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Research Article

An L^p-L^q-Version of Morgan's Theorem for the n-Dimensional Euclidean Motion Group

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We establish an L^p-L^q-version of Morgan's theorem for the group Fourier transform on the n-dimensional Euclidean motion group $M(n)$.

1. Introduction

An aspect of uncertainty principle in real classical analysis asserts that a function f and its Fourier transform $\widehat{f}$ cannot decrease simultaneously very rapidly at infinity. As illustrations of this, one has Hardy's theorem [1], Morgan's theorem [2], and Beurling-Hörmander's theorem [3–5]. These theorems have been generalized to many other situations; see, for example, [6–10].

In 1983, Cowling and Price [11] have proved an L^p-L^q-version of Hardy's theorem. An L^p-L^q-version of Morgan's theorem has been also proved by Ben Farah and Mokni [7].

To state the L^p-L^q-versions of Hardy's and Morgan's theorems more precisely, we propose the following.

Let $a,b>0$, $p,q\in[1,+\infty]$, $\alpha\geq 2$, and β such that $1/\alpha+1/\beta=1$.

If we consider measurable functions f on $\mathbb{R}$ such that

$$e^{a|x|^\alpha} f\in L^p(\mathbb{R}),\qquad e^{b|y|^\beta}\widehat{f}\in L^q(\mathbb{R}),\tag{1.1}$$

we obtain the following.

(i) If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta}>(\sin(\pi/2)(\beta-1))^{1/\beta}$, then $f=0$ a.e.

(ii) If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta}\leq(\sin(\pi/2)(\beta-1))^{1/\beta}$, then one has infinitely many such f.

The case $\alpha=\beta=2$, $p=q=+\infty$ corresponds to Hardy's theorem.

The case $\alpha=\beta=2, 1\leq p, q<+\infty$ corresponds to the Cowling-Price theorem.

The case $\alpha>2$, $p=q=+\infty$ corresponds to Morgan's theorem.

The case $\alpha>2, 1\leq p, q<+\infty$ corresponds to the Ben Farah-Mokni theorem.

We remark that for each one of those cases there are further requirements for f if $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = (\sin(\pi/2)(\beta-1))^{1/\beta}$.

In this paper, we give an L^p-L^q-version of Morgan's theorem for the n-dimensional Euclidean motion group $M(n)$, $n \geq 2$.

We can note that for the motion group, theorems of Beurling and Hardy have been studied by Sarkar and Thangavelu [12]. For example, the condition in Theorem 1.1 below for $f = 0$ a.e. for the case $\alpha = 2$ follows from their work.

The motion group $M(n)$ is the semidirect product of $\mathbb{R}^n$ with $K = \mathrm{SO}(n)$. As a set $M(n) = \mathbb{R}^n \times K$, and the group law is given by

$$(x,k)(x',k') = (x + k\cdot x', kk'), \tag{1.2}$$

here $k \cdot x'$ is the naturel action of K on $\mathbb{R}^n$. The Haar measure of $M(n)$ is $dx\,dk$, where dx is the Lebesgue measure on $\mathbb{R}^n$ and dk is the normalized Haar measure on K.

Denote by $\widehat{M}(n)$ the unitary dual of the motion group. The abstract Plancherel theorem asserts that there is a unique measure μ on $\widehat{M}(n)$ such that for all $f \in L^1(M(n)) \cap L^2(M(n))$,

$$\int_{M(n)} |f(x,k)|^2 dx\,dk = \int_{\widehat{M}(n)} \operatorname{tr}(\pi(f)\pi(f)^*)\,d\mu(\pi), \tag{1.3}$$

where $\pi(f) = \int_{M(n)} f(x,k)\pi(x,k)dx\,dk$ is the group Fourier transform of f at $\pi \in \widehat{M}(n)$.

It is well known that μ is supported by the set of infinite-dimensional elements of $\widehat{M}(n)$, which is parametrized by $(r,\lambda) \in]0,\infty[\times \widehat{U}$, where $U = \mathrm{SO}(n-1)$ is the subgroup of $\mathrm{SO}(n)$ leaving fixed $\varepsilon_n = (0,\dots,0,1)$ in $\mathbb{R}^n$. As such an element $\pi_{r,\lambda}$ is realized in a Hilbert space H_λ, we note that for $f \in L^1(M(n)) \cap L^2(M(n))$, $\pi_{r,\lambda}(f)$ is a Hilbert-Schmidt operator on H_λ, moreover the restriction of the Plancherel measure on the part $]0,\infty[\times\{\lambda\}$ is given up to a constant depending only on n, by $r^{n-1}dr$.

For the analogue of Morgan's theorem on $M(n)$ we propose the following version, where we use the notation $\widehat{f}(r,\lambda) = \pi_{r,\lambda}(f)$.

Theorem 1.1. *Let $p,q \in [1,+\infty]$, $a,b \in]0,+\infty[$, and α, β positive real numbers satisfying $\alpha > 2$ and $1/\alpha + 1/\beta = 1$.*

Suppose that f is in $L^2(M(n))$ such that

(i) $e^{a\|x\|^\alpha} f(x,k) \in L^p(M(n))$,

(ii) $e^{br^\beta} \|\widehat{f}(r,\lambda)\|_{HS} \in L^q(\mathbb{R}^+, r^{n-1}dr)$ *for all fixed λ in $\widehat{U}$.*

If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi/2)(\beta-1))^{1/\beta}$, then f is null a.e.

If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} \leq (\sin(\pi/2)(\beta-1))^{1/\beta}$, then there are infinitely many such f.

This paper is organized as follows.

In Section 2, we give a description of the unitary dual of the n-dimensional Euclidean motion group $M(n)$. Section 3 is devoted to the above version of Morgan's theorem for $M(n)$.

2. Description of the unitary dual of $M(n)$

We are going to describe the infinite-dimensional elements of $\widehat{M}(n)$, which are sufficient for the Plancherel formula. We start by some notations.

For any integer m, let $\langle\cdot,\cdot\rangle$ denote the Hermitian (resp., Euclidian) product on $\mathbb{C}^m$ (resp., on $\mathbb{R}^m$) and let $\|\cdot\|$ be the corresponding norm. For $y \neq 0$ in $\mathbb{R}^n$ let U_y be the stabilizer of y in K under its natural action on $\mathbb{R}^n$. U_y is conjugate to the subgroup $U = \mathrm{SO}(n-1)$ of $\mathrm{SO}(n)$ leaving fixed $\varepsilon_n = (0,\dots,0,1)$ in $\mathbb{R}^n$.

We remark that $\widehat{\mathbb{R}}^n$, the set of unitary characters of $\mathbb{R}^n$, is identified with $\mathbb{R}^n$. In fact any such character is of the form χ_y, $y \in \mathbb{R}^n$, and is defined for all $x \in \mathbb{R}^n$ by $\chi_y(x) = e^{i\langle x,y\rangle}$. The trivial character corresponds to $y = 0$.

To construct an infinite-dimensional irreducible unitary representation of the motion group $M(n)$, we use the following steps.

Step 1. Take a nontrivial element χ_y in $\widehat{\mathbb{R}}^n$. It is stabilized under the action of K by U_y.

Step 2. Take $\lambda \in \widehat{U}_y$ and consider $\chi_y \otimes \lambda$ as a representation of the semidirect product of $\mathbb{R}^n$ by U_y denoted by $\mathbb{R}^n \ltimes U_y$.

Step 3. Induce $\chi_y \otimes \lambda$ from $\mathbb{R}^n \ltimes U_y$ to $M(n)$ to obtain a representation $T_{y,\lambda}$ of $M(n)$.

We have then the following properties (see [13, 14] for details).

(a) For $y \neq 0$ and any $\lambda \in \widehat{U}_y$, the representation $T_{y,\lambda}$ is unitary and irreducible.

(b) Every infinite-dimensional irreducible unitary representation of $M(n)$ is equivalent to $T_{y,\lambda}$ for some y and λ as above.

(c) The representations T_{y_1,λ_1} and T_{y_2,λ_2} are equivalent if and only if $\|y_1\| = \|y_2\|$ and λ_1 is equivalent to λ_2 under the obvious identification of U_{y_1} with U_{y_2}.

In particular, when $\|y\| = r > 0$, $T_{y,\lambda}$ is equivalent to $T_{r\varepsilon_n,\lambda}$, so the different classes of infinite-dimensional representations of $M(n)$ can be parametrized by $(r,\lambda) \in]0,\infty[\times\widehat{U}$. We use the notation $\pi_{r,\lambda}$ for $T_{r\varepsilon_n,\lambda}$ and for its equivalence class in $\widehat{M}(n)$. Let us make this representation explicit.

λ is an irreducible unitary representation of $U = \mathrm{SO}(n-1)$, it is of finite dimension d_λ and acts on $\mathbb{C}^{d_\lambda}$. Let H_λ be the vector space of all measurable function $\psi : K \to \mathbb{C}^{d_\lambda}$ such that $\int_K \|\psi(k)\|^2 dk < \infty$ and $\psi(uk) = \lambda(u)(\psi(k))$ for all $u \in U$, $k \in K.H_\lambda$ is a Hilbert space with respect to the inner product defined by

$$(\psi_1 \mid \psi_2) = d_\lambda \int_K \langle \psi_1(k), \psi_2(k)\rangle dk. \tag{2.1}$$

$\pi_{r,\lambda}$ acts on H_λ via

$$[\pi_{r,\lambda}(a,k)\psi](k_0) = e^{i\langle k_0^{-1}\cdot r\varepsilon_n, a\rangle}\psi(k_0k), \quad \psi \in H_\lambda, \tag{2.2}$$

for $a \in \mathbb{R}^n$, $k, k_0 \in K$.

The Plancherel measure μ is then supported by the subset of $\widehat{M}(n)$ given by $\{\pi_{r,\lambda} : \lambda \in \widehat{U}, r \in \mathbb{R}^+\}$, and on each "piece" $\{\pi_{r,\lambda} : r \in \mathbb{R}^+\}$ with λ fixed in $\widehat{U}$, it is given by $C_n r^{n-1} dr$, where C_n is a constant depending only on n.

The Fourier transform of a function f in $L^1(M(n))$ is denoted as above by $\widehat{f}$. It is defined for $(r,\lambda) \in]0,\infty[\times \widehat{U}$ by

$$\widehat{f}(r,\lambda) = \pi_{r,\lambda}(f) = \int_{\mathbb{R}^n}\int_K f(a,k)\pi_{r,\lambda}(a,k)dk\,da \tag{2.3}$$

(the integral being interpreted suitably, see [15]).

By the Plancherel theorem we know that for $f \in L^1(M(n)) \cap L^2(M(n))$, $\widehat{f}(r,\lambda)$ is a Hilbert-Schmidt operator. Let $\|\widehat{f}(r,\lambda)\|_{HS}$ be its Hilbert-Schmidt norm.

3. Morgan's theorem for the motion group

Before giving Morgan's theorem for the motion group $M(n)$, we state the following complex analysis lemma proved by Ben Farah and Mokni [7]. This lemma plays a crucial role in the proof of our main theorem.

Lemma 3.1. *Suppose* $\rho \in]1,2[$, $q \in [1,+\infty]$, $\sigma > 0$, *and* $B > \sigma \sin(\pi/2)(\rho - 1)$.

If g is an entire function on $\mathbb{C}$ *satisfying the conditions*

$$\begin{gathered} |g(x+iy)| \leq \text{const}\, e^{\sigma|y|^\rho} \quad \text{for any } x,y \in \mathbb{R}, \\ e^{B|x|^\rho} g_{|\mathbb{R}} \in L^q(\mathbb{R}), \end{gathered} \tag{3.1}$$

then $g = 0$.

We now give the L^p-L^q-version of Morgan's theorem.

Theorem 3.2. *Let* $p,q \in [1,+\infty]$, $a,b \in]0,+\infty[$, *and* α,β *positive real numbers satisfying* $\alpha > 2$ *and* $1/\alpha + 1/\beta = 1$.

Suppose that f *is a measurable function on* $M(n)$ *such that*

(i) $e^{a\|x\|^\alpha} f(x,k) \in L^p(M(n))$,

(ii) $e^{br^\beta} \|\widehat{f}(r,\lambda)\|_{HS} \in L^q(\mathbb{R}^+, r^{n-1}dr)$ *for all fixed* λ *in* $\widehat{U}$.

If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi/2)(\beta-1))^{1/\beta}$, *then* f *is null a.e.*

Proof. To prove that $f = 0$, we are going to prove that $\widehat{f}(r,\lambda) = 0$. For this, it suffices to show that for fixed $\lambda \in \widehat{U}$ and for any fixed K-finite vectors φ and ψ in H_λ, the condition $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi/2)(\beta-1))^{1/\beta}$ implies that $(\widehat{f}(r,\lambda)\varphi \mid \psi) \equiv 0$ as a function of r and λ.

Let $\lambda \in \widehat{U}$ and let φ, ψ be K-finite vectors in H_λ. We note that φ and ψ are continuous on K and thus bounded. On the other hand, for $r \in \mathbb{R}$,

$$(\widehat{f}(r,\lambda)\varphi \mid \psi) = \int_K\int_{\mathbb{R}^n} f(x,k)(\pi_{r,\lambda}(x,k)\varphi \mid \psi)dx\,dk. \tag{3.2}$$

Let $\Phi_r(x,k) = (\pi_{r,\lambda}(x,k)\varphi \mid \psi)$ for $r \in \mathbb{R}$ and $(x,k) \in M(n)$. Then, by definition of $\pi_{r,\lambda}$, we have

$$\begin{aligned}\Phi_r(x,k) &= d_\lambda \int_K \langle (\pi_{r,\lambda}(x,k)\varphi)(k_0), \psi(k_0)\rangle dk_0 \\ &= d_\lambda \int_K e^{i\langle k_0^{-1}\cdot r\varepsilon_n, x\rangle} \langle \varphi(k_0k), \psi(k_0)\rangle dk_0 \\ &= d_\lambda \int_K e^{i\langle r\varepsilon_n, k_0x\rangle} \langle \varphi(k_0k), \psi(k_0)\rangle dk_0.\end{aligned} \tag{3.3}$$

Note that the integral on the right-hand side makes sense even if $r \in \mathbb{C}$. Hence, with (x,k) fixed, the function $\Phi_r(x,k)$ of the variable r extends to the whole complex plane. One can easily see that for fixed (x,k), $z \mapsto \Phi_z(x,k)$ is an entire function on $\mathbb{C}$. Moreover, for $z \in \mathbb{C}$,

$$|\Phi_z(x,k)| \le d_\lambda \int_K |e^{i\langle z\varepsilon_n, k_0 a\rangle}| \cdot |\varphi(k_0k)| \cdot |\psi(k_0)| dk_0. \tag{3.4}$$

Then

$$|\Phi_z(x,k)| \le A \int_K e^{-\langle (\operatorname{Im} z)\varepsilon_n, k_0x\rangle} dk_0, \tag{3.5}$$

where A is a constant depending only on λ, φ, and ψ. (Note that φ and ψ are continuous functions on K and hence are bounded.)

Using the fact that dk_0 is a normalized measure on K, we obtain

$$|(\Phi_z(x,k))| \le A e^{|\operatorname{Im} z|\cdot\|x\|}. \tag{3.6}$$

By definition of $\Phi_z(x,k)$, we have

$$(\hat{f}(z,\lambda)\varphi \mid \psi) = \int_K \int_{\mathbb{R}^n} f(x,k)\Phi_z(x,k)\,dx\,dk. \tag{3.7}$$

Since f satisfies hypothesis (i) of Theorem 3.2 and $|(\Phi_z(x,k))| \le Ae^{|z|\cdot\|x\|}$, we conclude that the function $r \mapsto (\hat{f}(r,\lambda)\varphi \mid \psi)$ can be extended to the whole of $\mathbb{C}$ and indeed it can be proved that the function

$$z \longmapsto (\hat{f}(z,\lambda)\varphi \mid \psi) \quad \text{is an entire function.} \tag{3.8}$$

Further, from (3.6) and (3.7), we deduce that

$$|(\hat{f}(z,\lambda)\varphi \mid \psi)| \le A \int_K \int_{\mathbb{R}^n} |f(x,k)| e^{|\operatorname{Im} z|\cdot||x||} dx\,dk. \tag{3.9}$$

Let $I =](b\beta)^{-1/\beta}(\sin(\pi/2)(\beta-1))^{1/\beta}, (a\alpha)^{1/\alpha}[$, and $C \in I$. Applying the convex inequality $|ty| \le (1/\alpha)|t|^\alpha + (1/\beta)|y|^\beta$ to the positive numbers $C\|x\|$ and $|\operatorname{Im} z|/C$, we obtain

$$|\operatorname{Im} z| \cdot ||x|| \le \frac{C^\alpha}{\alpha}||x||^\alpha + \frac{1}{\beta C^\beta}|\operatorname{Im} z|^\beta, \tag{3.10}$$

thus

$$\left|\left(\widehat{f}(z,\lambda)\varphi \mid \psi\right)\right| \leq A e^{(1/\beta C^{\beta})|\operatorname{Im} z|^{\beta}} \int_K \int_{\mathbb{R}^n} |f(x,k)| e^{(C^{\alpha}/\alpha)\|x\|^{\alpha}} dx\,dk. \tag{3.11}$$

Then

$$\left|\left(\widehat{f}(z,\lambda)\varphi \mid \psi\right)\right| \leq A e^{(1/\beta C^{\beta})|\operatorname{Im} z|^{\beta}} \int_K \int_{\mathbb{R}^n} e^{a\|x\|^{\alpha}} |f(x,k)| e^{(C^{\alpha}/\alpha - a)\|x\|^{\alpha}} dx\,dk. \tag{3.12}$$

Using this inequality, hypothesis (i), the fact that dk is a normalized measure, and the inequality $a > c^{\alpha}/\alpha$, we obtain

$$\left|\left(\widehat{f}(z,\lambda)\varphi \mid \psi\right)\right| \leq \operatorname{const} e^{(1/\beta C^{\beta})|\operatorname{Im} z|^{\beta}}. \tag{3.13}$$

On the other hand, since $\pi_{-r,\lambda}$ and $\pi_{r,\lambda}$ are equivalent as representations of $M(n)$,

$$\|\widehat{f}(-r,\lambda)\|_{HS} = \|\widehat{f}(r,\lambda)\|_{HS}. \tag{3.14}$$

Hypothesis (ii) of Theorem 3.2 and the inequality (3.14) imply that the function

$$r \longmapsto e^{br^{\beta}} \|\widehat{f}(r,\lambda)\|_{\mathrm{HS}} \quad \text{belongs to } L^q(\mathbb{R}), \tag{3.15}$$

thus

$$r \longmapsto e^{br^{\beta}} \left(\widehat{f}(r,\lambda)\varphi \mid \psi\right)_{L^2(H_\lambda)} \quad \text{belongs to } L^q(\mathbb{R}). \tag{3.16}$$

It is clear from (3.8), (3.13), (3.16) that the function $z \mapsto (\widehat{f}(z,\lambda)\varphi,\psi)$ satisfies the hypothesis of Lemma 3.1, and so

$$\left(\widehat{f}(z,\lambda)\varphi \mid \psi\right) \equiv 0 \tag{3.17}$$

as a function of z.

Since φ, ψ, and λ are arbitrary, then $\widehat{f}(r,\lambda) \equiv 0$ for all $r \in \mathbb{R}_+$ and $\lambda \in \widehat{U}$. Hence, by the Plancherel formula, we get that $f = 0$ a.e. This completes the proof of the theorem. □

In order to prove that our version respects the analogy with Morgan's theorem, let us now establish the sharpness of the condition

$$(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > \left(\sin(\pi/2)(\beta-1)\right)^{1/\beta} \tag{3.18}$$

in Theorem 3.2.

Proposition 3.3. *Let $p,q \in [1,+\infty]$, $a,b \in]0,+\infty[$, and α,β positive real numbers satisfying $\alpha > 2$ and $1/\alpha + 1/\beta = 1$.*

If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} \leq (\sin(\pi/2)(\beta-1))^{1/\beta}$, then there are infinitely many measurable functions on $M(n)$ satisfying

(i) $e^{a\|x\|^{\alpha}} f(x,k) \in L^p(M(n))$,

(ii) $e^{br^{\beta}} \|\widehat{f}(r,\lambda)\|_{HS} \in L^q(\mathbb{R}^+, r^{n-1}dr)$ *for any λ fixed in $\widehat{U}$.*

To prove this proposition, we use the following lemma for a, b, α, β as above.

LEMMA 3.4. *If* $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = (\sin(\pi/2)(\beta-1))^{1/\beta}$, *then for all* $m \in \mathbb{R}$ *and* $m' = (2m + d(2-\alpha))/(2\alpha-2)$, *there exists a nonzero measurable function on* $M(n)$ *satisfying*

(i) $(1+\|x\|)^{-m}e^{a\|x\|^\alpha} f \in L^\infty(M(n))$,

(ii) $(1+r)^{-m'}e^{br^\beta}\|\widehat{f}(r,\lambda)\|_{HS} \in L^\infty(\mathbb{R}^+, r^{n-1}dr)$ *for any fixed* λ *in* $\widehat{U}$.

Proof. We put for (x,k) in $M(n)$

$$f(x,k) = -i\int_C z^\nu e^{z^q - qA\|x\|^2 z}dz, \tag{3.19}$$

where $q = \alpha/(\alpha-2)$, $A^\alpha = (1/4)((\alpha-2)a)^2$, $\nu = (2m+4-\alpha)/2(\alpha-2)$, and C is the path which lies in the half-plane $\mathrm{Re}\, z > 0$, and goes to infinity, in the directions $\arg z = \pm\theta_0$, $\pi/2q < \theta_0 < \pi/q$.

According to Morgan (see [2, page 190]), for $\|x\| \to \infty$, we have

$$f(x,k) \sim (\alpha-2)\left(\frac{(\alpha-2)a}{2}\right)^{m/\alpha}\sqrt{\left(\frac{\pi}{\alpha}\right)}\|x\|^m e^{-a\|x\|^\alpha}. \tag{3.20}$$

On the other hand, for λ fixed in $\widehat{U}$, $(\widehat{f}(r,\lambda)\varphi \mid \psi)$ is equal to

$$-id_\lambda\int_K\int_{\mathbb{R}^n}\int_C\int_K z^\nu e^{z^q - qA\|a\|^2 z}e^{i\langle r\varepsilon_n, k_0 a\rangle}\langle\varphi(k_0k),\psi(k_0)\rangle dk_0\,dz\,da\,dk, \tag{3.21}$$

which by a change of variables $x = k_0^{-1}a$ is equal to

$$-id_\lambda\int_K\int_{\mathbb{R}^n}\int_C\int_K z^\nu e^{z^q - qA\|x\|^2 z}e^{i\langle r\varepsilon_n, x\rangle}\langle\varphi(k_0k),\psi(k_0)\rangle dk_0\,dz\,dx\,dk. \tag{3.22}$$

Using this equality and Fubini's theorem, we obtain the following expression for $(\widehat{f}(r,\lambda)\varphi \mid \psi)$:

$$-id_\lambda\left(\iint_K\langle\varphi(k_0k),\psi(k_0)\rangle dk_0\,dk\right)\int_C\int_{\mathbb{R}^n} z^\nu e^{z^q - qA\|x\|^2 z}e^{i\langle r\varepsilon_n, x\rangle}dx\,dz. \tag{3.23}$$

Since

$$\int_{\mathbb{R}^n} e^{-qA\|x\|^2 z}e^{i\langle k_0^{-1}r\varepsilon_n, x\rangle}dx = \left(\frac{\pi}{qAz}\right)^{n/2}e^{-r^2/4qaz}, \tag{3.24}$$

we deduce that

$$(\widehat{f}(r,\lambda)\varphi \mid \psi) = -id_\lambda\left(\frac{\pi}{qA}\right)^{n/2}\left(\iint_K\langle\varphi(k_0k),\psi(k_0)\rangle dk_0\,dk\right)\int_C z^{\nu-n/2}e^{z^q - r^2/4aqz}dz. \tag{3.25}$$

Now, we fix an orthonormal basis $\{e_j; j \in \mathbb{N}\}$ of H_λ. Taking into account that $\widehat{f}(r,\lambda)$ is a Hilbert-Schmidt operator, we then replace φ by e_i, ψ by e_j and take the sum on $i, j \in \mathbb{N}$ to obtain

$$\|\widehat{f}(r,\lambda)\|_{HS} = \text{const.}\left|\int_C z^{\nu-n/2}e^{z^q-r^2/4aqz}dz\right| \quad a.e. \tag{3.26}$$

Adapting the method of Morgan (see [2, page 191]), we obtain

$$\|\widehat{f}(r,\lambda)\|_{HS} = O(r^{m'}e^{-br^\beta}) \tag{3.27}$$

with $m' = (2m + n(2-\alpha))/(2\alpha - 2)$. We conclude by using the estimations (3.20) and (3.27). □

Proof of Proposition 3.3. It suffices to prove the proposition for

$$(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = \left(sin\frac{\pi}{2}(\beta-1)\right)^{1/\beta}, \tag{3.28}$$

and the rest is a deduction. Let m be a real number verifying

$$m < \min\left(-\frac{n}{p}, \frac{n(1-\alpha)}{q} + \frac{n(\alpha-2)}{2}\right) \tag{3.29}$$

with the convention $1/r = 0$ when $r = \infty$. If $m' = (2m + n(2-\alpha))/(2\alpha - 2)$, then $m' < -n/q$.

For fixed λ in $\widehat{U}$, Lemma 3.4 gives a nonzero measurable function f on $M(n)$ satisfying the inequalities

$$\begin{aligned} e^{a\|x\|^\alpha}\,|\,f(x,k)\,| &\le \text{const.}(1+\|x\|)^m, \\ e^{br^\beta}\|\widehat{f}(r,\lambda)\|_{\text{HS}} &\le \text{const.}(1+r)^{m'}. \end{aligned} \tag{3.30}$$

The conditions $m < -n/p$ and $m' < -n/q$ and the fact that dk is a normalized measure imply that $e^{a\|x\|^\alpha}f$ belongs to $L^p(M(n))$ and $e^{br^\beta}\|\widehat{f}(r,\lambda)\|_{HS}$ belongs to $L^q(\mathbb{R}^+, C_n r^{n-1}dr)$ for fixed λ in $\widehat{U}$. □

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