\tau) \in |L\text{-}\mathbf{Top}|$, in which case (X,L,τ) is a *topological space* and τ is a *topology* on (X,L).

Definition 1.25 (cf. [2]). Fixing $\mathscr{C} \subset \mathbf{SQuant}$ and $\mathbf{C} = \mathscr{C}^{\mathrm{op}}$, the category **C-QFTop** has ground category $\mathbf{Set} \times \mathbf{C}$ and comprises the following data:

(1) *Objects*: $(X,L,\mathcal{T})$, where $(X,L) \in |\mathbf{Set} \times \mathbf{C}|$ and $(X,\mathcal{T}) \in |L\text{-}\mathbf{QFTop}|$. The object $(X,L,\mathcal{T})$ is a *q(uasi)-fuzzy topological space* and $\mathcal{T}$ is a *q-fuzzy topology* on (X,L).

(2) *Morphisms*: $(f,\phi) : (X,L,\mathcal{T}) \to (Y,M,\mathcal{S})$, where $(f,\phi) : (X,L) \to (Y,M)$ is from $\mathbf{Set} \times \mathbf{C}$ and $\mathcal{T} \circ (f,\phi)^{\leftarrow} \ge \phi^{\mathrm{op}} \circ \mathcal{S}$ on M^Y.

(3) *Composition, identities*: from $\mathbf{Set} \times \mathbf{C}$.

If $\mathscr{C} \subset$ **USQuant**, the category **C-FTop** is defined as **C-QFTop** with the additional condition that objects $(X, L, \mathscr{T})$ satisfy the condition that $(X, \mathscr{T}) \in |L\text{-}\mathbf{FTop}|$, in which case $(X, L, \mathscr{T})$ is a *fuzzy topological space* and $\mathscr{T}$ is a *fuzzy topology* on (X, L).

The above categories arise naturally in the sequel: for example, the L-**Top**'s and **C-Top**'s [L-**FTop**'s and **C-FTop**'s, resp.] are generated in Section 3.4 as topological [fuzzy topological] theories arising from certain topological powerset theories in their respective grounds; and characterizations of standard construction algebraic theories (Example 3.15) generating these powerset theories are given in Sections 5 and 6. The following theorem, which generalizes analogous theorems in [1, 2], follows from Theorems 3.10–3.12, 3.26 (whose proofs generalize those in [1, 2]).

Theorem 1.26 (cf. [1, 2]). *For each* $L \in |\mathbf{SQuant}|$ *and* $\mathbf{C} \subset \mathbf{LoSQuant}$ [$L \in |\mathbf{USQuant}|$ *and* $\mathbf{C} \subset \mathbf{LoUSQuant}$], L-**QTop** *and* **C-QTop** *[L-**Top** and **C-Top**, resp.] are topological over their specified ground categories with respect to the usual forgetful functors; and for each* $L \in |\mathbf{OSQuant}|$ *and* $\mathbf{C} \subset \mathbf{LoOSQuant}$ [$L \in |\mathbf{UOSQuant}|$ *and* $\mathbf{C} \subset \mathbf{LoUOSQuant}$], L-**QFTop** *and* **C-QFTop** *[L-**FTop** and **C-FTop**, resp.] are topological over their specified ground categories with respect to the usual forgetful functors.*

Example 1.27. There is a rich example inventory extant in the literature of both objects and morphisms of the above categories; see [14], topology chapters of [36, 53, 38, 54, 55], and the bibliographies of these citations.

2. Motivating examples for algebraic, powerset, topological theories

2.1. Semigroups. The study of semigroups is considered a part of mathematics called "algebra," and one could therefore think of this part of mathematics as comprising a family of "algebraic" theories.

How should one specify in what sense semigroups are algebraic theories? One way would be to examine whether the category **SGroup** is an algebraic category over **Set**; and in fact this is the case—see [6, Examples 23.34(1)]. But another way would be to examine in what sense semigroups are "equational" and decide if this sense is indeed what we would like to call "algebraic." We now explore this second option, adapting the following paragraphs and Proposition 2.1 below from [3, Section 1].

A semigroup $S \equiv (S, \bullet)$ is a set S equipped with an associative binary operation $\bullet$. Now let A be any set, which may be thought of as a set of variables which take values in S. Then given variables $a, b, c \in A$, the two variable expressions $a \bullet (b \bullet c)$ and $(a \bullet b) \bullet c$, for any instantiation with values in S, are equal; and hence these expressions may be viewed as equivalent. In particular, the "equivalence class" of these strings may be identified with the ordered, grouping-symbol-free string abc.

More generally, for a set A, we consider the family of all finite expressions with variable names in A, place on this family an equivalence relation—two expressions are equivalent if and only if they use the same ordered, grouping-symbol-free string, and note that the quotient set of all equivalence classes is bijective with the set of all finite, ordered, grouping-symbol-free strings of the form

$$a_1 \cdots a_n, \quad n > 0. \tag{2.1}$$

Given set A, call the associated family of all finite, ordered, grouping-symbol-free strings $T(A)$ ("T" for "theory"). Then $T : |\mathbf{Set}| \to |\mathbf{Set}|$ is an object function on the category **Set**. We now collect some facts and observations involving this theory T which provide a template for the axiomatization of an algebraic theory (or monad) in clone form.

(1) Given sets B, C, each function $g : B \to T(C)$ extends to $g^{\#} : T(B) \to T(C)$ by

$$b_1 \cdots b_n \longmapsto g(b_1) \cdots g(b_n), \tag{2.2}$$

that is, by insertion of g determined expressions with variables in C for variables in expressions with variables in B, in short, by concatenation of strings.

(2) Given sets A, B, C and functions $f : A \to T(B)$, $g : B \to T(C)$, we may form a "clone composition"

$$g \diamond f : A \longrightarrow T(C) \tag{2.3}$$

by setting

$$g \diamond f = g^{\#} \circ f. \tag{2.4}$$

(3) Given set A, there is the "insertion of variables as singular expressions" or "variables as expressions" map $\eta_A : A \to T(A)$ given by

$$a \longmapsto a. \tag{2.5}$$

We put $\mathbf{T} \equiv (T, \eta, \diamond)$ and speak of $\mathbf{T}$ as the "algebriac theory in **Set** induced by semigroup S". We note several additional facts.

Proposition 2.1. *The following hold:*

(1) *The clone composition is associative.*

(2) *The components of η yield identities on both sides for the clone composition.*

(3) *If the objects of* **Set**, *functions of the form $f : A \to T(B)$, the clone composition, and the identities of the clone composition are taken, a new category $\mathbf{Set}_{\mathbf{T}}$ is obtained.*

(4) *For each function $f : A \to B$, there is a mapping $f^{\Delta} : A \to T(B)$ induced by*

$$f^{\Delta} = \eta_B \circ f. \tag{2.6}$$

(5) *For each mapping $f : A \to B$, there is a mapping $T(f) : T(A) \to T(B)$ induced by*

$$T(f) = f^{\Delta} \diamond \mathrm{id}_{T(A)}. \tag{2.7}$$

2.2. Traditional powerset operators. Let X be a set. Then to X we may associate

$$\wp(X) = \{A : A \subset X\}, \tag{2.8}$$

called the "powerset" of X, as well as an "insertion map" $\eta_X : X \to \wp(X)$ defined by

$$\eta_X(x) = \{x\}. \tag{2.9}$$

Further, a function $f : X \to Y$ determines mappings $f^{\to} : \wp(X) \to \wp(Y)$, $f^{\leftarrow} : \wp(X) \leftarrow \wp(Y)$ defined by Definition 1.19, respectively called the "forward/image" and "backward/preimage" powerset operators of f. Then we know that the "adjunction condition"

$$f^{\to} \dashv f^{\leftarrow} \tag{2.10}$$

holds, where these operators are viewed as isotone mappings between presets, and that the "lifting condition"

$$f^{\to} \circ \eta_X = n_Y \circ f \tag{2.11}$$

also holds.

The ordered tuple $(\wp, \to, \leftarrow)$ yields a "powerset theory in **Set**" which provides a template for the axiomatization of powerset theories considered later in this paper, especially because of these well-known features of $(\wp, \to, \leftarrow)$.

Proposition 2.2. *Let X, Y, Z be sets and $f : X \to Y$, $g : Y \to Z$ be functions. The following hold:*

(1) *adjunction condition: $f^{\to} \dashv f^{\leftarrow}$.*

(2) *Concreteness, naturality conditions: there is an "insertion map" $\eta_X : X \to \wp(X)$ defined by*

$$\eta_X(x) = \{x\} \tag{2.12}$$

such that

$$f^{\to} \circ \eta_X = n_Y \circ f. \tag{2.13}$$

(3) *Topological conditions:*

(T1) $f^{\leftarrow} \in$ **CBool** *(and so $f^{\leftarrow}$ preserves universal and empty sets).*

(T2) $(g \circ f)^{\leftarrow} = f^{\leftarrow} \circ g^{\leftarrow}$.

(T3) $(\mathrm{id}_X)^{\leftarrow} = \mathrm{id}_{\wp(X)}$.

Other important facts are given in the following remark.

Remark 2.3. Under the assumptions of Proposition 2.2, the following hold:

(1) $f^{\to} \in$ **CSLat**$(\bigvee)$ and so $f^{\to}$ preserves arbitrary $\bigcup$ (and $\varnothing$), while $f^{\leftarrow}$ preserves arbitrary $\bigcup$ and $\bigcap$ as well as complements.

(2) $f^{\to} : \wp(X) \to \wp(Y)$, $f^{\leftarrow} : \wp(X) \leftarrow \wp(Y)$ can be reformulated as follows:

$$\begin{aligned} f^{\to}(A) &= \bigcap \{B \in \wp(Y) : A \subset f^{\leftarrow}(B)\}, \\ f^{\leftarrow}(B) &= \bigcup \{A \in \wp(X) : f^{\to}(A) \subset B\}. \end{aligned} \tag{2.14}$$

(3) The adjunction $f^{\to} \dashv f^{\leftarrow}$ is a fundamental part of the foundations of analysis. It is the essence of the proof that continuity preserves compactness, which together with the Heine-Borel theorem proves the extreme value theorem, which in turn leads to the Rolle-Fermat lemma and the mean-value theorem upon which the differential calculus rests.

(4) The concreteness condition of Proposition 2.2(2) suggests that powersets and powerset operators may form a theory related to algebraic theories as motivated by semigroups; see Section 3.

(5) The topological conditions of Proposition 2.2(3), along with the existence of complete fibres, are crucial to the proof that **Top** is a topological construct; and conditions of this type will prove critical to the formulation of powerset theories which yield topological categories and hence topological theories; see Section 3.

Formalizing the axioms of a powerset theory should capture many of these traditional properties and thereby enable us to recognize many powerset theories in **Set** and other "ground categories" which have powerful consequences similar to those of the powerset theory described above, including recognizing those which are generated from algebraic theories and those which build topological theories. The interesting question is to formulate powerset theories which are *both* generated from algebraic theories *and* which build topological theories.

2.3. Topological spaces and continuous maps. Since **Top** is categorically isomorphic to 2-**Top** (choosing $L = 2$ in Definition 1.22), it behooves us to review some basic facts concerning **Top** in order to gain insights into the categories of Section 1.5 and their relationship with lattice-valued powersets. Recall the following definition of the category **Top**:

(1) *Objects*: (X,τ), where $X \in |\mathbf{Set}|$ and $\tau \subset \wp(X)$ is closed under arbitrary $\bigcup$ and finite $\cap$.

(2) *Morphisms*: $f : (X,\tau) \to (Y,\sigma)$, where $f : X \to Y$ is from **Set** and $\tau \supset (f^{\leftarrow})^{\rightarrow}(\sigma)$.

(3) *Composition, identities*: from **Set**.

The notion of a topological category is found and extensively developed in [6] and midrashed and exampled in [2, Section 1].

Theorem 2.4. **Top** *is a fibre-small category topological over* **Set** *with respect to the forgetful functor* $V : \mathbf{Top} \to \mathbf{Set}$ *defined by* $V(X,\tau) = X$, $V(f) = f$.

The key ingredients in the proof that **Top** is a category are (T2) and (T3) of Proposition 2.2(3), one of the key ingredients in the proof that the expected lift of a V-structured source is in fact initial is the equivalence of continuity with subbasic continuity, and this equivalence rests squarely on (T1) of Proposition 2.2(3), and the key ingredient in the proof that the standard lift of a V-structured source is in fact unique rests on (T3) of Proposition 2.2(3) above. A fundamental ingredient not accounted for in the topological conditions of Proposition 2.2(3) is that the fibre of topologies on each set is a complete lattice ordered by inclusion (in the double powerset), but this is an object issue and not a morphism issue (unless we replace objects by identity morphisms and repackage the notion of fibres in terms of morphisms) and hence not a preimage operator issue. These observations justify calling the conditions of Proposition 2.2(3) "topological conditions." Further, it would seem that if a powerset theory has a preimage operator with conditions similar to Proposition 2.2(3) *and* such a theory produces in an appropriate way a category which has complete fibres with respect to its "ground category" and a "forgetful" functor, then such a category ought to be a topological category over its ground with respect to such a functor. Such thinking will guide our formulation of "topological powerset theories" and "(fuzzy) topological theories" in the next section, and it will be seen (Sections 3.2–3.3, 3.6) that our axiomatizations are somewhat "dual" to that of [6, Exercise 22.B].

3. Axioms for algebraic, powerset, topological, fuzzy topological theories

3.1. Axioms for algebraic theories

Definition 3.1 (axioms for algebraic theories, [3, Definition 1.3.1]). Let $\mathbf{K}$ be a category, called the *base* or *ground* category. An *algebraic theory (in clone form) in* $\mathbf{K}$ is an ordered triple $\mathbf{T} \equiv (T, \eta, \diamond)$ which is specified by the following data and axioms:

(D1) $T : |\mathbf{K}| \to |\mathbf{K}|$ is an object function on $\mathbf{K}$.

(D2) η assigns to each $A \in |\mathbf{K}|$ a $\mathbf{K}$ morphism $\eta_A : A \to T(A)$.

(D3) $\diamond$ assigns to each pair of $\mathbf{K}$ morphisms, $f : A \to T(B)$, $g : B \to T(C)$, a $\mathbf{K}$ morphism $g \diamond f : A \to T(C)$.

(A1) $\diamond$ is associative, that is, for each $f : A \to T(B)$, $g : B \to T(C)$, $h : C \to T(D)$,

$$h \diamond (g \diamond f) = (h \diamond g) \diamond f. \tag{3.1}$$

(A2) η furnishes left-identities, that is, for each $f : A \to T(B)$,

$$\eta_B \diamond f = f. \tag{3.2}$$

(A3) $\diamond$ is compatible with the composition $\circ$ of $\mathbf{K}$ morphisms, that is, given $f : A \to B$, $g : B \to T(C)$, and setting $f^{\Delta} : A \to T(B)$ by

$$f^{\Delta} = \eta_B \circ f, \tag{3.3}$$

then it is the case that

$$g \diamond f^{\Delta} = g \circ f. \tag{3.4}$$

In comparison to [6, 3], the phrase "ground category" is preferred in this paper to "base category" for the following reasons: our first goal in this paper is applications to powerset theories in which the words "base", "basis" refer to the underlying lattice of membership values; our second goal is applications to lattice-valued topology in which the words "base", "basis" can also refer to a generating collection of lattice-valued open sets; and finally, from [30–32] on, the term "ground category" has been used for the category underlying the topological concrete categories in lattice-valued mathematics, and as shown this paper, ground categories of certain algebraic theories are indeed the ground category of certain topological categories in lattice-valued mathematics.

Remark 3.2 (cf. *infra* [3, Definition 1.3.1]). (1) Axiom (A2) only specifies that η gives left-hand identities. But in fact the following holds:

(A2') η furnishes identities on both sides for $\diamond$, that is, for each $f : A \to T(B)$,

$$\eta_B \diamond f = f, \qquad f \diamond \eta_A = f. \tag{3.5}$$

(The proof replaces f in (A3) by id_A and g by f.)

(2) An algebraic theory $\mathbf{T}$ induces a new category $\mathbf{K_T}$, the *Kleisli category of T*, as follows:

(a) *objects*: the same as $|\mathbf{K}|$;
(b) *morphisms*: $\mathbf{K}$ morphisms of the form $f : A \to T(B)$;
(c) *composition*: the clone composition $\diamond$;
(d) *identities*: the components of η.

(3) Each $\mathbf{K}$ morphism $f : A \to B$ induces a $\mathbf{K}$ morphism $T(f) : T(A) \to T(B)$, lifting f, by

$$T(f) = f^{\Delta} \diamond \mathrm{id}_{T(A)}. \tag{3.6}$$

In fact, $T : \mathbf{K} \to \mathbf{K}$ is a functor and η is a natural transformation from $\mathrm{Id}_{\mathbf{K}}$ to T.

Convention 3.3. It is convenient to call $\mathbf{T} \equiv (T, \eta, \diamond)$ satisfying (D1)–(D3) a *theory*, usually in the context of it being a candidate for an algebraic theory.

3.2. Axioms for powerset theories

Definition 3.4 (inventory of conditions related to powerset theories). Let a category $\mathbf{K}$ be given, called a *ground category*, and let $\mathscr{C} \subset \mathbf{SQuant}$. Consider the following conditions:

(P1) *powerset generator*: $P : |\mathbf{K}| \to |\mathscr{C}|$ is an object-mapping.

(P2) *forward/image powerset operator*: assuming (P1), there is an operator $\to$ such that for each $f : A \to B$ in $\mathbf{K}$, there exists $f_{\mathbf{P}}^{\to} : P(A) \to P(B)$ in **PreSet**.

(P3) *backward/preimage powerset operator*: assuming (P1), there is an operator $\leftarrow$ such that for each $f : A \to B$ in $\mathbf{K}$, there exists $f_{\mathbf{P}}^{\leftarrow} : P(A) \leftarrow P(B)$ in **PreSet**.

(Ad) *adjunction*: assuming (P1)–(P3), for each $f : A \to B$ in $\mathbf{K}$,

$$f_{\mathbf{P}}^{\to} \dashv f_{\mathbf{P}}^{\leftarrow}; \tag{3.7}$$

(C) *concreteness*: assuming (P1), (P2), there exists a *concrete functor* $V : \mathbf{K} \to \mathbf{Set}$ and an *insertion map* η which determines for each $A \in |\mathbf{K}|$ a **Set** morphism

$$\eta_A : V(A) \longrightarrow P(A). \tag{3.8}$$

(N) *naturality*: assuming (P1), (P2), (C), and $f : A \to B$ in $\mathbf{K}$, then in **Set**

$$f_{\mathbf{P}}^{\to} \circ \eta_A = n_B \circ V(f). \tag{3.9}$$

(QT) *q(uasi)-topological criterion*: assuming (P1), (P3), this criterion comprises the following conditions:

(QT1) for each $f : A \to B$ in $\mathbf{K}$, $f_{\mathbf{P}}^{\leftarrow} : P(A) \leftarrow P(B)$ is in $\mathscr{C}$.

(T2) for each $f : A \to B$ in $\mathbf{K}$, for each $g : B \to C$ in $\mathbf{K}$, $(g \circ f)_{\mathbf{P}}^{\leftarrow} = f_{\mathbf{P}}^{\leftarrow} \circ g_{\mathbf{P}}^{\leftarrow}$.

(T3) for each A in $\mathbf{K}$, $(\mathrm{id}_A)_{\mathbf{P}}^{\leftarrow} = \mathrm{id}_{P(A)}$.

(T) *topological criterion*: this criterion comprises the same conditions as in (QT), but assumes that $\mathscr{C} \subset \mathbf{USQuant}$ and relabels (QT1) as (T1).

Definition 3.5 (axioms for powerset theories). Let a category $\mathbf{K}$ be given, called a *ground category*, and let $\mathscr{C} \subset \mathbf{SQuant}$.

(1) $\mathbf{P} \equiv (P, \rightarrow)$ is a *forward $\mathscr{C}$-powerset theory in K* if (P1), (P2) are satisfied.
(2) $\mathbf{P} \equiv (P, \leftarrow)$ is a *backward $\mathscr{C}$-powerset theory in K* if (P1), (P3) are satisfied.
(3) $\mathbf{P} \equiv (P, \rightarrow, \leftarrow)$ is a *balanced $\mathscr{C}$-powerset theory in K* if (P1)–(P3) are satisfied.
(4) $\mathbf{P} \equiv (P, \rightarrow, \leftarrow)$ is an *adjunctive $\mathscr{C}$-powerset theory in K* if (P1)–(P3), (Ad) are satisfied.
(5) $\mathbf{P} \equiv (P, \rightarrow, V, \eta)$ is a *concrete $\mathscr{C}$-powerset theory in K* if (P1), (P2), (C) are satisfied; and $\mathbf{P}$ is *natural* if additionally (N) is satisfied.
(6) $\mathbf{P} \equiv (P, \leftarrow)$ is a *q(uasi)-topological* [*topological*] *$\mathscr{C}$-powerset theory in K* if (P1), (P3), (QT) [(P1), (P3), (T), resp.] are satisfied.

"$\mathscr{C}$", "$\mathbf{K}$", "forward", "backward", and so forth may be dropped if they are understood. Modifiers may be combined to define many other powerset theories; for example, an *adjunctive natural topological powerset theory* satisfies all the conditions of Definition 3.4. A label is applied to a powerset theory when the appropriate parts of its structure satisfy the axioms for that label; for example, if $\mathbf{P} \equiv (P, \rightarrow, \leftarrow, V, \eta)$ is a topological $\mathscr{C}$-powerset theory in $\mathbf{K}$, this means that $(P, \leftarrow)$ is a topological $\mathscr{C}$-powerset theory in $\mathbf{K}$, in which case $\mathbf{P} \equiv (P, \leftarrow)$ may also be written if there is no confusion. If one part of $\mathbf{P}$'s structure creates another part of $\mathbf{P}$'s structure, $\mathbf{P}$ may be written with both parts or with only the generating part; for example, if $\rightarrow$ creates $\leftarrow$, then $\mathbf{P} \equiv (P, \rightarrow)$ or $\mathbf{P} \equiv (P, \rightarrow, \leftarrow)$ may be written. These conventions apply throughout this paper and particularly to the following proposition.

Proposition 3.6. *Let* $\mathbf{K}$ *be a ground category,* $\mathscr{C} \subset$ **SQuant** [**USQuant**], *and P satisfy (P1). The following hold:*
(1) *If (P2) is satisfied, then* $\langle$*for each* $f : A \to B$ *in* $\mathbf{K}$, $f_{\mathbf{P}}^{\rightarrow} : P(A) \to P(B)$ *is in* $\mathbf{CSLat}(\bigvee)\rangle$ *if and only if* $\langle$*for each* $f : A \to B$ *in* $\mathbf{K}$, $f_{\mathbf{P}}^{\leftarrow} : P(A) \leftarrow P(B)$ *is uniquely determined such that* $\mathbf{P} \equiv (P, \rightarrow, \leftarrow)$ *is an adjunctive powerset theory*$\rangle$.
(2) *If (P3) is satisfied, then* $\langle$*for each* $f : A \to B$ *in* $\mathbf{K}$, $f_{\mathbf{P}}^{\leftarrow} : P(A) \leftarrow P(B)$ *is in* $\mathbf{CSLat}(\bigwedge)\rangle$ *if and only if* $\langle$*for each* $f : A \to B$ *in* $\mathbf{K}$, $f_{\mathbf{P}}^{\rightarrow} : P(A) \to P(B)$ *is uniquely determined such that* $\mathbf{P} \equiv (P, \rightarrow, \leftarrow)$ *is an adjunctive powerset theory*$\rangle$.
(3) *There is an operator* $\leftarrow$ *such that* $\mathbf{P} \equiv (P, \leftarrow)$ *is a q-topological [topological] powerset theory if and only if the object mapping P extends to a contravariant functor* $P_{\leftarrow} : \mathbf{K} \to \mathscr{C}$.

Proof. Both (1) and (2) are a corollary of AFT Theorem 1.10. For necessity in (3), assume that $\mathbf{P}$ is a q-topological [topological] $\mathscr{C}$-powerset theory in $\mathbf{K}$, define $P_{\leftarrow} : \mathbf{K} \to \mathscr{C}$ by

$$P_{\leftarrow}(A) = P(A), \qquad P_{\leftarrow}(f : A \longrightarrow B) = f_{\mathbf{P}}^{\leftarrow} : P(A) \longleftarrow P(B), \tag{3.10}$$

and note that $P_{\leftarrow}$ being a contravariant functor is immediate from (QT) [(T)]; and for sufficiency in (3), assume that the object mapping P extends to a contravariant functor $P_{\leftarrow} : \mathbf{K} \to \mathscr{C}$, define the operator $\leftarrow$ for $f : A \to B$ in $\mathbf{K}$ by

$$f_{\mathbf{P}}^{\leftarrow} = P_{\leftarrow}(f), \tag{3.11}$$

and note that $\mathbf{P} \equiv (P, \leftarrow)$ being a q-topological [topological] $\mathscr{C}$-powerset theory in $\mathbf{K}$ is immediate from $P_{\leftarrow}$ being a contravariant functor. $\square$

Comparing Proposition 3.6(3) with Remark 3.2(3), a simple observation is that algebraic theories are associated with covariant functors, and topological powerset theories in our sense (which give rise to topological theories in Definition 3.7 and Theorem 3.10) are essentially contravariant functors; and in Section 3.6 we compare our approach to q-topological powerset theories with topological theories in the sense of [6], the latter using covariant functors. The role of q-topological and topological powerset theories, and hence the justification of the modifiers "q-topological" and "topological" in Definitions 3.4 and 3.5, is clarified in the next two sections.

3.3. Axioms for topological and fuzzy topological theories

Definition 3.7 (axioms for quasi-topological and topological theories). Let $\mathscr{C} \subset \mathbf{SQuant}$. A *q(uasi)-topological theory* $\mathbf{T_{KP}}$ of a backward $\mathscr{C}$-powerset theory $\mathbf{P}$ in a ground category $\mathbf{K}$ is a collection of objects and morphisms satisfying the following axioms:

(1) *Objects*: (A,τ), where $A \in |\mathbf{K}|$ and the *q(uasi)-topology* $\tau \subset P(A)$ is closed under arbitrary $\bigvee$ and binary $\otimes$.

(2) *Morphisms*: $f : (A,\tau) \to (B,\sigma)$, where $f \in \mathbf{K}(A,B)$ and $\tau \supset (f_{\mathbf{P}}^{\leftarrow})^{\to}(\sigma)$ (i.e., f is *continuous*), where $(f_{\mathbf{P}}^{\leftarrow})^{\to} : \wp(P(A)) \leftarrow \wp(P(B))$ indicates the traditional forward powerset operator of $f_{\mathbf{P}}^{\leftarrow} : P(A) \leftarrow P(B)$.

(3) *Composition*: inherited from $\mathbf{K}$, that is, $f \in \mathbf{T_{KP}}((A,\tau),(B,\sigma))$ and $g \in \mathbf{T_{KP}}((B,\sigma),(C,\upsilon)) \Rightarrow g \circ f \in \mathbf{T_{KP}}((A,\tau),(C,\upsilon))$.

(4) *Identities*: inherited from $\mathbf{K}$, that is,

$$(A,\tau) \in |\mathbf{T_{KP}}| \Longrightarrow \mathrm{id}_A \in \mathbf{T_{KP}}((A,\tau),(A,\tau)). \tag{3.12}$$

A q-topological theory $\mathbf{T_{KP}}$ is a *topological theory* if $\mathscr{C} \subset \mathbf{USQuant}$ and each object (A,τ) additionally has the property that $e \in \tau$, where e is the unit of $P(A)$.

Definition 3.8 (axioms for quasi-fuzzy topological and fuzzy topological theories). Let $\mathscr{C} \subset \mathbf{SQuant}$. A *q(uasi)-fuzzy topological theory* $\mathbf{T}_{F\mathbf{KP}}$ of a backward $\mathscr{C}$-powerset theory $\mathbf{P}$ in a *fuzzy ground category* $\mathbf{K}$ of the form

$$\mathbf{K} = \mathbf{K}_s \times \mathbf{K}_l, \tag{3.13}$$

where $\mathbf{K}_s$, $\mathbf{K}_l$ are categories and $\mathbf{K}_l \subset \mathscr{C}^{\mathrm{op}}$, is a collection of objects and morphisms satisfying the following axioms:

(1) *Objects*: $(A,\mathscr{T})$, where $A \equiv (A_s,A_l) \in |\mathbf{K}|$ and the *q(uasi)-fuzzy topology* $\mathscr{T} : P(A) \to A_l$ satisfies the following conditions:

(O1) *Union condition*: for each indexing set J, for each $\{u_j : j \in J\} \subset P(A)$,

$$\bigwedge_{j\in J} \mathscr{T}(u_j) \leq \mathscr{T}\left(\bigvee_{j\in J} u_j\right), \tag{3.14}$$

(O2) *Intersection condition*: for each indexing set J with $|J| = 2$, for each $\{u_j : j \in J\} \subset P(A)$,

$$\bigotimes_{j\in J} \mathcal{T}(u_j) \leq \mathcal{T}\left(\bigotimes_{j\in J} u_j\right). \tag{3.15}$$

(2) *Morphisms*: $(f,\phi) : (A,\mathcal{T}) \to (B,\mathcal{S})$, where $(f,\phi) \in \mathbf{K}(A,B)$ and

$$\mathcal{T} \circ (f,\phi)^{\leftarrow}_{\mathbf{P}} \geq \phi^{\mathrm{op}} \circ \mathcal{S} \tag{3.16}$$

on $P(B)$ (i.e., (f,ϕ) is *fuzzy continuous*), where $(f,\phi)^{\leftarrow}_{\mathbf{P}} : P(A) \leftarrow P(B)$ is the preimage powerset operator given by $\mathbf{P}$.

(3) *Composition*: inherited from $\mathbf{K}$, that is, $(f,\phi) \in \mathbf{T}_{F\mathbf{KP}}((A,\mathcal{T}),(B,\mathcal{S}))$ and $(g,\psi) \in \mathbf{T}_{F\mathbf{KP}}((B,\mathcal{S}),(C,\mathcal{U}))$ imply that

$$(g,\psi) \circ (f,\phi) \in \mathbf{T}_{F\mathbf{KP}}((A,\mathcal{T}),(C,\mathcal{U})). \tag{3.17}$$

(4) *Identities*: inherited from $\mathbf{K}$, that is,

$$(A,\mathcal{T}) \in |\mathbf{ß}| \Longrightarrow \mathrm{id}_A \in \mathbf{T}_{F\mathbf{KP}}((A,\mathcal{T}),(A,\mathcal{T})). \tag{3.18}$$

A q-fuzzy topological theory $\mathbf{T}_{F\mathbf{KP}}$ is a *fuzzy topological theory* in $\mathbf{K}$ if $\mathcal{C} \subset$ **USQuant** and each object $(A,\mathcal{T})$ in (1) additionally satisfies the following:

(O3) *Space condition*: $\mathcal{T}(e_{P(A)}) = e_{A_l}$, where $e_{P(A)}$ is the unit of $P(A)$ and e_{A_l} is the unit of A_l.

In Definition 3.8, the "s" in the subscript stands for "set" since A_s is "acting like" a set, and the "l" in the subscript stands for "lattice" since A_l is "acting like" a lattice of membership values for A_s.

How are topological theories related to fuzzy topological theories? If $\mathcal{C} \subset$ **USQuant** and $\mathbf{P}$ is a backward $\mathcal{C}$-powerset theory in $\mathbf{K}$, then each topological theory $\mathbf{T}_{\mathbf{KP}}$ in $\mathbf{K}$ is categorically isomorphic to the fuzzy topological theory $\mathbf{T}_{F(\mathbf{K}\times\mathbf{2})\mathbf{P}}$ in $\mathbf{K}\times\mathbf{2}^{\mathrm{op}}$, where $\mathbf{2}$ is the singleton category whose sole object is 2 and whose sole morphism is id_2, via $G_\chi : \mathbf{T}_{\mathbf{KP}} \to \mathbf{T}_{F(\mathbf{K}\times\mathbf{2}^{\mathrm{op}})\mathbf{P}}$ defined by

$$G_\chi(A,\tau) = (A,\chi_\tau), \qquad G_\chi(f) = (f,\mathrm{id}_2), \tag{3.19}$$

where $\chi_\tau : P(A) \to 2$ is the characteristic mapping associated with $\tau \subset P(A)$. For lattices more general than 2, G_χ may be constructed along the lines of [2, 6.2.3(1)] (using the notion of subbasis introduced in the proof of Theorem 3.11 for L not a strictly two-sided quantale), but G_χ weakens to a bireflective embedding not necessarily an isomorphism. But the case briefly considered here gives the motivation behind fuzzy topological theories vis-a-vis topological theories, namely to capture the predicate of openness [37, 38, 1, 40–42, 2]. Such relationships are related to the next definition.

Definition 3.9 (axioms for L-quasi-fuzzy topological and L-fuzzy topological theories). Let $\mathcal{C} \subset$ **SQuant** [**USQuant**]. An *L-q(uasi)-fuzzy topological theory* [*L-fuzzy topological theory*] $\mathbf{T}_{LF\mathbf{KP}}$ of a backward $\mathcal{C}$-powerset theory $\mathbf{P}$ in a ground category $\mathbf{K}$ is a q-fuzzy topological theory [fuzzy topological theory] of **P** in the fuzzy ground category $\mathbf{K} \times \mathbf{L}^{\mathrm{op}}$ in the sense of Definition 3.8, where $L \in |\mathscr{C}|$ and **L** is the singleton category whose sole object is L and whose sole morphism is id_L (from $\mathscr{C}$).

For reasons that will be apparent in Section 3.4, it is *easier* to define the variable-basis version of a fuzzy topological theory *first*, and *then* define the *fixed-basis* version in terms of the variable-basis, than the other way round.

3.4. Topological and fuzzy topological theories as topological categories. The basic theme of this section is that topological theories of topological powerset theories are in fact topological categories, justifying the label "topological".

Theorem 3.10 ((q-)topological theories and topological categories). *Let* **K** *be a ground category, let* **P** *be a backward* $\mathscr{C}$*-powerset theory in* **K**, *and let* $\mathbf{T_{KP}}$ *be a* q*-topological [topological] theory of* **P** *in* **K**. *Then* $\mathbf{T_{KP}}$ *is a fibre-small category topological over* **K** *with respect to the forgetful functor* $V_{\mathbf{KP}} : \mathbf{T_{KP}} \to \mathbf{K}$ *defined by*

$$V_{\mathbf{KP}}(A,\tau) = A, \qquad V_{\mathbf{KP}}(f) = f \tag{3.20}$$

provided that **P** *is a* q*-topological [topological] powerset* $\mathscr{C}$*-powerset theory in* **K**.

Proof. Since the topological case is a corollary of the q-topological case, we prove only the latter. Axiom (T2) guarantees that the composition axiom of Definition 3.7 is satisfied; and (T3) guarantees that the identity axiom of Definition 3.7 is satisfied. Further,

$$\tau(A) \equiv \{\tau \subset P(A) : (A,\tau) \in |\mathbf{T_{KP}}|\} \subset \wp(P(A)). \tag{3.21}$$

Hence $\mathbf{T_{KP}}$ is a fibre-small category.

The proof that $\mathbf{T_{KP}}$ is topological over **K** with respect to $V_{\mathbf{KP}}$ is analogous to the proof that **Top** is a topological construct, so we only comment on the most important steps, leaving the details to the reader.

Step 1. $\mathbf{T_{KP}}$ *has complete fibres*. For each $A \in \mathbf{K}$, it can be shown that the fibre $\tau(A)$ is a complete sublattice of $\wp(P(A))$ with respect to inclusion as the order and with meets equal to intersections.

Step 2. Subbasic continuity holds if and only if continuity holds. For each $f : V_{\mathbf{KP}}(A,\tau_1) \to V_{\mathbf{KP}}(B,\tau_2)$ in **K** with $\sigma \subset P(B)$ and

$$\tau_2 = \langle\langle\sigma\rangle\rangle \equiv \bigcap\{\tau \in \tau(B) : \sigma \subset \tau\}, \tag{3.22}$$

σ may be called a *subbasis for* τ_2, we have that $f : (A,\tau_1) \to (B,\tau_2)$ is in $\mathbf{T_{KP}}$ (i.e., f is continuous) if and only if $\tau_1 \supset (f^{\leftarrow})^{\rightarrow}(\sigma)$ (i.e., f is *subbasic continuous*). The proof hinges around the fact that

$$\{v \in P(B) : f_{\mathbf{P}}^{\leftarrow}(v) \in \tau_1\} \tag{3.23}$$

is a topology on B containing σ and hence containing τ_2, and the proof of this fact follows directly from Step 1 and axiom (QT1).

Step 3. Each $V_{\mathbf{KP}}$-structured source $(f_i : A \to V_{\mathbf{P}}(A_i,\tau_i))_{i\in I}$ has a unique, initial lift in $\mathbf{T}_{\mathbf{KP}}$. The lift $(f_i : (A,\tau) \to (A_i,\tau_i))_{i\in I}$ comes from choosing

$$\bigcup_{i\in I} \left((f_i)^{\leftarrow}_{\mathbf{P}} \right)^{\to} (\tau_i) \tag{3.24}$$

as the subbasis (using Step 2) for the topology τ on A: clearly all the f_i's are continuous. That this lift is initial is a consequence of Step 2 and (T2); and its uniqueness is a consequence of axiom (T3).

□

Theorem 3.11 ((q-)fuzzy topological theories and topological categories). *Let $\mathbf{K} \equiv \mathbf{K}_s \times \mathbf{K}_l$ be a fuzzy ground category in the sense of Definition 3.8, let $\mathbf{P}$ be a backward $\mathscr{C}$-powerset theory in $\mathbf{K}$, and let $\mathbf{T}_{F\mathbf{KP}}$ be a q-fuzzy topological [fuzzy topological] theory of $\mathbf{P}$. Then $\mathbf{T}_{F\mathbf{KP}}$ is a fibre-small category topological over $\mathbf{K}$ with respect to the forgetful functor $V_{F\mathbf{KP}} : \mathbf{T}_{F\mathbf{KP}} \to \mathbf{K}$ defined by*

$$V_{F\mathbf{KP}}(A,\mathscr{T}) = A, \qquad V_{F\mathbf{KP}}(f,\phi) = (f,\phi) \tag{3.25}$$

*provided that $\mathscr{C} \subset$ **OSQuant** [**UOSQuant**] and $\mathbf{P}$ is a q-topological [topological] $\mathscr{C}$-powerset theory in $\mathbf{K}$.*

Proof. We prove only the q-fuzzy topological case. Axiom (T2) guarantees that the composition axiom of Definition 3.8 is satisfied; and (T3) guarantees that the identity axiom of Definition 3.8 is satisfied. Further,

$$\mathscr{T}(A) \equiv \{\mathscr{T} : P(A) \longrightarrow A_l \mid (A,\mathscr{T}) \in |\mathbf{T}_{F\mathbf{KP}}|\} \subset A_l^{P(A)}. \tag{3.26}$$

Hence $\mathbf{T}_{F\mathbf{KP}}$ is a fibre-small category.

The proof that $\mathbf{T}_{F\mathbf{KP}}$ is topological over $\mathbf{K}$ with respect to $V_{F\mathbf{KP}}$ is analogous to the proof in [2] that $\mathbf{K}_L$-**FTop** is topological over $\mathbf{K}$ with respect to the usual forgetful functor (see [2, Theorem 3.3.9] and its supporting lemmas), so we only comment on the most important steps, leaving the details to the reader.

Step 1. $\mathbf{T}_{F\mathbf{KP}}$ has complete fibres. For each $A \in \mathbf{K}$, it can be shown that the fibre $\mathscr{T}(A)$ is a complete sublattice of $A_l^{P(A)}$ with respect to the following: the order and meets are the liftings of those of A_l. The details are straightforward, analogous to the proof of [1, Proposition 2.3], and make necessary use of the isotonicity of $\otimes$.

Step 2. Fuzzy subbasic continuity holds if and only if fuzzy continuity holds. For each $(f,\phi) : V_{F\mathbf{KP}}(A,\mathscr{T}_1) \to V_{F\mathbf{P}}(B,\mathscr{T}_2)$ in $\mathbf{K}$ with $\mathscr{S} : P(B) \to B_l$ such that

$$\mathscr{T}_2 = \langle\langle \mathscr{S} \rangle\rangle \equiv \bigwedge \{\mathscr{T} \in \mathscr{T}(B) : \mathscr{S} \le \mathscr{T}\}, \tag{3.27}$$

$\mathscr{S}$ may be called a *subbasis for* $\mathscr{T}_2$, $(f,\phi) : (A,\mathscr{T}_1) \to (B,\mathscr{T}_2)$ is in $\mathbf{T}_{F\mathbf{KP}}$ (i.e., (f,ϕ) is fuzzy continuous) if and only if $\mathscr{T}_1 \circ (f,\phi)^{\leftarrow}_{\mathbf{P}} \ge \phi^{\mathrm{op}} \circ \mathscr{S}$ (i.e., (f,ϕ) is *subbasic continuous*). The proof hinges around the fact that

$$\phi^{\vdash} \circ \mathscr{T}_1 \circ (f,\phi)^{\leftarrow}_{\mathbf{P}}, \tag{3.28}$$

where $\phi^{\vdash} \equiv (\phi^{\mathrm{op}})^{\vdash}$ (see Definition 1.12), is a fuzzy topology on B containing $\mathcal{S}$ and hence containing $\mathcal{T}_2$. The proof of this step follows from Step 1 and axiom (QT1) and is analogous to the proofs of [2, Theorems 3.2.12(1), (2), (3), (4), and 3.2.13((3)⇒(1))].

Step 3. Each $V_{F\mathbf{KP}}$-structured source $((f_i,\phi_i) : A \to V_{F\mathbf{KP}}(A_i,\mathcal{T}_i))_{i\in I}$ has a unique, initial lift in $\mathbf{K}_{F\mathbf{P}}$. The lift $((f_i,\phi_i) : (A,\mathcal{T}) \to (A_i,\mathcal{T}_i))_{i\in I}$ comes from choosing a subbasis $\mathcal{S} : P(A) \to A_I$ for $\mathcal{T}$ (using Step 2) by defining $\mathcal{S}$ at $a \in P(A)$ as follows:

$$\mathcal{S}(a) = \begin{cases} \displaystyle\bigvee_{i\in I,\, b\in ((f_i,\phi_i)^{\overleftarrow{}}_{\mathbf{P}})^{\leftarrow}(\{a\})} \phi_i^{\mathrm{op}}(\mathcal{T}_i(b)), & \exists i \in I,\ \left((f_i,\phi_i)^{\overleftarrow{}}_{\mathbf{P}}\right)^{\leftarrow}(\{a\}) \neq \varnothing, \\ \bot, & \exists i \in I,\ \left((f_i,\phi_i)^{\overleftarrow{}}_{\mathbf{P}}\right)^{\leftarrow}(\{a\}) = \varnothing, \end{cases} \tag{3.29}$$

where $b \in ((f_i,\phi_i)^{\overleftarrow{}}_{\mathbf{P}})^{\leftarrow}(\{a\})$ means that $(f_i,\phi_i)^{\overleftarrow{}}_{\mathbf{P}}(b) = a$, that is, $((f_i,\phi_i)^{\overleftarrow{}}_{\mathbf{P}})^{\leftarrow}$ is the traditional backward powerset operator of the backward powerset operator $(f_i,\phi_i)^{\overleftarrow{}}_{\mathbf{P}}$ from powerset theory $\mathbf{P}$. It can now be shown that each (f_i,ϕ_i) is fuzzy continuous by a proof analogous to that given for [2, Lemma 3.3.5]. That this lift is initial is a consequence of Step 2 and (T2) and is analogous to the proof of [2, Lemma 3.3.6]; and the uniqueness of this lift is a consequence of axiom (T3) and is analogous to the proof of [2, Lemma 3.3.8]. □

THEOREM 3.12. *Let $\mathbf{K}$ be a ground category, let $\mathbf{P}$ be a backward $\mathcal{C}$-powerset theory in $\mathbf{K}$, let $L \in |\mathcal{C}|$, and let $\mathbf{K}_{LF\mathbf{KP}}$ be an L-q-fuzzy topological [L-fuzzy topological] theory of $\mathbf{P}$. Then $\mathbf{T}_{LF\mathbf{KP}}$ is a fibre-small category topological over $\mathbf{K}$ with respect to the forgetful functor $V_{LF\mathbf{KP}} : \mathbf{T}_{LF\mathbf{KP}} \to \mathbf{K}$ is defined by*

$$V_{LF\mathbf{KP}}(A,\mathcal{T}) = A, \qquad V_{LF\mathbf{KP}}(f) = f \tag{3.30}$$

*provided that $\mathcal{C} \subset$ **OSQuant** [**UOSQuant**] and $\mathbf{P}$ is a q-topological [topological] $\mathcal{C}$-powerset theory in $\mathbf{K}$.*

Proof. We prove only the L-q-fuzzy topological case. It is immediate from Theorem 3.11 that $\mathbf{T}_{LF\mathbf{KP}}$ is a fibre-small category. Now let the q-topological set theory $\mathbf{P}$ be denoted by $(P,\leftarrow)$ and define $\mathbf{Q} \equiv (Q,\leftarrow)$ in $\mathbf{K}\times\mathbf{L}^{\mathrm{op}}$ as follows:

(P1) $Q : |\mathbf{K}\times\mathbf{L}| \to |\mathcal{C}|$ by $Q(A,L) = P(A)$.

(P3) For each $(f,\mathrm{id}_L) : (A,L) \to (B,L)$ in $\mathbf{K}\times\mathbf{L}$, define $(f,\mathrm{id}_L)^{\overleftarrow{}}_{\mathbf{Q}} = f^{\leftarrow}_{\mathbf{P}} : Q(A,L) \leftarrow Q(B,L)$.

Since $\mathbf{P}$ is a backward $\mathcal{C}$-powerset theory in $\mathbf{K}$, it follows immediately that $\mathbf{Q}$ is a backward $\mathcal{C}$-powerset theory in $\mathbf{K}\times\mathbf{L}^{\mathrm{op}}$. Furthermore, it can be seen that since $\mathbf{P}$ is q-topological in $\mathbf{K}$, $\mathbf{Q}$ is q-topological in the fuzzy ground category $\mathbf{K}\times\mathbf{L}^{\mathrm{op}}$: the axiom (QT1) is immediately satisfied for $\mathbf{Q}$; concerning axiom (T2), we have

$$\begin{aligned} \left[(g,\mathrm{id}_L)\circ(f,\mathrm{id}_L)\right]^{\overleftarrow{}}_{\mathbf{Q}} &= (g\circ f,\mathrm{id}_L\circ\mathrm{id}_L)^{\overleftarrow{}}_{\mathbf{Q}} = (g\circ f,\mathrm{id}_L)^{\overleftarrow{}}_{\mathbf{Q}} \\ &= (g\circ f)^{\overleftarrow{}}_{\mathbf{P}} = f^{\leftarrow}_{\mathbf{P}}\circ g^{\leftarrow}_{\mathbf{P}} = (f,\mathrm{id}_L)^{\overleftarrow{}}_{\mathbf{Q}}\circ(g,\mathrm{id}_L)^{\overleftarrow{}}_{\mathbf{Q}}; \end{aligned} \tag{3.31}$$

and concerning axiom (T3), we have

$$\left(\mathrm{id}_{(A,L)}\right)^{\overleftarrow{}}_{\mathbf{Q}} = \left(\mathrm{id}_A,\mathrm{id}_L\right)^{\overleftarrow{}}_{\mathbf{Q}} = \left(\mathrm{id}_A\right)^{\overleftarrow{}}_{\mathbf{P}} = \mathrm{id}_{P(A)} = \mathrm{id}_{Q(A,L)}. \tag{3.32}$$

Since $\mathbf{Q}$ is a q-topological set theory in the fuzzy ground category $\mathbf{K} \times \mathbf{L}^{\mathrm{op}}$, it follows from Theorem 3.11 that $\mathbf{T}_{F(\mathbf{K}\times\mathbf{L}^{\mathrm{op}})\mathbf{Q}}$ is a category which is topological over $\mathbf{K} \times \mathbf{L}^{\mathrm{op}}$ with respect to the forgetful functor $V_{F(\mathbf{K}\times\mathbf{L}^{\mathrm{op}})\mathbf{Q}} : \mathbf{T}_{F(\mathbf{K}\times\mathbf{L}^{\mathrm{op}})\mathbf{Q}} \to \mathbf{K} \times \mathbf{L}^{\mathrm{op}}$ defined by

$$V_{F(\mathbf{K}\times\mathbf{L}^{\mathrm{op}})\mathbf{Q}}((A,L),\mathcal{T}) = (A,L), \qquad V_{F(\mathbf{K}\times\mathbf{L}^{\mathrm{op}})\mathbf{Q}}(f,\mathrm{id}_L) = (f,\mathrm{id}_L). \tag{3.33}$$

Now we observe the following: the obvious functor H is a categorical isomorphism from $\mathbf{K} \times \mathbf{L}^{\mathrm{op}}$ to $\mathbf{K}$; the obvious functor J is a categorical isomorphism from $\mathbf{T}_{F(\mathbf{K}\times\mathbf{L}^{\mathrm{op}})\mathbf{Q}}$ to $\mathbf{T}_{LF\mathbf{KP}}$; and H, J yield that

$$V_{LF\mathbf{KP}} = H \circ V_{F(\mathbf{K}\times\mathbf{L}^{\mathrm{op}})\mathbf{Q}} \circ J^{-1}. \tag{3.34}$$

Since $\mathbf{T}_{F(\mathbf{K}\times\mathbf{L}^{\mathrm{op}})\mathbf{Q}}$ is topological over $\mathbf{K} \times \mathbf{L}^{\mathrm{op}}$ with respect to $V_{F(\mathbf{K}\times\mathbf{L}^{\mathrm{op}})\mathbf{Q}}$, it follows from [2, Proposition 1.3.1] that $\mathbf{T}_{LF\mathbf{KP}}$ is topological over $\mathbf{K}$ with respect to $V_{LF\mathbf{KP}}$. This concludes the proof of the theorem. □

Comment 3.13. The extent to which topological powerset theories are generated by algebraic theories may be viewed as one measure of the extent to which topology (and fuzzy topology) has an algebraic foundation. The question of determining this extent is part of the larger question as to the extent to which powerset theories are generated by algebraic theories, a question studied at length in Sections 5 and 6 after some preliminaries in Section 4. The rest of this section is concerned with examples (Section 3.5) of the various theories defined above as well the close relationship (Section 3.6) of topological theories as defined above with those defined in [6].

3.5. Examples of algebraic, powerset, topological, fuzzy topological theories

Example 3.14. Each semigroup induces an algebraic theory in $\mathbf{K} \equiv \mathbf{Set}$. See Section 2.1 for details and [3] for many other such examples.

Example 3.15 (cf. [3, Examples 1.3.5, 4.3.3]). Traditional powersets collectively form an algebraic theory in $\mathbf{K} = \mathbf{Set}$. Define $\mathbf{T} = (T, \eta, \diamond)$ as follows:

(D1) For each $X \in |\mathbf{Set}|$, define $T(X) = \wp(X)$.

(D2) For each $X \in |\mathbf{Set}|$, define $\eta_X : X \to \wp(X)$ by

$$\eta_X(x) = \{x\}. \tag{3.35}$$

(D3) For each $f : X \to \wp(Y)$, $g : Y \to \wp(Z)$, define $g \diamond f : X \to \wp(Z)$ by

$$(g \diamond f)(x) = \bigcup_{y \in f(x)} g(y). \tag{3.36}$$

Sequens, we may speak of a theory constructed in the manner of Example 3.15 as being of *standard construction*. The following theorem concerns the theory $\mathbf{T}$ constructed in Example 3.15.

Theorem 3.16. $\mathbf{T} = (T, \eta, \diamond)$ *is an algebraic theory in* $\mathbf{Set}$.

Proof. Ad(A1) Let $f : X \to \wp(Y), g : Y \to \wp(Z), h : Z \to \wp(W)$. Then $h \diamond (g \diamond f), (h \diamond g) \diamond f : X \to \wp(W)$. Now

$$\begin{aligned}
[h \diamond (g \diamond f)](x) &= \bigcup \Big\{ h(z) : z \in \bigcup \{ g(y) : y \in f(x) \} \Big\}, \\
[(h \diamond g) \diamond f](x) &= \bigcup \Big\{ \bigcup \{ h(z) : z \in g(y) \} : y \in f(x) \Big\}.
\end{aligned} \tag{3.37}$$

The reader can check that these are the same subset of W.

Ad(A2) Let $f : X \to \wp(Y)$ and $\eta_Y : Y \to \wp(Y)$. It is claimed that

$$\eta_Y \diamond f = f. \tag{3.38}$$

Now for $x \in X$,

$$(\eta_Y \diamond f)(x) = \bigcup_{y \in f(x)} \eta_Y(y) = \bigcup_{y \in f(x)} \{y\} = f(x). \tag{3.39}$$

Ad(A3) Let $f : X \to Y, g : Y \to \wp(Z)$. To see that

$$g \diamond f^{\Delta} = g \circ f, \tag{3.40}$$

let $x \in X$. Then

$$\begin{aligned}
(g \diamond f^{\Delta})(x) &= (g \diamond (\eta_{\wp(Y)} \circ f))(x) = \bigcup_{y \in \eta_{\wp(Y)}(f(x))} g(y) \\
&= \bigcup_{y \in \{f(x)\}} g(y) = \bigcup_{y = f(x)} g(y) = g(f(x)) = (g \circ f)(x).
\end{aligned} \tag{3.41}$$

This concludes the proof. □

Example 3.17. Traditional powerset theory in **Set**. Put $\mathbf{K} = \mathbf{Set}$, define $P : |\mathbf{K}| \to |\mathbf{CBool}|$ by

$$P(X) = \wp(X), \tag{3.42}$$

define $f_{\mathbf{P}}^{\rightarrow} : P(X) \to P(Y)$, $f_{\mathbf{P}}^{\leftarrow} : P(X) \leftarrow P(Y)$ to be the traditional powerset operators as in Section 1.2, define $V : \mathbf{K} \to \mathbf{Set}$ by $V = \mathrm{Id}_{\mathbf{Set}}$, and for $X \in |\mathbf{Set}|$, define $\eta_X : V(X) \to P(X)$ as in Section 2.1 and Example 3.15 by

$$\eta_X(x) = \{x\}. \tag{3.43}$$

The following lemma and theorem concern $\mathbf{P} \equiv (P, \rightarrow, \leftarrow, V, \eta)$ constructed in Example 3.17.

Lemma 3.18. $\mathbf{P} \equiv (P, \rightarrow, \leftarrow, V, \eta)$ *is an adjunctive, natural, topological* **CBool**-*powerset theory in* **Set**, *called the traditional powerset theory in* **Set**, *in which case*

$$\begin{aligned}
f_{\mathbf{P}}^{\rightarrow}(A) &= \bigcap \{ B \in \wp(Y) : A \subset f^{\leftarrow}(B) \}, \\
f_{\mathbf{P}}^{\leftarrow}(B) &= \bigcup \{ A \in \wp(X) : f^{\rightarrow}(A) \subset B \}.
\end{aligned} \tag{3.44}$$

Theorem 3.19. *The topological theory* $\mathbf{T}_{\mathbf{SetP}}$ *of* **P** *in* **Set** *is the topological construct* **Top** *of Section 2.3.*

Example 3.20. Fixed-basis powerset theories in **Set**. Let $\mathscr{C} \subset$ **SQuant**, put $\mathbf{K} = \mathbf{Set}$, and let $L \in |\mathscr{C}|$. Put $P : |\mathbf{K}| \to |\mathscr{C}|$ by

$$P(X) = L^X, \tag{3.45}$$

for $f : X \to Y$, put $f_{\mathbf{P}}^{\rightarrow} \equiv f_L^{\rightarrow} : L^X \to L^Y$, $f_{\mathbf{P}}^{\leftarrow} \equiv f_L^{\leftarrow} : L^X \leftarrow L^Y$ as in Definition 1.19(2); and when $L \in |\mathbf{USQuant}|$ with unit e_L, put $V : \mathbf{K} \to \mathbf{Set}$ by $V = \mathrm{Id}_{\mathbf{Set}}$, and for $X \in |\mathbf{Set}|$, define $\eta_X : V(X) \to P(X)$ by

$$\eta_X(x)(z) = \chi_{\{x\}}^{e_L}(z) \equiv \begin{cases} e_L, & z = x, \\ \perp, & z \neq x. \end{cases} \tag{3.46}$$

The following lemma and theorem concern $\mathbf{P} \equiv (P, \to, \leftarrow)$ constructed in Example 3.20.

Lemma 3.21. $\mathbf{P} \equiv (P, \to, \leftarrow)$ *is an adjunctive, q-topological* $\mathscr{C}$*-powerset theory in* **Set**, *called the L-powerset theory in* **Set**, *in which case*

$$f_{\mathbf{P}}^{\rightarrow}(a) = \bigwedge \{b \in L^Y : a \le f_L^{\leftarrow}(b)\}, \qquad f_{\mathbf{P}}^{\leftarrow}(b) = \bigvee \{a \in L^X : f_L^{\rightarrow}(a) \le b\}; \tag{3.47}$$

and if $\mathscr{C} \subset$ **USQuant**, $\mathbf{P} \equiv (P, \to, \leftarrow, V, \eta)$ *is natural and topological. Further, if* $\mathscr{C} \subset$ **Squant** [**USQuant**], *the q-topological [topological] theory* $\mathbf{T}_{\mathbf{SetP}}$ *of* **P** *in* **Set** *is* L-**QTop** [L-**Top**] *and the L-q-fuzzy [L-fuzzy] topological theory* $\mathbf{T}_{L F\mathbf{SetP}}$ *of* **P** *in* **Set** *is* L-**QFTop** [L-**FTop**].

Proof. Given the extensive literature on these operators, the only axioms needing comment are (N), (QT1), and (T1). Concerning (N), it is easy to check that $f_{\mathbf{P}}^{\rightarrow}(\chi_{\{x\}}^{e_L}) = \chi_{\{f(x)\}}^{e_L}$ for each $x \in \mathrm{dom}(f)$. Now $f_{\mathbf{P}}^{\leftarrow}$ preserves $\otimes$ (by evaluation); and additionally for the topological case, evaluation shows that

$$f_{\mathbf{P}}^{\leftarrow}(\underline{e_{L^Y}}) = \underline{e_{L^Y}} \circ f = \underline{e_{L^X}}. \tag{3.48}$$

□

Theorem 3.22 (cf. Theorem 1.26). *For* $\mathscr{C} \subset$ **Squant** [**USQuant**, **OSQuant**, **UOSQuant**] *and* $L \in |\mathscr{C}|$, L-**QTop** [L-**Top**, L-**QFTop**, L-**FTop**, *resp.*] *are topological constructs with respect to the usual forgetful functors.*

Proof. These statements are consequences of Theorems 3.10, 3.12, and Lemma 3.21. □

Example 3.23. Left-adjoint (variable-basis) powerset theory in $\mathbf{Set} \times \mathbf{C}$. Let $\mathscr{C} \subset$ **SQuant**, put $\mathbf{C} = \mathscr{C}^{\mathrm{op}}$, and put $\mathbf{K} = \mathbf{Set} \times \mathbf{C}$. The *left-adjoint (variable-basis) powerset theory*

$$\mathbf{P}(\dashv) \equiv (P, \longrightarrow, \longleftarrow) \tag{3.49}$$

in $\mathbf{Set} \times \mathbf{C}$ is given by the following data: define $P : |\mathbf{K}| \to |\mathscr{C}|$ by

$$P(X, L) = L^X, \tag{3.50}$$

and for $(f,\phi):(X,L)\to(Y,M)$, put

$$\begin{aligned}(f,\phi)^{\rightarrow}_{\mathbf{P}(\dashv)} &\equiv (f,\phi)^{\rightarrow}_{\dashv} \equiv (f,\phi)^{\rightarrow} : P(X,L) \longrightarrow P(Y,M),\\ (f,\phi)^{\leftarrow}_{\mathbf{P}(\dashv)} &\equiv (f,\phi)^{\leftarrow}_{\dashv} \equiv (f,\phi)^{\leftarrow} : P(X,L) \longleftarrow P(Y,M)\end{aligned} \tag{3.51}$$

from Definition 1.19(3), namely

$$(f,\phi)^{\rightarrow}_{\dashv}(a) = \bigwedge\{b : f_L^{\rightarrow}(a) \le \langle\phi^{\mathrm{op}}\rangle(b)\}, \qquad (f,\phi)^{\leftarrow}_{\dashv}(b) = \phi^{\mathrm{op}} \circ b \circ f, \tag{3.52}$$

or equivalently,

$$(f,\phi)^{\rightarrow}_{\dashv} = \langle\phi^{\dashv}\rangle \circ f_L^{\rightarrow}, \qquad (f,\phi)^{\leftarrow}_{\dashv} = \langle\phi^{\mathrm{op}}\rangle \circ f_M^{\leftarrow}. \tag{3.53}$$

Finally, given $\mathscr{C} \subset \mathbf{USQuant}$, put $V : \mathbf{K} \to \mathbf{Set}$ by $V = \Pi_1$ (the first projection functor), namely $V(X,L) = X$ and $V(f,\phi) = f$, and for $X \in |\mathbf{Set}|$, define $\eta_{(X,L)} : V(X,L) \to P(X,L)$ by

$$\eta_{(X,L)}(x) = \chi^{e_L}_{\{x\}} \tag{3.54}$$

as in Example 3.20; and in this case we also use the notation

$$\mathbf{P}(\dashv) \equiv (P, \longrightarrow, \longleftarrow, V, \eta). \tag{3.55}$$

The reason for the phrase "left-adjoint" in this example will be made clear in Section 6: the left-adjoint powerset theories just described arise under certain conditions from "left-adjoint" algebraic theories; and there are corresponding "right-adjoint" algebraic theories which generate new "right-adjoint" variable-basis powerset theories in $\mathbf{Set} \times \mathbf{C}$—using "right-adjoint" image and preimage operators $(f,\phi)^{\rightarrow}_{\vdash}$, $(f,\phi)^{\leftarrow}_{\vdash}$—different from the above left-adjoint powerset theories. Showing that these right-adjoint operators are distinct from the left-adjoint operators uses the *lower-left-adjoint image operator* $(f,\phi)^{\dashv}_{\rightarrow}$ guaranteed as the unique right-adjoint of $(f,\phi)^{\leftarrow}_{\dashv}$ by the AFT and the preservation of arbitrary $\bigvee$ by $(f,\phi)^{\leftarrow}_{\dashv}$ (cf. Definition 1.19(2))—see Definition 6.23 and Theorem 6.24.

The following lemma, proposition, theorem, and corollary concern $\mathbf{P} \equiv (P,\to,\leftarrow)$ constructed in Example 3.23.

Lemma 3.24. *The following hold:*

(1) *For each* $\mathscr{C} \subset \mathbf{SQuant}$ [$\mathbf{USQuant}$], $\mathbf{P}(\dashv) \equiv (P,\to,\leftarrow)$ *is a q-topological [topological]* $\mathscr{C}$*-powerset theory in* $\mathbf{Set} \times \mathbf{C}$.

(2) $\mathbf{P}(\dashv) \equiv (P,\to,\leftarrow)$ *is an adjunctive powerset theory in* $\mathbf{Set} \times \mathbf{C}$ *if and only if for each* $\phi \in \mathbf{C}$, ϕ^{op} *preserves arbitrary* $\bigwedge$.

(3) *For each* $\mathscr{C} \subset \mathbf{USQuant}$, $\mathbf{P}(\dashv) \equiv (P,\to,\leftarrow,V,\eta)$ *is a natural powerset theory in* $\mathbf{Set} \times \mathbf{C}$ *if and only if for each* $\phi \in \mathbf{C}$, $\phi^{\dashv}(e_L) = e_M$*; and if* $\mathscr{C} \subset \mathbf{ST\text{-}SQuant}$, *this latter condition holds if and only if for each* $\phi \in \mathbf{C}$, $\ker(\phi^{\mathrm{op}}) \equiv (\phi^{\mathrm{op}})^{\leftarrow}\{\top\} = \{\top\}$.

Proposition 3.25. *The predicate of the condition of Lemma 3.24(2) holds under each of the following conditions:*

(1) $\phi \in \mathbf{LoDmSQuant}(L,M)$.

(2) ϕ^{op} *is a backward Zadeh operator; that is, there exists* $N \in |\mathbf{SQuant}|$, *there exists* $g \in \mathbf{Set}(W,Z)$, $\phi^{\text{op}} = g_N^{\leftarrow}$.
(3) ϕ *is any of the examples constructed in [2, 7.1.7.2] or [34, 9.9(2(b),3)].*
(4) ϕ *is as given by* ϕ^{op} *in any of the examples of Example 1.17(1)–(10).*
(5) ϕ *is an isormorphism in* **LoSQuant**.

Lemma 3.26. *If* $\mathscr{C} \subset$ **SQuant** [**USQuant**], *the q-topological [topological] theory* $\mathbf{T}_{(\mathbf{Set}\times\mathbf{C})\mathbf{P}(\dashv)}$ *of* $\mathbf{P}(\dashv)$ *is the category* **C-QTop** [**C-Top**] *and the q-fuzzy [fuzzy] topological theory* $\mathbf{T}_{F(\mathbf{Set}\times\mathbf{C})\mathbf{P}(\dashv)}$ *of* $\mathbf{P}(\dashv)$ *is the category* **C-QFTop** [**C-FTop**].

Theorem 3.27 (cf. Theorem 1.26). *For* $\mathscr{C} \subset$ **Squant** [**USQuant**, **OSQuant**, **UOS-Quant**], **C-QTop** [**C-Top**, **C-QFTop**, **C-FTop**, *resp.*] *are topological over* $\mathbf{Set} \times \mathbf{C}$ *with respect to the usual forgetful functors.*

Proof. These statements are consequences of Theorems 3.10, 3.11, and Lemma 3.26. □

3.6. Topological theories *à la* Definition 3.7 vis-a-vis topological theories *à la* [6]. This section completely resolves the relationship between the approach to topological theories in Definition 3.7 and both that in [6] and a variation of the approach in [6].

Definition 3.28 [6]. A *topological theory* in a *ground category* **K** is a (covariant) functor $T : \mathbf{K} \to \mathbf{CSLat}(\bigvee)$.

(1) The category $\mathbf{Top}(T)$ is the concrete category over **K** defined as follows:
  (a) *Objects*: (X,t), where $X \in |\mathbf{K}|$ and $t \in T(X)$.
  (b) *Morphisms*: $f : (X,t) \to (Y,s)$, where $f : X \to Y$ in **K** and $T(f)(t) \le s$.
  (c) *Composition, identities*: from **K**.
(2) The category $\mathbf{Top}_{\text{alt}}(T)$ is the same as $\mathbf{Top}(T)$ except that the inequality defining morphisms is $T(f)(t) \ge s$. (The subscript "alt" stands for "alternative".)

It can be checked that each of $\mathbf{Top}(T)$ and $\mathbf{Top}_{\text{alt}}(T)$ is indeed a category.

Theorem 3.29. *Let T be a topological theory in* **K** *in the sense of Definition 3.28. The following hold.*

(1) $\mathbf{Top}(T)$ *is a fibre-small topological category over* **K** *with respect to the forgetful functor* $V : \mathbf{Top}(T) \to \mathbf{K}$ *given by* $V(X,t) = X$, $V(f) = f$ *[6].*
(2) $\mathbf{Top}_{\text{alt}}(T)$ *is a fibre-small topological category over* **K** *with respect to the forgetful functor* $V : \mathbf{Top}(T) \to \mathbf{K}$ *given by* $V(X,t) = X$, $V(f) = f$.

Proof. The proof of (1) is [6, Exercise 22.B(a)]; and the proof of (2) is similar except that given a source $(f_\gamma : A \to (B_\gamma, \sigma_\gamma))_{\gamma\in\Gamma}$, the lifting topology on the domain is

$$\tau = \bigwedge \left\{ t \in T(A) : T(f)(t) \ge \bigvee_{\gamma\in\Gamma} \sigma_\gamma \right\}. \tag{3.56}$$

That τ is the topology providing the unique initial lift is left to the reader. □

Lemma 3.30. *Let* **K** *be a ground category.*

(1) *Let* $\mathscr{C} = \mathbf{CSLat}(\bigvee)$ *with* $\otimes = \vee$ *(binary) for each L. Then each topological theory T in the sense of Definition 3.28 induces a topological* $\mathscr{C}$*-powerset theory* $\mathbf{P}_T$ *in the sense of Definition 3.7.*

(2) *Let* $\mathscr{C} = \mathbf{CSLat}(\bigvee)$ *with a* $\otimes$ *for each* L *(e.g.,* $\otimes$ *could be* $\vee$ *(binary) or* $\wedge$ *(binary)). Then each* q*-topological* $\mathscr{C}$*-powerset theory* $\mathbf{P}$ *in* $\mathbf{K}$ *in the sense of Definition 3.7 induces a topological theory* $T_{\mathbf{P}}$ *in the sense of Definition 3.28.*

Proof. Ad(1) Let $T : \mathbf{K} \to \mathbf{CSLat}(\bigvee)$ be a topological theory in the sense of Definition 3.28. Put $P_T : \mathbf{K} \to \mathscr{C}$ by

$$P_T(A) = T(A)^{\mathrm{op}}, \qquad P_T(f : A \longrightarrow B) = T(f)^{\vdash} : T(A)^{\mathrm{op}} \longleftarrow T(B)^{\mathrm{op}}, \tag{3.57}$$

where the superscripts "op" refer to the dual orders—which yield objects of $\mathbf{CSLat}(\bigvee)$—and $T(f)^{\vdash}$ is the right-adjoint of $T(f)$ guaranteed by AFT Theorem 1.10 with respect to the original orderings on $T(A)$ and $T(B)$. To show that

$$\mathbf{P}_T \equiv (P_T \text{ on objects}, P_T \text{ on morphisms}) \tag{3.58}$$

is a topological $\mathscr{C}$-powerset theory, it suffices by Proposition 3.6(3) to show that P_T is a contravariant functor. Clearly, with respect to the dual orders, each $P_T(f)$ preserves $\bigvee$ and $\otimes$ $(= \vee)$ and the unit $e = \perp$. Further, using Remark 1.13, we have the following:

$$\begin{aligned} P_T(g \circ f) &= T(g \circ f)^{\vdash} = \big(T(g) \circ T(f)\big)^{\vdash} \\ &= T(f)^{\vdash} \circ T(g)^{\vdash} = P_T(f) \circ P_T(g), \\ P_T\big(\mathrm{id}_A\big) &= T\big(\mathrm{id}_A\big)^{\vdash} = \big(\mathrm{id}_{T(A)}\big)^{\vdash} = \mathrm{id}_{T(A)} = \mathrm{id}_{P_T(A)}. \end{aligned} \tag{3.59}$$

Ad(2) Let $\mathbf{P} \equiv (P, \leftarrow)$ be a q-topological $\mathscr{C}$-powerset theory in $\mathbf{K}$ and put $T_{\mathbf{P}} : \mathbf{K} \to \mathbf{CSLat}(\bigvee)$ by

$$\begin{aligned} T_{\mathbf{P}}(A) &= \Big\{\tau \subset P(A) : \tau \text{ is closed under } \otimes \text{ and arbitrary } \bigvee\Big\} \subset \wp(P(A)), \\ T_{\mathbf{P}}(f : A \longrightarrow B) &= \big(f_{\mathbf{P}}^{\leftarrow}\big)^{\leftarrow}_{|T_{\mathbf{P}}(A)} : T_{\mathbf{P}}(A) \longrightarrow T_{\mathbf{P}}(B), \end{aligned} \tag{3.60}$$

where the inner arrow of $T(f)$ is the preimage operator $\leftarrow$ of $\mathbf{P}$ and the outer arrow of $T(f)$ is the traditional preimage operator of a function. To see that $T_{\mathbf{P}}(f)$ is well defined, we first note that $T_{\mathbf{P}}(f) \in \mathbf{CSLat}(\bigvee)$ since the traditional preimage operator preserves arbitrary $\bigvee$ $(= \bigcup)$. It must also be checked that $T_{\mathbf{P}}$ maps $T_{\mathbf{P}}(A)$ into $T_{\mathbf{P}}(B)$: given $\tau \in T_{\mathbf{P}}(A)$, then $T_{\mathbf{P}}(f)(\tau) \in T_{\mathbf{P}}(B)$ since

$$T_{\mathbf{P}}(f)(\tau) = \big(f_{\mathbf{P}}^{\leftarrow}\big)^{\leftarrow}_{|T_{\mathbf{P}}(A)}(\tau) = \{\nu \in P(B) : f_{\mathbf{P}}^{\leftarrow}(\nu) \in \tau\} \tag{3.61}$$

is a q-topology on $P(B)$, which holds because $f_{\mathbf{P}}^{\leftarrow}$ preserves arbitrary $\bigvee$ and $\otimes$ from (QT1) of $\mathbf{P}$ being a q-topological theory (cf. the proof of Step 2 in Theorem 3.10).

To see that T is a functor, we note from (T2), (T3) of $\mathbf{P}$ being a q-topological theory and the properties of the traditional preimage operator, ignoring the appropriate restrictions of domains, that

$$\begin{aligned} T_{\mathbf{P}}(g \circ f) &= \left((g \circ f)_{\mathbf{P}}^{\leftarrow}\right)^{\leftarrow} = \left(f_{\mathbf{P}}^{\leftarrow} \circ g_{\mathbf{P}}^{\leftarrow}\right)^{\leftarrow} \\ &= \left(g_{\mathbf{P}}^{\leftarrow}\right)^{\leftarrow} \circ \left(f_{\mathbf{P}}^{\leftarrow}\right)^{\leftarrow} = T_{\mathbf{P}}(g) \circ T_{\mathbf{P}}(f), \\ T_{\mathbf{P}}\left(\mathrm{id}_A\right) &= \left(\left(\mathrm{id}_A\right)_{\mathbf{P}}^{\leftarrow}\right)^{\leftarrow} = \left(\mathrm{id}_{P(A)}\right)^{\leftarrow} = \mathrm{id}_{\wp(P(A))} = \mathrm{id}_{T_{\mathbf{P}}(A)}. \end{aligned} \tag{3.62}$$

□

Theorem 3.31. *Let* $\mathbf{K}$ *be a ground category.*

(1) *Let* $\mathscr{C} = \mathbf{CSLat}(\bigvee)$ *with* $\otimes = \vee$ *(binary) for each* L, $T : \mathbf{K} \to \mathbf{CSLat}(\bigvee)$ *be a topological theory in the sense of Definition 3.28, and* $\mathbf{P}_T$ *be the topological* $\mathscr{C}$*-powerset theory in* $\mathbf{K}$ *in the sense of Definition 3.5 induced from* T *in Lemma 3.30*(1)*. Then there exists a functorial embedding* $F : \mathbf{Top}(T) \to \mathbf{T}_{\mathbf{KP}_T}$*, where the codomain is a topological theory in the sense of Definition 3.7.*

(2) *Let* $\mathscr{C} = \mathbf{CSLat}(\bigvee)$ *with a* $\otimes$ *for each* L *(e.g.,* $\otimes$ *could be* $\vee$ *(binary) or* $\wedge$ *(binary)),* $\mathbf{P}$ *be a* q*-topological* $\mathscr{C}$*-powerset theory in* $\mathbf{K}$ *in the sense of Definition 3.5, and* $T_{\mathbf{P}}$ *be the topological theory in the sense of Definition 3.28 induced from* $\mathbf{P}$ *in Lemma 3.30*(2)*. Then* $\mathbf{Top}_{\mathrm{alt}}(T_{\mathbf{P}}) = \mathbf{T}_{\mathbf{KP}}$*, the latter a* q*-topological theory in the sense of Definition 3.7.*

Proof. Ad(1) Let $(A,\tau) \in |\mathbf{Top}(T)|$. Then $\tau \in T(A)$ and the principal ideal $\downarrow(\tau) \subset T(A)^{\mathrm{op}}$, where we use the dual ordering $\leq^{\mathrm{op}}$ to construct $\downarrow(\tau)$. Now $T(A)^{\mathrm{op}}$ is a complete lattice and $\downarrow(\tau)$ is closed under $\bigvee$ and $\otimes$ (both in the dual order) and contains $e = \perp$. We write $\mathfrak{T} = \downarrow(\tau)$ and note that $\mathfrak{T} \subset P_T(A)$. It follows that $(A,\mathfrak{T}) \in |\mathbf{T}_{\mathbf{KP}_T}|$.

Now let $f : (A,\tau) \to (B,\sigma)$ be a morphism in $\mathbf{Top}(T)$. Then we have $T(f)(\tau) \leq \sigma$ in the original ordering of $T(B)$; and also we have

$$\mathfrak{T} = \downarrow(\tau), \qquad \mathfrak{S} = \downarrow(\sigma) \tag{3.63}$$

in the duals $T(A)^{\mathrm{op}}$ and $T(B)^{\mathrm{op}}$. We claim that $f : (A,\mathfrak{T}) \to (A,\mathfrak{S})$ is a morphism in $\mathbf{T}_{\mathbf{KP}_T}$. This means showing that

$$\forall s \in \downarrow(\sigma), \qquad P_T(f)(s) \in \downarrow(\tau). \tag{3.64}$$

Let $s \in \downarrow(\sigma)$. Then $s \leq^{\mathrm{op}} \sigma$, or $s \geq \sigma$. Using the adjunction properties of Definition 1.9, it follows in the original ordering of $T(A)$ that

$$\tau \leq T(f)^{\vdash}\left(T(f)(\tau)\right) \leq T(f)^{\vdash}(\sigma) \leq T(f)^{\vdash}(s). \tag{3.65}$$

Hence $\tau \leq T(f)^{\vdash}(s)$, which means in the dual ordering that we have

$$P_T(f)(s) = T(f)^{\vdash}(s) \leq^{\mathrm{op}} \tau, \tag{3.66}$$

so that $P_T(f)(s) \in \downarrow(\tau)$.

From above, we put $F : \mathbf{Top}(T) \to \mathbf{T}_{\mathbf{KP}_T}$ by

$$F(A,\tau) = (A,\mathfrak{T}), \qquad F\left(f : (A,\tau) \longrightarrow (B,\sigma)\right) = f : (A,\mathfrak{T}) \longrightarrow (A,\mathfrak{S}). \tag{3.67}$$

It follows that F is a functor (since T and P_T are functors) and faithful; and because $a \neq b \Rightarrow \downarrow(a) \neq \downarrow(b)$ in a poset, we have that F injects objects and hence is an embedding.

Ad(2) Let $(A,\tau) \in |\mathbf{Top}_{\mathrm{alt}}(T_{\mathbf{P}})|$. Then $\tau \in T_{\mathbf{P}}(A)$, so that $\tau \subset P(A)$ and τ is closed under $\otimes$ and arbitrary $\bigvee$. Hence τ is a q-topology on A, so that $(A,\tau) \in |\mathbf{T_{KP}}|$. This argument clearly reverses. Therefore $|\mathbf{Top}_{\mathrm{alt}}(T_{\mathbf{P}})| = |\mathbf{T_{KP}}|$.

Now let $f : (A,\tau) \to (B,\sigma)$ be a morphism in $\mathbf{Top}_{\mathrm{alt}}(T_{\mathbf{P}})$. Then in $\wp(P(B))$, we have

$$\sigma \subset T_{\mathbf{P}}(f)(\tau) = \left(f_{\mathbf{P}}^{\leftarrow}\right)^{\leftarrow}(\tau). \tag{3.68}$$

But by Proposition 2.2,

$$\left(\left(f_{\mathbf{P}}^{\leftarrow}\right)^{\leftarrow}\right)^{\dashv} = \left(f_{\mathbf{P}}^{\leftarrow}\right)^{\rightarrow}, \tag{3.69}$$

and using AFT Theorem 1.10 and the properties of Definition 1.9, it is the case that

$$\sigma \subset \left(f_{\mathbf{P}}^{\leftarrow}\right)^{\leftarrow}(\tau) \iff \left(f_{\mathbf{P}}^{\leftarrow}\right)^{\rightarrow}(\sigma) \subset \tau. \tag{3.70}$$

This implies that $f : (A,\tau) \to (B,\sigma)$ is a morphism in $\mathbf{T_{KP}}$. Now the reverse argument also holds. Hence $\mathbf{Top}_{\mathrm{alt}}(T_{\mathbf{P}}) = \mathbf{T_{KP}}$ with respect to morphisms. Therefore, $\mathbf{Top}_{\mathrm{alt}}(T_{\mathbf{P}}) = \mathbf{T_{KP}}$. □

Remark 3.32. We state some consequences of the above results. When referring to Definition 3.28, we may not distinguish between Definitions 3.28(1) and 3.28(2).

(1) The approach of Definition 3.7 rests (because of Proposition 3.6(3)) on a contravariant functor, while that of Definition 3.28 rests on a covariant functor.

(2) The approach of Definition 3.7 is essentially a "first-order" or single-powerset theory and a preimage operator, while that of Definition 3.28 is a "second-order" or fibre theory resting on a double-powerset theory and a double preimage operator. This could be the most important point of comparison.

(3) The phrase "topological theory" is applied in Definition 3.7 to the category of spaces generated by a powerset theory, while this phrase in Definition 3.28 is applied to the functor motivated by the double-powerset theory behind the category of spaces.

(4) The approach of Definition 3.7 has an extremely general lattice-theoretic foundation, while that of Definition 3.28 (either (1) or (2)) is restricted to $\mathbf{CSLat}(\bigvee)$ with typically two choices of tensors; however, there seems no reason not to generalize $\mathbf{CSLat}(\bigvee)$ in Definition 3.28 to more general lattice-theoretic categories $\mathscr{C}$ (see inventory in Definition 1.1 and [1, 2]).

(5) While the approach of Definition 3.7 has an extension in Definition 3.8 to fuzzy topological theories, it may be possible to meaningfully extend Definition 3.28 in this direction as well.

(6) As to which of Definitions 3.7 or 3.28 is more general, we have a mixed situation even if we restrict $\mathscr{C}$ to be $\mathbf{CSLat}t(\bigvee)$ with appropriate choice of tensor. On one hand, for $\mathscr{C} = \mathbf{CSLat}(\bigvee)$ with $\otimes = \vee$ (binary), the approach of Definition 3.28(1) is a special case of Definition 3.7, due to the categorical embedding of Theorem 3.31(1); more precisely, for each category of spaces $\mathbf{Top}(T)$ of a topological theory T in the sense of Definition 3.28(1), there is a topological theory $\mathbf{T_{KP_{\mathit{T}}}}$ in the sense of Definition 3.7 into

which $\mathbf{Top}(T)$ categorically embeds. On the other hand, for $\mathscr{C} = \mathbf{CSLat}(\bigvee)$ with a $\otimes$ for each L (e.g., $\otimes$ could be $\vee$ (binary) or $\wedge$ (binary)), the approach of Definition 3.7 may be regarded as a special case of Definition 3.28(2) due to Theorem 3.31(2); more precisely, for each topological theory $\mathbf{T_{KP}}$ in the sense of Definition 3.7, there is a topological theory T in the sense of Definition 3.28(2) so that $\mathbf{T_{KP}} = \mathbf{Top}(T)$.

(7) One advantage of a single-powerset theory approach in the sense of Definition 3.7 is that its syntax lends itself to an examination of when the underlying single powerset theory arises from an algebraic theory in the sense of [3]. In resolving this question for topology in the sense of Definition 3.7, and thereby giving one answer concerning the degree to which topology has an algebraic foundation, the results of this section motivate the open question concerning the extent to which topology in the sense of [6] and Definition 3.28 has an algebraic foundation.

4. Algebraic generation of powerset theories: preliminary notions

4.1. Motivating example: algebraic generation of traditional powerset theory in Set

Referring to the algebraic theory $\mathbf{T}$ of standard construction in Example 3.15, Theorem 3.16, and to the traditional powerset theory $\mathbf{P}$ of Example 3.17, we have the following result.

Lemma 4.1. *Given $f : X \to Y$ in* **Set**, *the operator $f_{\mathbf{T}}^{\rightarrow} : \wp(X) \to \wp(Y)$ induced from* $\mathbf{T}$ *by setting*

$$f_{\mathbf{T}}^{\rightarrow} = T(f) = f^{\Delta} \diamond \mathrm{id}_{\wp(X)} \tag{4.1}$$

is the same as the image operator $f_{\mathbf{P}}^{\rightarrow} \equiv f^{\rightarrow}$ from traditional powerset theory $\mathbf{P}$. *In this sense, the algebraic theory* $\mathbf{T}$ *generates the traditional powerset theory in the sense of* $\mathbf{P} \equiv (P, \rightarrow)$.

Proof. Let $A \in \wp(X)$. Then

$$\begin{aligned} f_{\mathbf{T}}^{\rightarrow}(A) &= [T(f)](A) = [f^{\Delta} \diamond \mathrm{id}_{\wp(X)}](A) = [(\eta_Y \circ f) \diamond \mathrm{id}_{\wp(Y)}](A) \\ &= \bigcup_{x \in \mathrm{id}_{\wp(X)}(A)} (\eta_Y \circ f)(x) = \bigcup_{x \in A} \eta_Y(f(x)) = \bigcup_{x \in A} \{f(x)\} \\ &= \{f(x) : x \in A\} = f_{\mathbf{P}}^{\rightarrow}(A). \end{aligned} \tag{4.2}$$

□

Given that the AFT applies to the traditional $f^{\rightarrow}$ to obtain the traditional $f^{\leftarrow}$ and that η comes from $\mathbf{T}$ and that V is trivial, then in the sense of Section 4.2, we have the following theorem.

Theorem 4.2. *The traditional powerset theory $\mathbf{P} \equiv (P, \rightarrow, \leftarrow, V, \eta)$ in* **Set** *is algebraically generated from* $\mathbf{T}$ *and so is algebraic.*

Remark 4.3. In light of Theorem 3.10 and Example 3.17, Theorem 4.2 says that traditional topology has an algebraic foundation (of standard construction). This is related to the general question motivating this paper: determine the extent to which lattice-valued topology is algebraic.

4.2. Definition of algebraic generation of powerset theories

Definition 4.4. Let $\mathscr{C} \subset \mathbf{SQuant}$. An algebraic theory $\mathbf{T} = (T, \hat{\eta}, \diamond)$ in a category $\mathbf{K}$ *generates* a concrete $\mathscr{C}$-powerset theory $\mathbf{P} \equiv (P, \to, V, \eta)$ if the following are satisfied:

(G1) *Compatibility of objective functions*: for each $A \in |\mathbf{K}|$, $V(T(A)) = P(A)$.

(G2) *Compatibility of insertion morphisms*: for each $A \in |\mathbf{K}|$, $V(\hat{\eta}_A) = \eta_A$.

(G3) *Generation of forward/image powerset operator*: for each $f : A \to B$, the operator $f_{\mathbf{T}}^{\to} : V(T(A)) \to V(T(B))$ defined by setting

$$f_{\mathbf{T}}^{\to} = V(T(f)) \tag{4.3}$$

is precisely the image operator

$$f_{\mathbf{P}}^{\to} : P(A) \longrightarrow P(B) \tag{4.4}$$

of $\mathbf{P}$, where $T(f) : T(A) \to T(B)$ is the arrow induced by $\mathbf{T}$ (Remark 3.2(3)).

If $\mathbf{P} \equiv (P, \to, \leftarrow, V, \eta)$ is a balanced, concrete $\mathscr{C}$-powerset theory, then $\mathbf{P}$ is generated from an algebraic theory $\mathbf{T}$ if $\mathbf{T}$ generates $(P, \to, V, \eta)$ and the following additional condition holds:

(G4) *Generation of backward/preimage powerset operator*: $f_{\mathbf{P}}^{\leftarrow}$ is always uniquely determined from $f_{\mathbf{P}}^{\to}$ so that (P3) and (Ad) are satisfied.

In any case, if there is an algebraic theory $\mathbf{T}$ which generates $\mathbf{P}$, then $\mathbf{P}$ is *algebraically generated* or *algebraic.*

Proposition 4.5. *Let* $\mathscr{C} \subset \mathbf{SQuant}$ [$\mathbf{USQuant}$] *and suppose that* $\mathbf{P} \equiv (P, \to, V, \eta)$ *is an algebraic powerset theory in* $\mathbf{K}$. *The following hold:*

(1) $\mathbf{P} \equiv (P, \to, \leftarrow, V, \eta)$ *is algebraic if the antecedent of the consequent of Proposition 3.6*(1) *holds.*

(2) $\mathbf{P} \equiv (P, \to, \leftarrow, V, \eta)$ *is algebraic and* (QT1) *[*(T1)*] holds if and only if* $\mathbf{P}$ *is q-topological [topological].*

Remark 4.6 (formulating when topology has algebraic foundation). (1) The traditional powerset theory in **Set** is algebraic (cf. Theorem 4.2) of standard construction.

(2) For algebraic concrete powerset theories, the issue between being a balanced algebraic powerset theory and a (q-)topological powerset theory is precisely whether $f_{\mathbf{P}}^{\leftarrow} \in \mathscr{C}$.

*Proof.*By definition, $f_{\mathbf{P}}^{\leftarrow} \in \mathscr{C}$ is necessary. To see that $f_{\mathbf{P}}^{\leftarrow} \in \mathscr{C}$ is sufficient, it suffices to see that the other conditions imply (T2), (T3). We leave (T3) to the reader and show that (T2) holds. Since T and V are functors, their composition is a functor and preserves composition. It follows that $(g \circ f)_{\mathbf{P}}^{\to} = g_{\mathbf{P}}^{\to} \circ f_{\mathbf{P}}^{\to}$. Then applying Remark 1.13 and the fact (Ad) is satisfied, we have

$$(g \circ f)_{\mathbf{P}}^{\leftarrow} = \left[(g \circ f)_{\mathbf{P}}^{\to}\right]^{\vdash} = \left[g_{\mathbf{P}}^{\to} \circ f_{\mathbf{P}}^{\to}\right]^{\vdash} = \left(f_{\mathbf{P}}^{\to}\right)^{\vdash} \circ \left(g_{\mathbf{P}}^{\to}\right)^{\vdash} = f_{\mathbf{P}}^{\leftarrow} \circ g_{\mathbf{P}}^{\leftarrow}. \tag{4.5}$$

□

As we know from Section 3.4, under rather general lattice-theoretic conditions, topological powerset theories produce topological and fuzzy topological theories which are topological categories over their grounds with the expected forgetful functors.

(3) This paper is concerned with the overall question of how much algebra is needed in topology. The question of whether a known topological category has an algebraic foundation can be reformulated in this way: is this topological category the topological or fuzzy topological theory of a topological powerset theory which is algebraic?

Remark 4.6(3) leads us to Section 4.3 in which we pose specific questions concerning the topological and fuzzy topological theories inventoried in Section 3.5.

4.3. Questions concerning algebraic generation of topological powerset theories

Question 4.7. Let $\mathscr{C} \subset$ **SQuant**.

(1) Are there necessary and sufficient conditions under which the (q-topological) L-powerset theory in **Set** is algebraic of standard construction (where $L \in |\mathscr{C}|$)? This question was open prior to this paper except for the case $L = 2$ (which is "isomorphic" to the traditional powerset theory in **Set**); and a sufficient condition of L being a frame is given in [3]—a condition shown in Section 5 to be not necessary. Restating the question, does there exist a theory $\mathbf{T}$ of standard construction such that there are necessary and sufficient conditions under which the (topological) L-powerset theory in **Set** is algebraically generated from $\mathbf{T}$ as an algebraic theory in **Set**? The answer to this question is *yes*, details for which are given in Section 5.

(2) Are there necessary and sufficient conditions under which the (q-topological) left-adjoint powerset theory in $\mathbf{Set} \times \mathbf{C}$ is algebraic of standard construction (where $\mathbf{C} = \mathscr{C}^{\mathrm{op}}$)? This question was completely open prior to this paper except for the case $\mathbf{C} = \mathbf{2}$ (the latter being the category with 2 as the one object and id_2 as the one morphism). Restating the question, does there exist a theory $\mathbf{T}$ of standard construction such that there are necessary and sufficient conditions under which the left-adjoint topological set theory in $\mathbf{Set} \times \mathbf{C}$ is algebraically generated from $\mathbf{T}$ as an algebraic theory in $\mathbf{Set} \times \mathbf{C}$? The answer to this question is *yes*, details for which are given in Section 6.

5. Algebraic generation of topological powerset theories: fixed-basis case

Throughout this section, $L \equiv (L, \leq, \otimes)$ is a semi-quantale (Section 1) with possibly additional conditions considered. The conditions of an s-quantale [us-quantale, os-quantale, uos-quantale] suffice to guarantee that each L-**QTop** [L-**Top**, L-**QFTop**, L-**FTop**, resp.] is a topological category over **Set** (Section 3.4, cf. [1, 2]).

This section constructs two theories $\mathbf{T}_1$ and $\mathbf{T}_2$ of standard construction (see [3], Convention 3.3, Example 3.15) and finds a necessary and sufficient condition on L for these theories to be algebraic theories in **Set**, theories from which the (topological) L-powerset theory in **Set** (Example 3.20, Lemma 3.21) is generated. To contextualize these results, the proofs (Theorems 3.10, 3.12, 3.22) that each L-**QTop**, L-**Top**, L-**QFTop**, L-**FTop** is topological rest on the Zadeh preimage operator (Definition 1.19(2), Example 3.20), this operator comes from the q-topological L-powerset theory in **Set**, this L-powerset theory is generated from each of $\mathbf{T}_1$ and $\mathbf{T}_2$ under a certain condition which is both necessary and sufficient for $\mathbf{T}_1$ and $\mathbf{T}_2$ to be algebraic; and so under this certain condition, the topological behavior of L-**QTop**, L-**Top**, L-**QFTop**, L-**FTop** rests on algebraic theories $\mathbf{T}_1$ and $\mathbf{T}_2$ of standard construction.

This certain condition is that $(L, \leq, \otimes)$ is a u-quantale (1.2.1.8). Section 5.1 furnishes the critical Lemma 5.1 showing that a particular theory $\mathbf{T}_1$ of standard construction is an algebraic theory in **Set** if and only L is a u-quantale. A simple modification of the clone composition of $\mathbf{T}_1$ yields a second theory $\mathbf{T}_2$ of standard construction with the corresponding result that $\mathbf{T}_2$ is an algebraic theory in **Set** if and only L is a u-quantale. A restriction Corollary 5.4 of Lemma 5.1 using st-quantales is also given, this case being first given in [4]; sufficiency in this restriction generalizes the result essentially appearing in [3, Example 4.3.3] that assumes that L is a frame with $\otimes = \wedge$ (binary); but necessity even in this restriction of Lemma 5.1 is new.

Section 5.2 uses Lemma 5.1 to give the theorem that the (topological) L-powerset theory in **Set** (Example 3.20), under the assumption that L is a u-quantale, is generated from each of $\mathbf{T}_1$ and $\mathbf{T}_2$ of Section 5.1. The corollary then follows that the (topological) L-powerset theory is algebraically generated from $\mathbf{T}_1$ and $\mathbf{T}_2$ if and only if L is a u-quantale.

5.1. Necessary and sufficient condition for algebraic theories $\mathbf{T}_1$ and $\mathbf{T}_2$ of standard construction in Set. Initially, we will give a necessary and sufficient condition for $L \in |\mathbf{SQuant}|$ under which a theory $\mathbf{T}$ (defined below) is algebraic in **Set**. We then "double" this theory to obtain two theories $\mathbf{T}_1$ and $\mathbf{T}_2$ for each of which this same necessary and sufficient condition on L makes that theory algebraic in **Set**. In the following section, we show that these theories algebraically generate the (same) L-powerset theory in **Set**.

Lemma 5.1 (characterization lemma). *Let $(L, \leq, \otimes) \in |\mathbf{SQuant}|$, $e \in L$ be some fixed element, and $\mathbf{T} = (T, \eta, \diamond)$ be as follows:*

(D1) $T : |\mathbf{Set}| \to |\mathbf{Set}|$ *by* $T(X) = L^X$.

(D2) *For each $X \in |\mathbf{Set}|$, the component $\eta_X : X \to L^X$ of η is defined by*

$$\eta_X(x)(z) = \chi^e_{\{x\}}(z) \equiv \begin{cases} e, & z = x, \\ \perp, & z \neq x. \end{cases} \tag{5.1}$$

(D3) *For each $f : X \to L^Y$, for each $g : Y \to L^Z$ in* **Set**, *define $g \diamond f : X \to L^Z$ by*

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [(f(x))(y) \otimes (g(y))(z)]. \tag{5.2}$$

Then $\mathbf{T}$ is an algebraic theory in **Set** *(of standard construction) if and only if $(L, \leq, \otimes)$ is a unital quantale with unit e, that is, $\otimes$ satisfies each of the following conditions:*

(L1) $\otimes$ *distributes across arbitrary $\bigvee$ from both the left and right.*

(L2) $\otimes$ *is associative.*

(L3) *e is a two-sided unit for $\otimes$.*

In either case, $\otimes$ also satisfies the following condition:

(L4) $\perp$ *is a two-sided zero for $\otimes$.*

Proof. Sufficiency. We first note from Definition 1.1(6) that (L1) implies (L4). Now the goal is to demonstrate the sufficiency of (L1)–(L3), using (L4), for $\mathbf{T}$ to be an algebraic theory. To that end, we verify (A1)–(A3) of Definition 3.1.

Ad(A1) Let $f : X \to L^Y$, $g : Y \to L^Z$, $h : Z \to L^W$. Then $h \diamond (g \diamond f), (h \diamond g) \diamond f : X \to L^W$. Let $x \in X$. But to show that

$$[h \diamond (g \diamond f)](x) = [(h \diamond g) \diamond f](x) \tag{5.3}$$

as L-subsets of W, let $w \in W$. It follows that

$$([h \diamond (g \diamond f)](x))(w) = \bigvee_{z \in Z} \left[\left(\bigvee_{y \in Y} [(f(x))(y) \otimes (g(y)(z))] \right) \otimes (h(z))(w) \right], \tag{5.4}$$

$$([(h \diamond g) \diamond f](x))(w) = \bigvee_{y e Y} \left[(f(x))(y) \otimes \left(\bigvee_{z \in Z} [(g(y))(z) \otimes (h(z))(w)] \right) \right]. \tag{5.5}$$

One can use (L1) to rewrite both (5.4) and (5.5) as double joins, and these joins commute by the associativity of joins. Finally, to make the terms of these double joins the same, we apply the associativity of $\otimes$ (L2).

Ad(A2) Let $f : X \to L^Y$, and recall that $\eta_Y : Y \to L^Y$ is defined by $\eta_Y(y) = \chi^e_{\{y\}}$. To show that $\eta_Y \diamond f = f$, let $x \in X$ and $y \in Y$. Then

$$\begin{aligned}
[(\eta_Y \diamond f)(x)](y) &= \bigvee_{z \in Y} [(f(x))(z) \otimes (\eta_Y(z))(y)] = \bigvee_{z \in Y} [(f(x))(z) \otimes \chi^e_{\{z\}}(y)] \\
&= \bigvee_{z \in Y} \left[(f(x))(z) \otimes \begin{cases} e, & z = y \\ \perp, & z \neq y \end{cases} \right] = \bigvee_{z \in Y} \begin{cases} (f(x))(z) \otimes e, & z = y, \\ (f(x))(z) \otimes \perp, & z \neq y, \end{cases} \\
&= \bigvee_{z \in Y} \begin{cases} (f(x))(y), & z = y, \\ \perp, & z \neq y, \end{cases} \\
&= (f(x))(y),
\end{aligned} \tag{5.6}$$

where we have used that e is a right-sided identity and $\perp$ is a right-sided zero for $\otimes$ from (L3), (L4).

Ad(A3) Let $f : X \to Y$ and let $g : Y \to L^Z$. To show that $g \diamond f^\Delta = g \circ f$, we let $x \in X$ and $z \in Z$. Then

$$\begin{aligned}
[(g \diamond f^\Delta)(x)](z) &= \bigvee_{y \in Y} [(f^\Delta(x))(y) \otimes (g(y))(z)] = \bigvee_{y \in Y} [(\eta_Y(f(x)))(y) \otimes (g(y))(z)] \\
&= \bigvee_{y \in Y} [\chi^e_{\{f(x)\}}(y) \otimes (g(y))(z)] = \bigvee_{y \in Y} \left[\begin{cases} e, & y = f(x) \\ \perp, & y \neq f(x) \end{cases} \otimes (g(y))(z) \right] \\
&= \bigvee_{y \in Y} \begin{cases} (g(f(x)))(z), & y = f(x), \\ \perp, & y \neq f(x), \end{cases} \\
&= (g(f(x)))(z),
\end{aligned} \tag{5.7}$$

where we have used that e is a left-sided identity and $\perp$ is a left-sided zero for $\otimes$ from (L3), (L4).

Necessity. The goal is to demonstrate the necessity of (L1)–(L3) for **T** to be an algebraic theory. The general trick is to assume (A1)–(A3), make the appropriate choices of sets and functions, and then force each of the conditions (L1)–(L3).

Ad(L3) Let $a \in L$. Choose $X = \{x\}$, $Y = \{y\}$, and let $f : X \to L^Y$ by $f(x) = \underline{a}$. Then applying (A2)—$\eta_Y \diamond f = f$—we have

$$\begin{aligned} a \otimes e = (f(x))(y) \otimes e &= \bigvee_{z \in Y} [(f(x))(z) \otimes (\eta_Y(z))(y)] \\ &= [(\eta_Y \diamond f)(x)](y) = (f(x))(y) = a. \end{aligned} \tag{5.8}$$

So $a \otimes e = a$, and e is a right-hand identity. But (A3), as noted above, implies (for any algebraic theory **T**) that η gives right-hand identities for $\diamond$; hence, in particular, we have $f \diamond \eta_X = f$, and this equation for our chosen f forces $e \otimes a = a$ by a symmetric argument.

Ad(L1) For the nonempty case, let $a \in L$ and $\{b_\gamma\}_{\gamma \in \Gamma} \subset L$. Choose $X = \{x\}$, $Y = \{y\}$, $Z = \Gamma$, $W = \{w\}$—X, Y, W are singletons and Z is the indexing set Γ—and choose $f : X \to L^Y$, $g : Y \to L^Z$, $h : Z \to L^W$ by $f(x) = \underline{a}$ (constant map), $g(y)(\gamma) = b_\gamma$, $h(\gamma) = \underline{e}$ (constant map). Then applying (L3) established above and (A1), we have

$$\begin{aligned} a \otimes \left(\bigvee_{\gamma \in \Gamma} b_\gamma \right) = a \otimes \left(\bigvee_{\gamma \in \Gamma} (b_\gamma \otimes e) \right) &= [((h \diamond g) \diamond f)(x)](w) \\ &= [(h \diamond (g \diamond f))(x)](w) = \bigvee_{\gamma \in \Gamma} [(a \otimes b_\gamma) \otimes e] = \bigvee_{\gamma \in \Gamma} (a \otimes b_\gamma), \end{aligned} \tag{5.9}$$

which yields that $\otimes$ distributes across arbitrary $\bigvee$ from the right. Now if we keep the same mapping g, but rechoose f by $f(x) = \underline{e}$ and rechoose h by $h(z) = \underline{a}$, then a symmetric argument yields that $\otimes$ distributes across $\bigvee$ from the left in the nonempty case.

Now for the empty case (the same as (L4)), choose X, Y, W, f as in the nonempty case, choose $Z = \varnothing$, choose h to be the empty function, and choose $g : Y \to L^{\varnothing}$ to be the constant function. Then (5.4) in this context becomes

$$[(h \diamond (g \diamond f))(x)](w) = \perp, \tag{5.10}$$

while (5.5) in this context becomes

$$[((h \diamond g) \diamond f)(x)](w) = a \otimes \perp, \tag{5.11}$$

in which case (A1) forces

$$a \otimes \perp = \perp, \tag{5.12}$$

showing that $\perp$ is a right zero for $\otimes$. To show that $\perp$ is a left zero, rechoose $Y = \varnothing$, $Z = \{z\}$, $f : X \to L^{\varnothing}$ to be the constant function, g to be the empty function, and $h : Z \to L^W$ by $h(z) = \underline{a}$. Then (5.4), (5.5), and (A1) force

$$\perp \otimes a = \perp, \tag{5.13}$$

finishing the proof of (L1) in the empty case.

Ad(L2) Let $a,b,c \in L$, choose $X = \{x\}$, $Y = \{y\}$, $Z = \{z\}$, $W = \{w\}$, and choose $f : X \to L^Y$, $g : Y \to L^Z$, $h : Z \to L^W$ by $f(x) = \underline{a}$, $g(y) = \underline{b}$, $h(z) = \underline{c}$. Then applying (A1) yields

$$a \otimes (b \otimes c) = [((h \diamond g) \diamond f)(x)](w) = [(h \diamond (g \diamond f))(x)](w) = (a \otimes b) \otimes c. \tag{5.14}$$

This concludes the proof of the lemma. □

Remark 5.2 (doubling theories). Since $\otimes$ is generally not commutative, it follows that the tensor products appearing in the definition of the clone composition in (D3) of Lemma 5.1 are ordered according to our choice. Restated, the clone composition

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [(f(x))(y) \otimes (g(y))(z)] \tag{5.15}$$

could also be chosen as

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [(g(y))(z) \otimes (f(x))(y)]. \tag{5.16}$$

This yields an alternative clone composition and therefore an alternative theory $\mathbf{T}$. Let us denote the theory presented in Lemma 5.1 by $\mathbf{T}_1 = (T, \eta, \diamond_1)$ and the alternative theory by $\mathbf{T}_2 = (T, \eta, \diamond_2)$. We have then the following corollary.

Corollary 5.3. *The following are equivalent:*
(1) $\mathbf{T}_1 = (T, \eta, \diamond_1)$ *is an algebraic theory in* **Set***;*
(2) $(L, \le, \otimes)$ *is a u-quantale with unit e;*
(3) $\mathbf{T}_2 = (T, \eta, \diamond_2)$ *is an algebraic theory in* **Set***.*

Proof. The proof of (1)⇔(2) follows from Lemma 5.1; and clearly the proof of Lemma 5.1 may be modified—given that e is a two-sided unit and $\perp$ is a two-sided zero ((L3), (L4)) in the proof of sufficiency—to yield that $\mathbf{T}_2 = (T, \eta, \diamond_2)$ is an algebraic theory in **Set** if and only if $(L, \le, \otimes)$ is a u-quantale, thus verifying (2)⇔(3). □

The following corollary to Lemma 5.1 is a special case of that lemma in two closely related ways: first, we restrict the setting of the necessary and sufficient condition of the lemma from $|\mathbf{UQuant}|$ to $|\mathbf{STQuant}|$; and second, the components of η of the algebraic theory of the lemma are adjusted to use characteristic mappings based on $\top$ instead of some element e. This incorporates [4] as a special case.

Corollary 5.4 (special restriction of Lemma 5.1). *Let* $(L, \le, \otimes) \in |\mathbf{SQuant}|$ *and* $\mathbf{T} = (T, \eta, \diamond)$ *be as follows:*
(D1) $T : |\mathbf{Set}| \to |\mathbf{Set}|$ *by* $T(X) = L^X$.
(D2) *For each* $X \in |\mathbf{Set}|$*, the component* $\eta_X : X \to L^X$ *of* η *is defined by*

$$\eta_X(x)(z) = \chi_{\{x\}}. \tag{5.17}$$

(D3) *For each* $f : X \to L^Y$, *for each* $g : Y \to L^Z$ *in* **Set**, *define* $g \diamond f : X \to L^Z$ *by*

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [(f(x))(y) \otimes (g(y))(z)]. \tag{5.18}$$

Then **T** *is an algebraic theory in* **Set** *if and only if* $(L, \le, \otimes)$ *is an st-quantale.*

Remark 5.5. The corresponding restrictions of Remark 5.2 and Corollary 5.3 can now be stated by the reader and follow using st-quantales with $e = \top$.

5.2. Algebraic generation of L-powerset theories

Theorem 5.6. *Let* $(L, \le, \otimes)$ *be a u-quantale with unit* e. *Then* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *are algebraic theories in* **Set**, *in which case each* $f : X \to Y$ *in* **Set** *lifts to* $f_{\mathbf{T}}^{\to} \equiv T(f) : L^X \to L^Y$ *in each of* $\mathbf{T}_1$ *and* $\mathbf{T}_2$, *and each* $f_{\mathbf{T}}^{\to} = f_{\mathbf{P}}^{\to} \equiv f_L^{\to}$. *Hence each of* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *algebraically generates the adjunctive natural topological L-powerset theory* $\mathbf{P} \equiv (P, \to, \leftarrow, V, \eta)$ *in* **Set** *(Example 3.20).*

Proof. Working with Definition 4.4, we first discuss the $\mathbf{T}_1$ case. Comparing Example 3.20 and Lemma 5.1 shows that (G1), (G2) of Definition 4.4 are satisfied. To show that (G3) of Definition 4.4 is satisfied, we first recall

$$\begin{aligned} f^{\Delta} : X \longrightarrow L^Y \quad &\text{by } f^{\Delta} = \eta_Y \circ f, \\ T(f) : L^X \longrightarrow L^Y \quad &\text{by } T(f) = f^{\Delta} \diamond \mathrm{id}_{T(X)}. \end{aligned} \tag{5.19}$$

Let $a \in L^X$. Then $f_{\mathbf{T}}^{\to}(a) \in L^Y$. To compute this L-subset of Y, let $y \in Y$. Then using (L3), (L4), we have

$$\begin{aligned} (f_{\mathbf{T}}^{\to}(a))(y) &= [(T(f))(a)](y) = [(f^{\Delta} \diamond \mathrm{id}_{T(X)})(a)](y) \\ &= \bigvee_{x \in X} [(\mathrm{id}_{T(X)}(a))(x) \otimes (f^{\Delta}(x))(y)] \\ &= \bigvee_{x \in X} [a(x) \otimes (\eta_Y(f(x)))(y)] \\ &= \bigvee_{x \in X} [a(x) \otimes \chi^{e}_{\{f(x)\}}(y)] \\ &= \bigvee_{x \in X} \left[a(x) \otimes \begin{cases} e, & f(x) = y \\ \perp, & f(x) \ne y \end{cases} \right] \\ &= \bigvee_{x \in X} \begin{cases} a(x) \otimes e, & f(x) = y, \\ \perp, & f(x) \ne y, \end{cases} \\ &= \bigvee \{a(x) : x \in f^{\leftarrow}\{y\}\} \\ &= (f_L^{\to}(a))(y) = (f_{\mathbf{P}}^{\to}(a))(y). \end{aligned} \tag{5.20}$$

This shows that $f_{\mathbf{T}}^{\to} = f_{\mathbf{P}}^{\to}$ and so (G3) holds. As for (G4) of Definition 4.4, we apply the AFT to $f_{\mathbf{P}}^{\to}$ to obtain $f_{\mathbf{P}}^{\leftarrow}$—see details in [20, 21]. Hence $\mathbf{T}_1$ algebraically generates the

L-powerset theory $\mathbf{P} \equiv (P, \rightarrow, \leftarrow, V, \eta)$. The proof for the $\mathbf{T}_2$ case is symmetric using (L3), (L4). □

Corollary 5.7. *The following are equivalent:*

(1) *The adjunctive q-topological L-powerset theory $\mathbf{P} \equiv (P, \rightarrow, \leftarrow, V, \eta)$ is algebraically generated from the algebraic theory $\mathbf{T}_1$, in which case $\mathbf{P}$ is also natural and topological.*

(2) *$(L, \leq, \otimes)$ is a u-quantale.*

(3) *The adjunctive q-topological L-powerset theory $\mathbf{P} \equiv (P, \rightarrow, \leftarrow, V, \eta)$ is algebraically generated from the algebraic theory $\mathbf{T}_2$, in which case $\mathbf{P}$ is also natural and topological.*

Remark 5.8. Theorem 5.6 and Corollary 5.7 show that two different algebraic theories in **Set** can algebraically generate the same powerset theory in **Set**.

Remark 5.9. (1) Note that the derivation of $f_{\mathbf{P}}^{\rightarrow}$ needs only the properties (L3), (L4). Moreover, the proof of $f_{\mathbf{P}}^{\rightarrow} \dashv f_{\mathbf{P}}^{\leftarrow}$ uses only that L is an s-quantale (and in fact ignores $\otimes$ altogether). Further, the various characterizations [21] of these operators, restated for the more general lattices of this paper and replacing χ by χ^e, only require that L be an *integral s-quantale*, namely L is an s-quantale which satisfies (L4) and has e as a right-hand identity of $\otimes$. Thus, the powerset theories constructed in the above results from algebraic theories account for a significant subclass of fixed-basis powerset theories in **Set**, but *do not account* for a significant subclass of fixed-basis powerset theories in **Set**; that is, there are significant L-powerset theories in **Set** which do not arise from algebraic theories constructed in Lemma 5.1 even though they *behave* syntactically exactly like powerset theories arising from such algebraic theories. Thus a significant part of fixed-basis lattice-valued topology has an algebraic foundation in the sense of Lemma 5.1 *and* a significant part of fixed-basis lattice-valued topology does *not* have an algebraic foundation in the sense of Lemma 5.1.

(2) We augment the comments of (1). Some preimage powerset operators which guarantee that L-**QTop**, L-**Top**, L-**QFTop**, L-**FTop** are topological constructs arise from adjunctive natural topological L-powerset theories and some do not arise from adjunctive natural topological L-powerset theories, though the syntax of many examples of the latter is identical to that of the former. This means that a significant part of fixed-basis lattice-valued topology has adjunctive natural topological powerset theory foundation in the sense of Sections 3.4 and 3.5 *and* a significant part of fixed-basis lattice-valued topology does *not* have adjunctive natural topological powerset theory foundation in the sense of Sections 3.4 and 3.5; topology and fuzzy topology in the latter case are therefore twice removed from having an algebraic foundation in the sense of Lemma 5.1.

6. Algebraic generation of topological powerset theories: variable-basis case

This section gives lattice-theoretic and category-theoretic conditions which are both necessary and sufficient for variable-basis powerset theories in a ground category to be algebraically generated from theories of standard construction. Throughout this section, $\mathbf{C} \subset \mathbf{LoSQuant}$ with additional restrictions as indicated.

Section 6.1, by Lemmas 6.1–6.3 analogous to Lemma 5.1 and identifies four variable-basis theories of standard construction—two right-adjoint theories $\mathbf{T}_1(\vdash)$, $\mathbf{T}_2(\vdash)$ and two left-adjoint theories $\mathbf{T}_1(\dashv)$, $\mathbf{T}_2(\dashv)$—and proves that there are necessary and sufficient conditions guaranteeing that they are algebraic in $\mathbf{Set} \times \mathbf{C}$, as well as two variable-basis theories $\mathbf{T}_1(*)$, $\mathbf{T}_2(*)$ of standard construction for which there are sufficient conditions guaranteeing that they are algebraic. A restriction of these lemmas to st-quantales, analogous to Corollary 5.4, is stated, this restriction being first given in [4].

Section 6.2 proves that whenever $\mathbf{T}_1(\dashv)$, $\mathbf{T}_2(\dashv)$ are algebraic, they generate the well-known left-adjoint (topological) powerset theory $\mathbf{P}(\dashv)$ in $\mathbf{Set} \times \mathbf{C}$ (hence the label "left-adjoint" for $\mathbf{T}_1(\dashv), \mathbf{T}_2(\dashv)$)—see Example 3.23. To contextualize the results of Section 6.2, we note the following:

(1) Lemma 3.26 says that the categories **C-QTop**, **C-Top**, **C-QFTop**, **C-FTop** are topological over $\mathbf{Set} \times \mathbf{C}$ (with respect to the usual forgetful functor) for *any* $\mathbf{C} \subset$ **LoSQuant**, **LoUSQuant**, **LoOSQuant**, **LoUOSQuant**, respectively; and

(2) Lemma 3.24(2) says that the preimage operators underneath these categories can be recovered from the image operator of the left-adjoint (q-)topological powerset theory—making this theory adjunctive—precisely for those subcategories $\mathbf{C}$ in which the dual morphisms also preserve arbitrary $\bigwedge$.

Thus Section 6.2 finds a condition necessary and sufficient for $\mathbf{P}(\dashv)$ to be algebraically generated from certain left-adjoint algebraic theories in $\mathbf{Set} \times \mathbf{C}$ of standard construction; and together, Sections 6.1 and 6.2 identify those subcategories $\mathbf{C}$ of **LoSQuant** for which the topological behavior of **C-QTop**, **C-Top**, **C-QFTop**, **C-FTop** rests on algebraic theories in $\mathbf{Set} \times \mathbf{C}$.

Section 6.3 proves that whenever $\mathbf{T}_1(\vdash)$ and $\mathbf{T}_2(\vdash)$ are algebraic, they generate a *new* variable-basis powerset theory $\mathbf{P}(\vdash)$ in $\mathbf{Set} \times \mathbf{C}$, which we dub the *right-adjoint* (*variable-basis*) *powerset theory* in $\mathbf{Set} \times \mathbf{C}$; and indeed the syntax of such powerset theories, even when there is no algebraic foundation (in the sense of $\mathbf{T}_1(\vdash)$ and $\mathbf{T}_2(\vdash)$ being algebraic), builds new right-adjoint topological powerset theories on which new topological and fuzzy topological theories rest, the right-adjoint counterparts to **C-QTop**, **C-Top**, **C-QFTop**, **C-FTop**. In particular, we show these right-adjoint powerset theories are not redundant compared with the left-adjoint powerset theories—so they really are new—and hence they induce new preimage operators, new topological and fuzzy topological theories, and new topological categories, that is, new kinds of variable-basis topology and fuzzy topology.

6.1. Necessary and sufficient conditions for $\mathbf{T}_{1,2}(\vdash)$ and $\mathbf{T}_{1,2}(\dashv)$ to be algebraic theories in $\mathbf{Set} \times \mathbf{C}$. This section gives a necessary and sufficient condition on $\mathbf{C} \subset$ **LoSQuant** for theories $\mathbf{T}(\vdash)$ and $\mathbf{T}(\dashv)$ (defined below) to be algebraic in $\mathbf{Set} \times \mathbf{C}$; and then a sufficient condition is given on $\mathbf{C} \subset$ **LoSQuant** for theory $\mathbf{T}(*)$ (also defined below) to be algebraic in $\mathbf{Set} \times \mathbf{C}$. Then, analogous to the fixed-basis case, each of these theories is "doubled" to obtain four theories $\mathbf{T}_{1,2}(\vdash)$ and $\mathbf{T}_{1,2}(\dashv)$ for which there is a necessary and sufficient condition on $\mathbf{C} \subset$ **LoSQuant** making these theories algebraic in $\mathbf{Set} \times \mathbf{C}$, as well as two theories $\mathbf{T}_{1,2}(*)$ for which there is a sufficient condition on $\mathbf{C} \subset$ **LoSQuant** making these theories algebraic in $\mathbf{Set} \times \mathbf{C}$. One important distinction between this section and Section 5.1 is the role of choice functions.

Lemma 6.1 (right-adjoint theory characterization). *Let* $\mathbf{C} \subset \mathbf{LoSQuant}$ *and let* $\mathbf{T}(\vdash) \equiv (T, \eta, \diamond)$ *be as follows:*

(D1) $T : |\mathbf{Set} \times \mathbf{C}| \to |\mathbf{Set} \times \mathbf{C}|$ *by*

$$T(X,L) = (L^X, L). \tag{6.1}$$

(D2) *For each* $(X,L) \in |\mathbf{Set} \times \mathbf{C}|$, *the component*

$$\eta_{(X,L)} : (X,L) \longrightarrow (L^X, L) \tag{6.2}$$

of η *is* $\eta_{(X,L)} = (\eta_X, \mathrm{id}_L)$, *where* $\mathrm{id}_L : L \to L$ *is the identity morphism of* L *in* $\mathbf{C}$,

$$e : |\mathbf{C}| \longrightarrow \bigcup_{M \in |\mathbf{C}|} M \tag{6.3}$$

is a choice function with $e_M \equiv e(M) \in M$ *for each* $M \in |\mathbf{C}|$, *and*

$$\eta_X(x)(z) = \chi_{\{x\}}^{e_L}(z) \equiv \begin{cases} e_L, & z = x, \\ \perp, & z \neq x. \end{cases} \tag{6.4}$$

(D3) *For each* $(f,\phi) : (X,L) \to (M^Y, M)$, $(g,\psi) : (Y,M) \to (N^Z, N)$ *in* $\mathbf{Set} \times \mathbf{C}$, *define*

$$(g,\psi) \diamond (f,\phi) : (X,L) \longrightarrow (N^Z, N) \tag{6.5}$$

by $(g,\psi) \diamond (f,\phi) = (g \diamond f, \phi \diamond \psi)$, *where* $\phi \diamond \psi = \phi \circ \psi$ *and*

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [\psi^{\vdash}((f(x))(y)) \otimes (g(y))(z)]. \tag{6.6}$$

Then the following are equivalent:

(1) *there exists a choice function* e *such that* $\mathbf{T}(\vdash)$ *is an algebraic theory in* $\mathbf{Set} \times \mathbf{C}$;

(2) $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$.

Lemma 6.2 (left-adjoint theory characterization). *The same statement as Lemma 6.1 except that* $\mathbf{C} \subset \mathbf{LoUQuant}(\dashv)$ *replaces* $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$, $\diamond$ *is redefined by*

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [\psi^{\dashv}((f(x))(y)) \otimes (g(y))(z)], \tag{6.7}$$

$\mathbf{T}(\dashv)$ *replaces* $\mathbf{T}(\vdash)$, *and statement* (2) *adds that* $\mathbf{C}_{\dashv}(\dashv) \subset \mathbf{SQuant}$.

Lemma 6.3 (adjoint-like theory existence). *The same statement as the direction of Lemma 6.1 asserting the sufficiency of the condition* $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$, *except that* $\mathbf{C} \subset \mathbf{LoUQuant}^*$ *replaces* $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$, $\diamond$ *is redefined by*

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [\psi^*((f(x))(y)) \otimes (g(y))(z)] \tag{6.8}$$

provided that $\psi^* : L \to M$ *is chosen uniquely for each* ψ, *and* $\mathbf{T}(*)$ *replaces* $\mathbf{T}(\vdash)$.

The proofs of these lemmas are sufficiently similar that we prove in detail only Lemma 6.1 dealing with the right-adjoint case; but we explicitly indicate the one part where the proof of Lemma 6.2 is not analogous to the proof of Lemma 6.1.

Proof of Lemma 6.1 (right-adjoint case). (1)$\Leftarrow$(2). Referring to the statement of Lemma 5.1, we assume (L1)–(L4) as well as assuming that each morphism $\phi : L \to M$ in $\mathbf{C}$ has $\phi^{\vdash} : L \to M$ in **UQuant**, that is, $\phi^{\vdash}$ preserves arbitrary $\bigvee$, $\otimes$, e, the first property of which implies that $\phi^{\vdash}$ preserves $\perp$. Note that the obvious candidate for the choice function $e : |\mathbf{C}| \to \bigcup_{L\in|\mathbf{C}|} L$ needed in Lemma 6.1(1) is given by choosing $e(L)$ to be the unit e_L of $\otimes$ on L (L3). We now prove (A1)–(A3).

Ad(A1) Let $(f,\phi) : (X,L) \to (M^Y,M)$, $(g,\psi) : (Y,M) \to (N^Z,N)$, $(h,v) : (Z,N) \to (K^W,K)$. Then we are to show that

$$(h,v) \diamond ((g,\psi) \diamond (f,\phi)), \qquad ((h,v) \diamond (g,\psi)) \diamond (f,\phi) \tag{6.9}$$

are the same morphism from (X,L) to (K^W,K). Since the clone composition is defined component-wise, and since the clone composition in the second component is the composition from $\mathbf{C}$, then we have associativity in the second component; and thus we have only to show associativity in the first component, that is,

$$h \diamond (g \diamond f) = (h \diamond g) \diamond f \tag{6.10}$$

as maps from X to K^W. Let $x \in X$. But to show that

$$[h \diamond (g \diamond f)](x) = [(h \diamond g) \diamond f](x) \tag{6.11}$$

as K-subsets of W, let $w \in W$. It follows that

$$\begin{aligned} &([h \diamond (g \diamond f)](x))(w) \\ &\quad = \bigvee_{z\in Z} \left[v^{\vdash} \left(\bigvee_{y\in Y} [\psi^{\vdash}[(f(x))(y)] \otimes (g(y)(z))] \right) \otimes (h(z))(w) \right] \end{aligned} \tag{6.12}$$

and that

$$\begin{aligned} &([(h \diamond g) \diamond f](x))(w) \\ &\quad = \bigvee_{yeY} \left[(v \circ \psi)^{\vdash}[(f(x))(y)] \otimes \left(\bigvee_{z\in Z} [v^{\vdash}[(g(y))(z)] \otimes (h(z))(w)] \right) \right]. \end{aligned} \tag{6.13}$$

The verification that (6.12) and (6.13) are equal, similar to the corresponding verification that (5.4) and (5.5) in the proof of Lemma 5.1 are equal, uses these facts: $\otimes$ distributes from left and right over $\bigvee$; $\otimes$ is associative; each of $\psi^{\vdash}$ and $v^{\vdash}$ preserves $\bigvee$ and $\otimes$; and $(\psi \circ v)^{\vdash} = \psi^{\vdash} \circ v^{\vdash}$ (Remark 1.13).

Ad(A2) Let $(f,\phi) : (X,L) \to (M^Y,M)$, and recall that $\eta_{(Y,M)} : (Y,M) \to (M^Y,M)$ by $\eta_{(Y,M)} = (\eta_Y, \mathrm{id}_M)$, where $\eta_Y(y) = \chi_{\{y\}}^{e_M}$. Then

$$(\eta_Y, \mathrm{id}_M) \diamond (f,\phi) = (\eta_Y \diamond f, \mathrm{id}_M \diamond \phi) = (\eta_Y \diamond f, \mathrm{id}_M \circ \phi) = (\eta_Y \diamond f, \phi). \tag{6.14}$$

Now

$$[(\eta_Y \diamond f)(x)](y) = \bigvee_{y \in Y} [(\mathrm{id}_M)^{\vdash}((f(x))(y)) \otimes (\eta_Y(y))(z)]. \tag{6.15}$$

Since $\mathrm{id}_M^{\vdash} = \mathrm{id}_M^{\mathrm{op}}$,

$$[(\eta_Y \diamond f)(x)](y) = \bigvee_{y \in Y} [(f(x))(y) \otimes (\eta_Y(y))(z)]. \tag{6.16}$$

The completion of the proof now follows, using (L3) and (L4), from Ad(A2) in the proof of Lemma 5.1.

Ad(A3) Let $(f,\phi):(X,L) \to (Y,M)$ and $(g,\psi):(Y,M) \to (N^Z,N)$. Then

$$\begin{gathered}(f,\phi)^{\Delta} = \eta_{(Y,M)} \circ (f,\phi) = (\eta_Y \circ f,\phi),\\ (g,\psi) \diamond (f,\phi)^{\Delta} = (g \diamond (\eta_Y \circ f), \psi \circ \phi),\\ (g,\psi) \circ (f,\phi) = (g \circ f, \psi \circ \phi).\end{gathered} \tag{6.17}$$

So to show that the two previous lines are the same, it suffices to show that $g \diamond (\eta_Y \circ f) = g \circ f$. Note that

$$\begin{aligned}[(g \diamond (\eta_Y \circ f))(x)](y) &= \bigvee_{y \in Y} [\psi^{\vdash}((\eta_Y(f(x)))(y)) \otimes (g(y))(z)]\\ &= \bigvee_{y \in Y} \left[\psi^{\vdash}\begin{pmatrix} e_M, & x \in f^{\leftarrow}\{y\}\\ \perp, & x \notin f^{\leftarrow}\{y\}\end{pmatrix} \otimes (g(y))(z)\right].\end{aligned} \tag{6.18}$$

Since $\psi^{\vdash}$ preserves units and $\perp$, then the proof that the above is $(g(f(x)))(z)$ now follows, using (L3) and (L4), from Ad(A3) in the proof of Lemma 5.1.

(1)⇒(2). Our task now is to show—assuming we have a choice function e such that $\mathbf{T}(\vdash) \equiv (T,\eta,\diamond)$ as constructed in (D1), (D2), (D3) is algebraic in $\mathbf{Set} \times \mathbf{C}$—that $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$.

First, we show that each object of $\mathbf{C}$ is a unital quantale. Let $L \in |\mathbf{C}|$. Then for each $X \in |\mathbf{Set}|$, $(X,L) \in |\mathbf{Set} \times \mathbf{C}|$. If we choose the same sets and functions as in the proof of necessity of Lemma 5.1, always choose L as the lattice, and always choose $\mathrm{id}_L : L \to L$ in $\mathbf{C}$ as the lattice-theoretic component of each morphism, then the proof that (A1)–(A3) implies (L1)–(L3) for L is essentially that given in the proof of necessity of Lemma 5.1—we need only to add the trivial fact that $\mathrm{id}_L^{\vdash} = \mathrm{id}_L^{\mathrm{op}}$. To illustrate, the proof that $\otimes$ distributes across $\bigvee$ from the left in L uses the same X, Y, Z, f, g, h as in Lemma 5.1, along with $K = N = M = L$ and $\phi = \psi = \upsilon = \mathrm{id}_L$; and then the mechanics are essentially the same as for Lemma 5.1. Thus $|\mathbf{C}| \subset |\mathbf{LoUQuant}(\vdash)|$.

To finish the proof of (1)⇒(2), we must verify that the morphisms of $\mathbf{C}$ are in $\mathbf{LoUQuant}(\vdash)$. This verification, using the now established fact that each object of $\mathbf{C}$ satisfies (L1)–(L4), comprises the following series of claims.

Claim 1: For each $\mathbf{C}$-morphism ϕ, $\phi^{\vdash}$ preserves e. Let $\phi : M \to N$ in $\mathbf{C}$. We first show that (A3) implies that $\phi^{\vdash}(e_M)$ is a left-sided unit. Let $L = M$, N be objects in $\mathbf{C}$, pick $X = \{x\}$,

$Y = \{y\}$, $Z = \{z\}$, let $b \in N$, pick $f : X \to Y$ by $f(x) = y$ and $g : Y \to N^Z$ by $g(y) = \underline{b}$ (the constant map with value b). Then $(f, \mathrm{id}_L) : (X,L) \to (Y,M)$, $(g,\phi) : (Y,M) \to (N^Z,N)$, and

$$\begin{aligned}
(\mathrm{A3}) &\Longrightarrow g \diamond (\eta_Y \circ f) = g \circ f \\
&\Longrightarrow \bigvee_{y \in Y} [\phi^{\vdash}((\eta_Y(f(x)))(y)) \otimes (g(y))(z)] = (g(f(x)))(z) \\
&\Longrightarrow \phi^{\vdash}(e_M) \otimes b = b.
\end{aligned} \tag{6.19}$$

Now choosing $b = e_N$ implies that

$$\phi^{\vdash}(e_M) = \phi^{\vdash}(e_M) \otimes e_N = e_N. \tag{6.20}$$

Claim 2: For each **C**-morphism ϕ, $\phi^{\vdash}$ preserves $\otimes$. Let $\phi : L \to K$ in **C** and $a, b \in L$. Choose $X = \{x\}$, $Y = \{y\}$, $Z = \{z\}$, $L = M = N$, W any set, and define $f : X \to M^Y$, $g : Y \to N^Z$, $h : Z \to K^W$ by $f(x) = \underline{a}$, $g(y) = \underline{b}$, $h(z) = \underline{e_K}$. Then $(f, \mathrm{id}_L) : (X,L) \to (M^Y,M)$, $(g, \mathrm{id}_L) : (Y,M) \to (N^Z,N)$, $(h,\phi) : (Z,N) \to (K^W,K)$ in **Set** $\times$ **C**. Now (A1) implies that (6.12) and (6.13) agree. This means that

$$\begin{aligned}
&\bigvee_{z \in Z} \left[\phi^{\vdash} \left(\bigvee_{y \in Y} [\mathrm{id}_L^{\mathrm{op}} [(f(x))(y)] \otimes ((g(y))(z))] \right) \otimes (h(z))(w) \right] \\
&\quad = \bigvee_{yeY} \left[(\phi \circ \mathrm{id}_L^{\mathrm{op}})^{\vdash} [(f(x))(y)] \otimes \left(\bigvee_{z \in Z} [\phi^{\vdash} [(g(y))(z)] \otimes (h(z))(w)] \right) \right],
\end{aligned} \tag{6.21}$$

which reduces to

$$\begin{aligned}
&\phi^{\vdash}([(f(x))(y)] \otimes (g(y)(z))) \otimes (h(z))(w) \\
&\quad = \phi^{\vdash}[(f(x))(y)] \otimes [\phi^{\vdash}[(g(y))(z)] \otimes (h(z))(w)],
\end{aligned} \tag{6.22}$$

that is,

$$\phi^{\vdash}(a \otimes b) \otimes e_K = \phi^{\vdash}(a) \otimes (\phi^{\vdash}(b) \otimes e_K). \tag{6.23}$$

Hence $\phi^{\vdash}(a \otimes b) = \phi^{\vdash}(a) \otimes \phi^{\vdash}(b)$.

Claim 3: For each **C**-morphism ϕ, $\phi^{\vdash}$ preserves arbitrary $\bigvee$ (and hence $\perp$). Let $\phi : N \to K$ in **C**. For the nonempty case, let $\{a_\gamma : \gamma \in \Gamma\} \subset N$. Choose $X = \{x\}$ and $Y = \Gamma$, Z any set, W any set, choose $L = M = N$, and define $f : X \to M^Y$, $g : Y \to N^Z$, $h : Z \to K^W$, by $f(x) = \underline{e_M}$, $g(\gamma) = \underline{a_\gamma}$ (for each γ), $h(z) = \underline{e_K}$ (for each z). Then $(f, \mathrm{id}_L) : (X,L) \to (M^Y,M)$, $(g, \mathrm{id}_M) : (Y,M) \to (N^Z,N)$, $(h,\phi) : (Z,N) \to (K^W,K)$. Now (A1) implies that

(6.12) and (6.13) agree. This means that

$$\begin{aligned}&\bigvee_{z\in Z}\left[\phi^{\vdash}\left(\bigvee_{y\in Y}\left[\mathrm{id}_M^{\mathrm{op}}[(f(x))(y)]\otimes(g(y)(z))\right]\right)\otimes(h(z))(w)\right]\\&\quad=\bigvee_{yeY}\left[(\phi\circ\mathrm{id}_M)^{\vdash}[(f(x))(y)]\otimes\left(\bigvee_{z\in Z}\left[\phi^{\vdash}[(g(y))(z)]\otimes(h(z))(w)\right]\right)\right],\end{aligned}\tag{6.24}$$

which reduces to

$$\begin{aligned}&\phi^{\vdash}\left(\bigvee_{y\in Y}\left[[(f(x))(y)]\otimes(g(y)(z))\right]\right)\otimes(h(z))(w)\\&\quad=\bigvee_{yeY}\left[\phi^{\vdash}[(f(x))(y)]\otimes\left[\phi^{\vdash}[(g(y))(z)]\otimes(h(z))(w)\right]\right],\end{aligned}\tag{6.25}$$

that is,

$$\phi^{\vdash}\left(\bigvee_{y\in Y}[e_M\otimes a_y]\right)\otimes e_K=\bigvee_{yeY}\left(\phi^{\vdash}(e_M)\otimes\phi^{\vdash}(a_y)\right)\otimes e_K.\tag{6.26}$$

Since we have already established that $\phi^{\vdash}$ preserves units and tensors, it follows that

$$\phi^{\vdash}\left(\bigvee_{y\in Y}a_y\right)=\bigvee_{yeY}\phi^{\vdash}(a_y).\tag{6.27}$$

Now for the empty case, rechoose $Y=\varnothing$ in the nonempty case. Then from (6.12), (6.13), we have

$$\phi^{\vdash}(\perp)\otimes e_K=\perp\otimes e_K,\tag{6.28}$$

and hence

$$\phi^{\vdash}(\perp)=\perp.\tag{6.29}$$

This finishes the proof of the empty case, the claim, and Lemma 6.1. □

Proof of Lemma 6.2 (left-adjoint case). In comparison with the proof of Lemma 6.1 and in light of Proposition 1.15, we need only to add to the end of the proof of (1)⇒(2) the verification of the composition law—if $\phi: M\to N$, $\psi: N\to K$, then

$$(\psi\circ\phi)^{\dashv}=\psi^{\dashv}\circ\phi^{\dashv},\tag{6.30}$$

verifying that $\mathbf{C}_{\dashv}(\dashv)\subset\mathbf{SQuant}$. Given such ϕ, ψ, let $\alpha\in M$ and let some object x be given, and choose $L=M$, $X=Y=Z=W=\{x\}$,

$$\begin{aligned}f&:X\longrightarrow M^Y\quad\text{by } f(x)=\underline{\alpha},\\g&:Y\longrightarrow N^Z\quad\text{by } g(x)=\underline{e_N},\\h&:Z\longrightarrow K^W\quad\text{by } h(x)=\underline{e_K},\end{aligned}\tag{6.31}$$

and further consider the **Set** $\times$ **C** morphisms

$$\begin{aligned}(f,\mathrm{id}_L) &: (X,L) \longrightarrow (M^Y,M),\\ (g,\phi) &: (Y,M) \longrightarrow (N^Z,N),\\ (h,\psi) &: (Z,N) \longrightarrow (K^W,K).\end{aligned} \tag{6.32}$$

Then the associativity of $\diamond$ implies that (6.12) and (6.13) agree for these morphisms. But (6.12) in this context reduces to saying that

$$([h \diamond (g \diamond f)](x))(w) = \psi^{\dashv}(\phi^{\dashv}[\alpha] \otimes e_N) \otimes e_K = \psi^{\dashv}(\phi^{\dashv}(\alpha)) \tag{6.33}$$

and (6.13) in this context reduces to saying that

$$\begin{aligned}([(h \diamond g) \diamond f](x))(w) &= (\psi \circ \phi)^{\dashv}[\alpha] \otimes (\psi^{\dashv}[e_N] \otimes e_K)\\ &= (\psi \circ \phi)^{\dashv}[\alpha] \otimes (e_K \otimes e_K)\\ &= (\psi \circ \phi)^{\dashv}(\alpha).\end{aligned} \tag{6.34}$$

Associativity now says that $\psi^{\dashv}(\phi^{\dashv}(\alpha)) = (\psi \circ \phi)^{\dashv}(a)$. □

Remark 6.4 (doubling theories). Since $\otimes$ is generally not commutative, it follows that the tensor products appearing in the definition of the clone compositions in (D3) of Lemmas 6.1, 6.2, 6.3 are ordered according to our choice. As in Remark 5.2, different clone compositions could be chosen by reversing these tensor products; for example, in the case of Lemma 6.1,

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [\psi^{\vdash}((f(x))(y)) \otimes (g(y))(z)] \tag{6.35}$$

could also be chosen as

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [(g(y))(z) \otimes \psi^{\vdash}((f(x))(y))]. \tag{6.36}$$

This yields for each of $\mathbf{T}(\vdash)$, $\mathbf{T}(\dashv)$, $\mathbf{T}(*)$ an alternative clone composition and therefore an alternative theory. Let us denote the theory presented in Lemma 6.1 by $\mathbf{T}_1(\vdash) \equiv (T,\eta,\diamond_1)$, and the alternative theory by $\mathbf{T}_2(\vdash) \equiv (T,\eta,\diamond_2)$, the theory presented in Lemma 6.2 by $\mathbf{T}_1(\dashv) \equiv (T,\eta,\diamond_1)$, and the alternative theory by $\mathbf{T}_2(\dashv) \equiv (T,\eta,\diamond_2)$, and the theory presented in Lemma 6.3 by $\mathbf{T}_1(*) \equiv (T,\eta,\diamond_1)$ and the alternative theory by $\mathbf{T}_2(*) \equiv (T,\eta,\diamond_2)$. We have then the following corollary.

Corollary 6.5. *The following statements hold.*

(1) *The following statements are equivalent:*
  (a) *there is a choice function e such that $\mathbf{T}_1(\vdash)$ is an algebraic theory in* **Set** $\times$ **C***;*
  (b) **C** $\subset$ **LoUQuant**$(\vdash)$*;*
  (c) *there is a choice function e such that $\mathbf{T}_2(\vdash)$ is an algebraic theory in* **Set** $\times$ **C**.
The choice function in any case is given by $e(L)$ = unit of $\otimes$ on L.

(2) *The following statements are equivalent:*
 (a) *there is a choice function e such that* $\mathbf{T}_1(\dashv)$ *is an algebraic theory in* $\mathbf{Set}\times\mathbf{C}$*;*
 (b) $\mathbf{C}\subset\mathbf{LoUQuant}(\dashv)$ *with* $\mathbf{C}_{\dashv}(\dashv)\subset\mathbf{SQuant}$*;*
 (c) *there is a choice function e such that* $\mathbf{T}_2(\dashv)$ *is an algebraic theory in* $\mathbf{Set}\times\mathbf{C}$*.*
 The choice function in any case is given by $e(L)=$ *unit of* $\otimes$ *on* L*.*

(3) *The condition* $\mathbf{C}\subset\mathbf{LoUQuant}^*$ *is sufficient for each of the following statements:*
 (a) $\mathbf{T}_1(*)$ *is an algebraic theory in* $\mathbf{Set}\times\mathbf{C}$*;*
 (b) $\mathbf{T}_2(*)$ *is an algebraic theory in* $\mathbf{Set}\times\mathbf{C}$*.*

The proof is analogous to that for Corollary 5.3.

Remark 6.6. Note that Lemmas 6.1, 6.2 are independent of the axiom of choice because (i) choice functions are part of the construction of the theories in question, (ii) such theories being algebraic are consistent with only one allowable choice function! Due to this fact, we will not *sequens* refer to a choice function making a standard construction theory algebraic; rather, we will simply speak of such a theory being algebraic (in a ground category).

à la Corollary 5.4 and Remark 5.5, we have the following restriction of Lemmas 6.1–6.3, from which the reader can construct the corresponding restrictions of Remark 6.4 and Corollary 6.5 and which incorporates [4] as a special case.

Corollary 6.7 (special restriction of Lemmas 6.1–6.3). *If in Lemmas 6.1–6.3 the first coordinate function*

$$\eta_X(x)(z)=\chi_{\{x\}}^{e_L}(z)\equiv\begin{cases}e_L, & z=x,\\ \perp, & z\neq x,\end{cases} \tag{6.37}$$

of the insertion map component

$$\eta_{(X,L)}\equiv(\eta_X,\mathrm{id}_L):(X,L)\longrightarrow(L^X,L) \tag{6.38}$$

is replaced by

$$\eta_X(x)=\chi_{\{x\}} \tag{6.39}$$

(i.e., replace e_L *with* $\top$*), then each of* $\mathbf{T}_1(\vdash)$ $[\mathbf{T}_1(\dashv)]$ *and* $\mathbf{T}_2(\vdash)$ $[\mathbf{T}_2(\dashv)]$ *is an algebraic theory in* $\mathbf{Set}\times\mathbf{C}$ *if and only if* $\mathbf{C}\subset\mathbf{LoSTQuant}(\vdash)$ $[\mathbf{C}\subset\mathbf{LoSTQuant}(\dashv)$ *with* $\mathbf{C}_{\dashv}(\dashv)\subset\mathbf{SQuant}]$*, and* $\mathbf{T}_1(*)$ $[\mathbf{T}_2(*)]$ *is an algebraic theory in* $\mathbf{Set}\times\mathbf{C}$ *if* $\mathbf{C}\subset\mathbf{LoSTQuant}^*$.

6.2. Algebraic generation of left-adjoint topological powerset theories in Set × C. This section shows that whenever $\mathbf{T}_1(\dashv)$ and $\mathbf{T}_2(\dashv)$ are algebraic theories in $\mathbf{Set}\times\mathbf{C}$, each generates the left-adjoint natural topological powerset theory $\mathbf{P}(\dashv)$ in $\mathbf{Set}\times\mathbf{C}$ (Example 3.23 *et sequens*).

Theorem 6.8. *Let* $\mathbf{C}\subset\mathbf{LoUQuant}(\dashv)$ *with* $\mathbf{C}_{\dashv}(\dashv)\subset\mathbf{SQuant}$*. Then* $\mathbf{T}_1(\dashv)$ *and* $\mathbf{T}_2(\dashv)$ *are algebraic theories in* $\mathbf{Set}\times\mathbf{C}$*. In each case,* $(f,\phi):(X,L)\to(Y,M)$ *lifts to each of*

$T_1(\dashv)(f,\phi):(L^X,L)\to(M^Y,M)$ and $T_2(\dashv)(f,\phi):(L^X,L)\to(M^Y,M)$ and

$$(f,\phi)^{\rightarrow}_{\mathbf{T}_1(\dashv)}=V\big[T_1(\dashv)(f,\phi)\big]=(f,\phi)^{\rightarrow}_{\dashv}=V\big[T_2(\dashv)(f,\phi)\big]=(f,\phi)^{\rightarrow}_{\mathbf{T}_2(\dashv)},\tag{6.40}$$

where $T_1(\dashv)$ and $T_2(\dashv)$ are the functors arising from $\mathbf{T}_1(\dashv)$ and $\mathbf{T}_2(\dashv)$, respectively (Remark 3.2(3)). Hence each of $\mathbf{T}_1(\dashv)$ and $\mathbf{T}_2(\dashv)$ algebraically generates the same natural topological powerset theory $\mathbf{P}(\dashv)$ in $\mathbf{Set}\times\mathbf{C}$.

Proof. Using Definition 4.4, we first discuss the $\mathbf{T}_1(\dashv)$ case. Comparing Example 3.23 and Lemma 6.2 shows that (G1), (G2) are satisfied. To show that (G3) is satisfied, first recall

$$\begin{gathered}(f,\phi)^{\Delta}:(X,L)\longrightarrow(M^Y,M)\quad\text{by }(f,\phi)^{\Delta}=(\eta_Y,\mathrm{id}_{M^Y})\circ(f,\phi)=(\eta_Y\circ f,\phi),\\ T_1(\dashv)(f,\phi):(L^X,L)\longrightarrow(M^Y,M)\\ \text{by }T_1(\dashv)(f,\phi)=(f,\phi)^{\Delta}\diamond(\mathrm{id}_{L^X},\mathrm{id}_L)=((\eta_Y\circ f)\diamond\mathrm{id}_{L^X},\phi),\\ V\big[T_1(\dashv)(f,\phi)\big]:L^X\longrightarrow M^Y\quad\text{by }V\big[T_1(\dashv)(f,\phi)\big]=(\eta_Y\circ f)\diamond\mathrm{id}_{L^X}.\end{gathered}\tag{6.41}$$

Let $a\in L^X$. Then $(V[T_1(\dashv)(f,\phi)])(a)\in M^Y$. To compute this M-subset of Y, let $y\in Y$. Then using (L3), (L4) of Lemma 5.1, we have

$$\begin{aligned}\big[(f,\phi)^{\rightarrow}_{\mathbf{T}_1(\dashv)}(a)\big](y)&=\big[\big(V\big[T_1(\dashv)(f,\phi)\big]\big)(a)\big](y)=\big[\big((\eta_Y\circ f)\diamond\mathrm{id}_{L^X}\big)(a)\big](y)\\ &=\bigvee_{z\in X}\big[\phi^{\dashv}\big((\mathrm{id}_{L^X}(a))(z)\big)\otimes\big(\eta_Y(f(z))\big)(y)\big]\\ &=\bigvee_{z\in X}\big[\phi^{\dashv}(a(z))\otimes\chi_{\{f(z)\}}(y)\big]\\ &=\bigvee_{z\in X}\big\{\phi^{\dashv}(a(z)):z\in f^{\leftarrow}\{y\}\big\}\\ &=\phi^{\dashv}\big((f_L^{\rightarrow}(a))(y)\big).\end{aligned}\tag{6.42}$$

Hence

$$(f,\phi)^{\rightarrow}_{\mathbf{T}_1(\dashv)}(a)=\big(V\big[T_1(\dashv)(f,\phi)\big]\big)(a)=\phi^{\dashv}\big(f_L^{\rightarrow}(a)\big),\tag{6.43}$$

so that

$$V\big[T_1(\dashv)(f,\phi)\big]=\langle\phi^{\dashv}\rangle\circ f_L^{\rightarrow}=(f,\phi)^{\rightarrow}_{\dashv}.\tag{6.44}$$

Therefore (G3) holds. As for (G4), we apply the AFT to $(f,\phi)^{\rightarrow}_{\dashv}$ to obtain $(f,\phi)^{\leftarrow}_{\dashv}$ (see details in [20, 21]), noting that $(f,\phi)^{\rightarrow}_{\dashv}$ is a composition of arbitrary $\bigvee$ preserving maps. Hence $\mathbf{T}_1(\dashv)$ algebraically generates $\mathbf{P}(\dashv)\equiv(P,\to,\leftarrow,V,\eta)$. The proof for the $\mathbf{T}_2(\dashv)$ case is symmetric using (L3), (L4). □

Corollary 6.9. *The following are equivalent:*
(1) $\mathbf{P}(\dashv)$ *is algebraically generated from the algebraic theory* $\mathbf{T}_1(\dashv)$*;*
(2) $\mathbf{C}\subset\mathbf{LoUQuant}(\dashv)$ *with* $\mathbf{C}_{\dashv}(\dashv)\subset\mathbf{SQuant}$*;*
(3) $\mathbf{P}(\dashv)$ *is algebraically generated from the algebraic theory* $\mathbf{T}_2(\dashv)$*.*

Corollary 6.10. *Corollary 6.9 implies Corollary 5.7.*

Proof. Let $L \in |\mathbf{SQuant}|$ and put $\mathbf{C} = \mathbf{L} \equiv (\{L\}, \mathrm{id}_L)$. Then the three statements of Corollary 6.9 under this restriction become, respectively, logically equivalent to the three statements of Corollary 5.7. □

Remark 6.11. Theorem 6.8 and Corollary 6.9 show that two different left-adjoint algebraic theories in $\mathbf{Set} \times \mathbf{C}$ can algebraically generate the same powerset theory in $\mathbf{Set} \times \mathbf{C}$.

Remark 6.12. (1) Analogous to Remark 5.9, note that the derivation of $(f,\phi)^{\rightarrow}_{\dashv}$ needs only the properties (L3), (L4). Moreover, the proof of $(f,\phi)^{\rightarrow}_{\dashv} \dashv (f,\phi)^{\leftarrow}_{\dashv}$ requires only that the underlying L, M be s-quantales and that ϕ^{op} additionally preserve arbitrary $\bigwedge$ (cf. Proposition 1.15); indeed, given s-quantales L, M, this adjunction holding for all $f \in \mathbf{Set}$ is logically equivalent to ϕ being in $\mathbf{LoSQuant}$ such that ϕ^{op} preserves arbitrary $\bigwedge$—the proofs of [20, 21] trivially generalize to the s-quantalic case (cf. Lemma 3.24(2)). Thus, the algebraically generated powerset theories constructed in Theorem 6.8 account for a significant number of left-adjoint topological powerset theories, *but not for most of them, including most left-adjoint adjunctive (q-)topological powerset theories*; that is, most left-adjoint topological powerset theories in $\mathbf{Set} \times \mathbf{C}$ do not arise from the algebraic theories constructed in Lemma 6.2 even though their preimage operators topologically behave like those from powerset theories algebraically generated from such algebraic theories.

(2) To state some of the above comments more precisely, for each $\mathbf{C} \subset \mathbf{LoSQuant}$ [$\mathbf{LoOSQuant}, \mathbf{LoUSQuant}, \mathbf{LoUOSQuant}$], there is an appropriate preimage operator making $\mathbf{C}$-$\mathbf{QTop}$ [$\mathbf{C}$-$\mathbf{QFTop}$, $\mathbf{C}$-$\mathbf{Top}$, $\mathbf{C}$-$\mathbf{FTop}$, resp.] topological over $\mathbf{Set} \times \mathbf{C}$. Some of these preimage operators come from a left-adjoint topological powerset theory which is algebraic, but most of them come from nonalgebraic left-adjoint q-topological or topological powerset theories. However, the syntax of all these preimage operators is the *same*.

(3) For each $\mathbf{C} \subset \mathbf{LoSQuant}$, there is a right-adjoint variable-basis powerset theory in $\mathbf{Set} \times \mathbf{C}$ which does not arise from left-adjoint algebraic theories, and this brings us to the next section (see Theorem 6.24).

6.3. Algebraic generation of new right-adjoint topological powerset theories in $\mathbf{Set} \times \mathbf{C}$. This section creates new variable-basis powerset theories $\mathbf{P}(\vdash)$ in $\mathbf{Set} \times \mathbf{C}$, theories dubbed **right-adjoint** theories. Some of these powerset theories are algebraically generated from right-adjoint algebraic theories $\mathbf{T}_1(\vdash)$ and $\mathbf{T}_2(\vdash)$ in $\mathbf{Set} \times \mathbf{C}$, namely when $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$, and so this section partially parallels the previous section and Example 3.23. Under the restriction $\mathbf{C} \subset \mathbf{LoUSQuant}(\vdash\vdash)$, these new powerset theories become topological; and hence for $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash\vdash)$ [$\mathbf{LoUQuant}(\vdash\vdash)$], the new topological [fuzzy topological] theories generated by these powerset theories are topological categories, resulting in new categories for doing topology and fuzzy topology. These new categories for topology and fuzzy topology have an algebraic foundation when $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash) \cap \mathbf{LoUSQuant}(\vdash\vdash)$. "New" in the preceding statements is justified by proof of nonredundancy at the end of this section.

THEOREM 6.13. *Let* $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$. *Then* $\mathbf{T}_1(\vdash)$ *and* $\mathbf{T}_2(\vdash)$ *are algebraic theories in* $\mathbf{Set} \times \mathbf{C}$. *In each case,* $(f,\phi) : (X,L) \to (Y,M)$ *lifts to* $T_1(\vdash)(f,\phi) : (L^X, L) \to (M^Y, M)$ *and*

$T_2(\vdash)(f,\phi) : (L^X, L) \to (M^Y, M)$, and

$$(f,\phi)^{\rightarrow}_{\mathbf{T}_1(\vdash)} = V[T_1(\vdash)(f,\phi)] = \langle \phi^{\vdash} \rangle \circ f_L^{\rightarrow} = V[T_2(\vdash)(f,\phi)] = (f,\phi)^{\rightarrow}_{\mathbf{T}_2(\vdash)}, \tag{6.45}$$

where $T_1(\vdash)$ and $T_2(\vdash)$ are the functors arising from $\mathbf{T}_1(\vdash)$ and $\mathbf{T}_2(\vdash)$, respectively (Remark 3.2(3)).

The proof, analogous to that of Theorem 6.8, is omitted.

Definition 6.14 (right-adjoint forward/image operators). Let $(f,\phi) : (X,L) \to (Y,M) \in$ **Set** $\times$ **LoSQuant**. Then $(f,\phi)^{\rightarrow}_{\vdash} : L^X \to M^Y$ is the mapping defined by

$$(f,\phi)^{\rightarrow}_{\vdash} = \langle \phi^{\vdash} \rangle \circ f_L^{\rightarrow}, \tag{6.46}$$

that is, for each $a \in L^X$, for each $y \in Y$,

$$[(f,\phi)^{\rightarrow}_{\vdash}(a)](y) = \phi^{\vdash}[(f_L^{\rightarrow}(a))(y)]. \tag{6.47}$$

Lemma 6.15. *If $\mathbf{C} \subset$ **LoSQuant**$(\vdash)$ and $(f,\phi) : (X,L) \to (Y,M) \in$ **Set** $\times$ $\mathbf{C}$, then $(f,\phi)^{\rightarrow}_{\vdash} : L^X \to M^Y$ preserves arbitrary $\bigvee$.*

Proof. Since $\mathbf{C} \subset$ **LoSQuant**$(\vdash)$, $\phi^{\vdash}$ is in **SQuant** and hence preserves arbitrary joins; and it follows that $\langle \phi^{\vdash} \rangle$ preserves arbitrary joins. It is well known that the Zadeh forward operator $f_L^{\rightarrow}$ preserves arbitrary joins. Hence $(f,\phi)^{\rightarrow}_{\vdash}$ is a composition of maps preserving arbitrary joins, and so $(f,\phi)^{\rightarrow}_{\vdash}$ preserves arbitrary joins. $\square$

Theorem 6.16. *Let $\mathbf{C} \subset$ **LoSQuant**$(\vdash)$ and $(f,\phi) : (X,L) \to (Y,M) \in$ **Set** $\times$ $\mathbf{C}$.*

(1) *There exists a unique $(f,\phi)^{\leftarrow}_{\vdash} : L^X \leftarrow M^Y$ such that*

$$(f,\phi)^{\rightarrow}_{\vdash} \dashv (f,\phi)^{\leftarrow}_{\vdash}. \tag{6.48}$$

(2) *Further,*

$$(f,\phi)^{\leftarrow}_{\vdash} = f_L^{\leftarrow} \circ \langle \phi^{\vdash\vdash} \rangle. \tag{6.49}$$

(3) *$(f,\phi)^{\leftarrow}_{\vdash} \in$ **LoSQuant** if and only if $\phi \in$ **LoSQuant**$(\vdash\vdash)$, and $(f,\phi)^{\leftarrow}_{\vdash} \in$ **USQuant** if and only if $\phi \in$ **LoUSQuant**$(\vdash\vdash)$.*

Proof. Because of Lemma 6.15, the unique existence of a right-adjoint of $(f,\phi)^{\rightarrow}_{\vdash}$ is an immediate consequence of AFT Theorem 1.10, proving (1). As for (2), using Notation 1.11, Definition 1.12, and Remark 1.13, we have

$$(f,\phi)^{\leftarrow}_{\vdash} = ((f,\phi)^{\rightarrow}_{\vdash})^{\vdash} = [\langle \phi^{\vdash} \rangle \circ f_L^{\rightarrow}]^{\vdash} = (f_L^{\rightarrow})^{\vdash} \circ \langle \phi^{\vdash} \rangle^{\vdash} = f_L^{\leftarrow} \circ \langle \phi^{\vdash\vdash} \rangle. \tag{6.50}$$

As for (3), this follows from (2) and the properties of $f_L^{\leftarrow}$ tabulated in Proposition 1.21. $\square$

Definition 6.17 (right-adjoint backward/preimage operators). Let $(f,\phi) : (X,L) \to (Y,M)$ $\in$ **Set** $\times$ **LoSQuant**. Then $(f,\phi)^{\leftarrow}_{\vdash} : L^X \leftarrow M^Y$ is the mapping defined by

$$(f,\phi)^{\leftarrow}_{\vdash} = f_L^{\leftarrow} \circ \langle \phi^{\vdash\vdash} \rangle, \tag{6.51}$$

that is, for each $b \in M^Y$,

$$(f,\phi)_{\vdash}^{\leftarrow}(b) = \phi^{\vdash\vdash} \circ b \circ f. \tag{6.52}$$

Definition 6.18 (right-adjoint variable-basis powerset theories in $\mathbf{Set} \times \mathbf{C}$ (cf. Example 3.23)). Let $\mathscr{C} \subset \mathbf{SQuant}$, put $\mathbf{C} = \mathscr{C}^{\mathrm{op}}$, and put $\mathbf{K} = \mathbf{Set} \times \mathbf{C}$. The *right-adjoint (variable-basis) powerset theory*

$$\mathbf{P}(\vdash) \equiv (P, \longrightarrow, \longleftarrow) \tag{6.53}$$

in $\mathbf{Set} \times \mathbf{C}$ is given by the following data: define $P : |\mathbf{K}| \to |\mathscr{C}|$ by

$$P(X,L) = L^X, \tag{6.54}$$

and for $(f,\phi) : (X,L) \to (Y,M)$, put

$$\begin{aligned} (f,\phi)_{\mathbf{P}(\vdash)}^{\rightarrow} &\equiv (f,\phi)_{\vdash}^{\rightarrow} : P(X,L) \longrightarrow P(Y,M), \\ (f,\phi)_{\mathbf{P}(\vdash)}^{\leftarrow} &\equiv (f,\phi)_{\vdash}^{\leftarrow} : P(X,L) \longleftarrow P(Y,M) \end{aligned} \tag{6.55}$$

as given in Definitions 6.14 and 6.17, namely

$$(f,\phi)_{\vdash}^{\rightarrow} = \langle \phi^{\vdash} \rangle \circ f_L^{\rightarrow}, \qquad (f,\phi)_{\vdash}^{\leftarrow}(b) = f_L^{\leftarrow} \circ \langle \phi^{\vdash\vdash} \rangle. \tag{6.56}$$

Finally, given $\mathscr{C} \subset \mathbf{USQuant}$, put $V : \mathbf{K} \to \mathbf{Set}$ by $V = \Pi_1$ (the first projection functor)—$V(X,L) = X$ and $V(f,\phi) = f$, and for $X \in |\mathbf{Set}|$, define $\eta_{(X,L)} : V(X,L) \to P(X,L)$ by

$$\eta_{(X,L)}(x) = \chi_{\{x\}}^{e_L} \tag{6.57}$$

as in Example 3.23; and in this case also write

$$\mathbf{P}(\vdash) \equiv (P, \longrightarrow, \longleftarrow, V, \eta). \tag{6.58}$$

Theorem 6.19 (properties of right-adjoint powerset theories). *Let* $\mathscr{C} \subset \mathbf{SQuant}$ *and* $\mathbf{C} = \mathscr{C}^{\mathrm{op}}$. *The following hold:*

(1) $\mathbf{P}(\vdash) \equiv (P, \to, \leftarrow)$ *is a balanced* $\mathscr{C}$*-powerset theory in* $\mathbf{Set} \times \mathbf{C}$.

(2) *If* $\mathbf{C} \subset \mathbf{LoUSQuant}(\vdash)$, *then* $\mathbf{P}(\vdash) \equiv (P, \to, V, \eta)$ *is an adjunctive concrete* $\mathscr{C}$*-powerset theory in* $\mathbf{Set} \times \mathbf{C}$.

(3) *If* $\mathbf{C} \subset \mathbf{LoSQuant}(\vdash\vdash)$ $[\mathbf{LoUSQuant}(\vdash\vdash)]$, *then* $\mathbf{P}(\vdash) \equiv (P, \leftarrow)$ *is a q-topological [topological]* $\mathscr{C}$*-powerset theory in* $\mathbf{Set} \times \mathbf{C}$.

(4) *If* $\mathbf{C} \subset \mathbf{LoSQuant}(\vdash\vdash)$ $[\mathbf{LoUSQuant}(\vdash\vdash)]$, *then the q-topological [topological] theory* $\mathbf{T}_{(\mathbf{Set}\times\mathbf{C})\mathbf{P}(\vdash)}$ *of* $\mathbf{P}(\vdash)$ *is topological over* $\mathbf{Set} \times \mathbf{C}$ *with respect to the usual forgetful functor by Theorem 3.10.*

(5) *If* $\mathbf{C} \subset \mathbf{LoOSQuant}(\vdash\vdash)$ $[\mathbf{LoUOSQuant}(\vdash\vdash)]$, *then the q-fuzzy [fuzzy] topological theory* $\mathbf{T}_{F(\mathbf{Set}\times\mathbf{C})\mathbf{P}(\vdash)}$ *of* $\mathbf{P}(\vdash)$ *is topological over* $\mathbf{Set} \times \mathbf{C}$ *with respect to the usual forgetful functor.*

(6) *If* $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$, *then* $\mathbf{P}(\vdash) \equiv (P, \to, V, \eta)$ *is algebraic, that is, algebraically generated by each of* $\mathbf{T}_1(\vdash)$ *and* $\mathbf{T}_2(\vdash)$.

(7) *If* $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$, *then* $\mathbf{P}(\vdash) \equiv (P, \rightarrow, \leftarrow, V, \eta)$ *is algebraic, that is, algebraically generated by each of* $\mathbf{T}_1(\vdash)$ *and* $\mathbf{T}_2(\vdash)$.

(8) *If* $\mathbf{C} \subset \mathbf{LoSQuant}(\vdash\vdash) \cap \mathbf{LoUQuant}(\vdash)$ $[\mathbf{LoUSQuant}(\vdash\vdash) \cap \mathbf{LoUQuant}(\vdash)]$, *then* $\mathbf{P}(\vdash) \equiv (P, \rightarrow, \leftarrow, V, \eta)$ *is an adjunctive, concrete, q-topological [topological]* $\mathscr{C}$*-powerset theory in* $\mathbf{Set} \times \mathbf{C}$ *which is algebraic; and if* $\mathbf{C} \subset \mathbf{LoSQuant}(\vdash\vdash) \cap \mathbf{LoSQuant}(\vdash)$ $[\mathbf{LoUSQuant}(\vdash\vdash) \cap \mathbf{LoSQuant}(\vdash)]$, *then* $\mathbf{P}(\vdash) \equiv (P, \rightarrow, \leftarrow, V, \eta)$ *is an adjunctive concrete q-topological [topological]* $\mathscr{C}$*-powerset theory in* $\mathbf{Set} \times \mathbf{C}$ *which need not be algebraic.*

Proof. Ad(1) is immediate from Definition 6.18.

Ad(2) This is straightforward, needing the facts that $f_L^{\rightarrow}(\chi_{\{x\}}^{e_L}) = \chi_{\{f(x)\}}^{e_L}$ and $\langle \phi^{\vdash} \rangle (\chi_{\{f(x)\}}^{e_L}) = \chi_{\{f(x)\}}^{e_M}$, the latter following from $\phi^{\vdash} \in \mathbf{USQuant}$.

Ad(3) We show that Definition 3.4 (QT) [resp. (T)] is satisfied. (QT1) [resp. (T1)] follows from Theorem 6.16(3). For (T2), let $(f, \phi) : (X, L) \to (Y, M)$, $(g, \psi) : (Y, M) \to (Z, N)$ in $\mathbf{Set} \times \mathbf{C}$ be given and let $c \in N^Z$; it is to be checked that $[(g, \psi) \circ (f, \phi)]_{\vdash}^{\leftarrow}(c) = [(f, \phi)_{\vdash}^{\leftarrow} \circ (g, \psi)_{\vdash}^{\leftarrow}](c)$. Repeatedly applying Definition 6.17 in conjunction with Remark 1.13 yields the following:

$$\begin{aligned}
[(g, \psi) \circ (f, \phi)]_{\vdash}^{\leftarrow}(c) &= (g \circ f, \psi \circ \phi)_{\vdash}^{\leftarrow}(c) = (\psi \circ \phi)^{\vdash\vdash} \circ c \circ g \circ f \\
&= (\psi^{\vdash} \circ \phi^{\vdash})^{\vdash} \circ c \circ g \circ f = \phi^{\vdash\vdash} \circ \psi^{\vdash\vdash} \circ c \circ g \circ f \\
&= \phi^{\vdash\vdash} \circ (\psi^{\vdash\vdash} \circ c \circ g) \circ f = \phi^{\vdash\vdash} \circ ((g, \psi)_{\vdash}^{\leftarrow}(c)) \circ f \\
&= (f, \phi)_{\vdash}^{\leftarrow}((g, \psi)_{\vdash}^{\leftarrow}(c)) = [(f, \phi)_{\vdash}^{\leftarrow} \circ (g, \psi)_{\vdash}^{\leftarrow}](c).
\end{aligned} \tag{6.59}$$

And for (T3), we observe that

$$\begin{aligned}
(\mathrm{id}_{(X,L)})_{\mathbf{P}(\vdash)}^{\leftarrow} &= (\mathrm{id}_{(X,L)})_{\vdash}^{\leftarrow} = (\mathrm{id}_X, \mathrm{id}_L)_{\vdash}^{\leftarrow} = (\mathrm{id}_X)_L^{\leftarrow} \circ \langle \mathrm{id}_L^{\vdash\vdash} \rangle \\
&= \mathrm{id}_{L^X} \circ \mathrm{id}_{L^X} = \mathrm{id}_{L^X} = \mathrm{id}_{P(X,L)}.
\end{aligned} \tag{6.60}$$

Ad(4) This is a consequence of (3) and Theorem 3.10.

Ad(5) This is a consequence of (3) and Theorem 3.11.

Ad(6) This is a corollary of (2) and Theorem 6.13.

Ad(7) This follows from (6) and Lemma 6.15 and Theorem 6.16.

Ad(8) The first statement is immediate since (3) and (7) force the satisfaction of Definition 4.4(G4) in addition to (G1)–(G3); and the second statement follows from (2), (3), and Lemma 6.1. □

Corollary 6.20 (cf. Corollary 6.9). *The following are equivalent:*

(1) $\mathbf{P}(\vdash)$ *is algebraically generated from the algebraic theory* $\mathbf{T}_1(\vdash)$*;*

(2) $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$*;*

(3) $\mathbf{P}(\vdash)$ *is algebraically generated from the algebraic theory* $\mathbf{T}_2(\vdash)$.

Corollary 6.21. *Corollary 6.20 implies Corollary 5.7.*

The proof is *à la* the proof of Corollary 6.10.

Remark 6.22 (relationships of right-adjoint (fuzzy) topology to algebraic theories). (1) Theorem 6.19 and Corollary 6.20 show that two different algebraic theories in $\mathbf{Set} \times \mathbf{C}$ can algebraically generate the same powerset theory in $\mathbf{Set} \times \mathbf{C}$.

(2) All of the algebraic, powerset, topological, and fuzzy topological theories being addressed in Theorem 6.19 are nonempty in that all of the categories in question have both nontrivial objects and nontrivial morphisms—this is a consequence of Example 1.17. Further, all such theories are nonredundant compared with left-adjoint theories, that is, cannot be recaptured using left-adjoint theories—see Definition 6.23 and Theorem 6.24.

(3) The powerset theories addressed do not all have an algebraic foundation in the sense of being algebraically generated by $\mathbf{T}_1(\vdash)$ and $\mathbf{T}_2(\vdash)$. Hence, some right-adjoint theories are algebraic while *most are not*, but they all have the *same* syntax. So $\mathbf{T}_1(\vdash)$ and $\mathbf{T}_2(\vdash)$ provide an algebraic check that we have the correct syntax for the new powerset theories created in this section (cf. Remark 6.12(1)).

(4) To continue the previous comment, some of the new topological powerset theories created in this section have an algebraic foundation and most do not, and hence some of the new (q-)topological and (q-)fuzzy topological theories created in this section have an algebraic foundation in the sense of $\mathbf{T}_1(\vdash)$ and $\mathbf{T}_2(\vdash)$ and most do not. But the syntax of all these topological categories is the same and ultimately is algebraically checked by $\mathbf{T}_1(\vdash)$ and $\mathbf{T}_2(\vdash)$. This algebraic check assures us that in the new kinds of variable-basis topology and fuzzy topology created in this section, the syntax (especially for the all-important preimage operator) is correct.

(5) Whether the new categories $\mathbf{T}_{(\mathbf{Set}\times\mathbf{C})\mathbf{P}(\vdash)}$ and $\mathbf{T}_{F(\mathbf{Set}\times\mathbf{C})\mathbf{P}(\vdash)}$ for variable-basis (fuzzy) topology constructed above are syntactically generated from within the context of algebraic theories, it is possible to give an external description of these new categories as well as a more conventional notation. Since the examples of variable-basis categories for topology and fuzzy topology given in Section 1.5 and Example 3.23 are shown in Section 6.2 to exemplify the left-adjoint categories $\mathbf{T}_{(\mathbf{Set}\times\mathbf{C})\mathbf{P}(\dashv)}$ and $\mathbf{T}_{F(\mathbf{Set}\times\mathbf{C})\mathbf{P}(\dashv)}$ for (fuzzy) topology, we will modify for these comments the notation for these left-adjoint categories to be the following: $\mathbf{C}$-$\mathbf{QTop}(\dashv)$, $\mathbf{C}$-$\mathbf{Top}(\dashv)$, $\mathbf{C}$-$\mathbf{QFTop}(\dashv)$, $\mathbf{C}$-$\mathbf{FTop}(\dashv)$. Thus we now give an analogous external description and conventional notation for $\mathbf{T}_{(\mathbf{Set}\times\mathbf{C})\mathbf{P}(\vdash)}$ and $\mathbf{T}_{F(\mathbf{Set}\times\mathbf{C})\mathbf{P}(\vdash)}$ as follows:

(a) $\mathbf{C}$-$\mathbf{QTop}(\vdash)$, $\mathbf{C}$-$\mathbf{Top}(\vdash)$ are defined exactly as $\mathbf{C}$-$\mathbf{QTop}(\dashv)$, $\mathbf{C}$-$\mathbf{Top}(\dashv)$, respectively, in Definition 1.24, but with this modification: $(f,\phi):(X,L)\to(Y,M)$ is a morphism from (X,L,τ) to (Y,M,σ) if the condition for continuity in Definition 1.24(2) is changed to read

$$\tau \subset \left((f,\phi)_{\vdash}^{\leftarrow}\right)^{\rightarrow}(\sigma), \tag{6.61}$$

namely for each $\nu \in \sigma$, $(f,\phi)_{\vdash}^{\leftarrow}(\nu) \equiv \phi^{\vdash\vdash} \circ \nu \circ f \in \tau$.

(b) $\mathbf{C}$-$\mathbf{QFTop}(\vdash)$, $\mathbf{C}$-$\mathbf{FTop}(\vdash)$ are defined exactly as $\mathbf{C}$-$\mathbf{QFTop}(\dashv)$, $\mathbf{C}$-$\mathbf{FTop}(\dashv)$, respectively, in Definition 1.25, but with this modification: $(f,\phi):(X,L)\to(Y,M)$ is a morphism from $(X,L,\mathcal{T})$ to $(Y,M,\mathcal{S})$ if the condition for continuity in

Definition 1.25(2) is changed, in accordance with Definition 3.8(2), to read

$$\mathcal{T} \circ (f,\phi)^{\leftarrow}_{\vdash} \geq \phi^{\mathrm{op}} \circ \mathscr{S}, \tag{6.62}$$

namely for each $b \in M^Y$, $\mathcal{T}(f_L^{\leftarrow}(\langle \phi^{\vdash\vdash}\rangle(b))) \geq \phi^{\mathrm{op}}(\mathscr{S}(b))$.

The following definition and theorem resolve the question: to what extent are right-adjoint theories redundant; namely, when are the powerset operators of a right-adjoint variable-basis powerset theories already given by some left-adjoint variable-basis powerset theory. More precisely, it suffices to have the following definition.

Definition 6.23 (criteria of redundancy of right-adjoint powerset theories). A right-adjoint variable-basis powerset theory $\mathbf{P}(\vdash)$ in $\mathbf{Set} \times \mathbf{C}$ is *redundant* if at least one of the following statements holds:

(1) For each $(f,\phi)^{\rightarrow}_{\vdash}$ in $\mathbf{P}(\vdash)$, $(f,\phi)^{\leftarrow}_{\dashv} \dashv (f,\phi)^{\rightarrow}_{\vdash}$ (i.e., the standard right-adjoint image operator $(f,\phi)^{\rightarrow}_{\vdash}$ is the lower-left-adjoint image operator $(f,\phi)^{\dashv}_{\rightarrow}$ of Example 3.23).

(2) For each $(f,\phi)^{\rightarrow}_{\vdash}$ in $\mathbf{P}(\vdash)$, $(f,\phi)^{\rightarrow}_{\vdash} \dashv (f,\phi)^{\leftarrow}_{\dashv}$ (i.e., the standard right-adjoint image operator $(f,\phi)^{\rightarrow}_{\vdash}$ is the standard left-adjoint image operator $(f,\phi)^{\rightarrow}_{\dashv}$).

Theorem 6.24. *If* $\mathbf{C} \subset \mathbf{LoSQuant}$, *then* $\mathbf{P}(\vdash)$ *is not redundant. In particular, for* $\mathbf{C} \subset \mathbf{LoUQuant}(\vdash)$, *the right-adjoint variable-basis powerset theories generated by* $\mathbf{T}_1(\vdash)$ *and* $\mathbf{T}_2(\vdash)$ *are not redundant.*

Proof. We must be able to deny each of Definition 6.23(1) and (2). The needed counterexamples given below suffice for both claims of the theorem.

Denial of (1). Let $f : X \equiv \{x_1,x_2\} \to Y \equiv \{y\}$, set $L = M = \mathbb{I}$, and put $\phi^{\mathrm{op}} = \mathrm{id}_L$. Now consider the L-subset $a \in L^X$ given by $a(x_1) = 1/3$, $a(x_2) = 2/3$. Then

$$\begin{aligned}(f,\phi)^{\leftarrow}_{\dashv}[(f,\phi)^{\rightarrow}_{\vdash}(a)](x_1) &= \langle \phi^{\mathrm{op}}\rangle(\langle \phi^{\vdash}\rangle[(f_L^{\rightarrow}(a))(f(x_1))]) \\ &= \bigvee \{a(z) : z \in f^{\leftarrow}\{y\}\} \\ &= a(x_2) = \frac{2}{3} > a(x_1).\end{aligned} \tag{6.63}$$

Hence

$$(f,\phi)^{\leftarrow}_{\dashv}[(f,\phi)^{\rightarrow}_{\vdash}(f_L^{\rightarrow}(a))] \nleq a, \tag{6.64}$$

which implies

$$\neg[(f,\phi)^{\leftarrow}_{\dashv} \dashv (f,\phi)^{\rightarrow}_{\vdash}]. \tag{6.65}$$

Denial of (2). Let L, M, ϕ be as in Example 1.17(4), let $f : X \equiv \{x\} \to Y \equiv \{y\}$, and let $d \in L^X$ by $d = \underline{b}$. Recall that

$$\phi^{\vdash}(b) = \perp, \tag{6.66}$$

and hence

$$\phi^{\mathrm{op}}(\phi^{\vdash}(b)) = \phi^{\mathrm{op}}(\perp) = \perp < b. \tag{6.67}$$

Thus

$$(f,\phi)_{\dashv}^{\leftarrow}[(f,\phi)_{\vdash}^{\rightarrow}(d)](x) = \langle\phi^{\mathrm{op}}\rangle(\langle\phi^{\vdash}\rangle[(f_L^{\rightarrow}(d))(f(x))]) = \phi^{\mathrm{op}}(\phi^{\vdash}(b)) < b. \tag{6.68}$$

It follows

$$\neg[(f,\phi)_{\vdash}^{\rightarrow} \dashv (f,\phi)_{\dashv}^{\leftarrow}]. \tag{6.69}$$

This concludes the proof of the theorem. □

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