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Research Article
Certain Coefficient Bounds for p-Valent Functions

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In the present paper, the authors obtain sharp bounds for certain subclasses of p-valent functions. The results are extended to functions defined by convolution.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

defined on the *open* unit disk

$$\Delta = \{z : z \in \mathbb{C} : |z| < 1\}, \tag{1.2}$$

and let $\mathcal{A}_1 := \mathcal{A}$. For $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n, \tag{1.3}$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n. \tag{1.4}$$

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Then we say that the function f is *subordinate* to g if there exists a Schwarz function $\omega(z)$, analytic in Δ with

$$\omega(0)=0, \qquad |\omega(z)|<1 \quad (z\in\Delta), \tag{1.5}$$

such that

$$f(z)=g(\omega(z)) \quad (z\in\Delta). \tag{1.6}$$

We denote this subordination by

$$f\prec g \quad \text{or} \quad f(z)\prec g(z) \quad (z\in\Delta). \tag{1.7}$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0)=g(0), \qquad f(\Delta)\subset g(\Delta). \tag{1.8}$$

Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0)=1$, $\phi'(0)>0$ which maps the open unit disk Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Ali et al. [1] defined and studied the class $S^*_{b,p}(\phi)$ to be the class of functions in $f\in\mathcal{A}_p$ for which

$$1+\frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)}-1\right)\prec\phi(z) \quad (z\in\Delta,\ b\in\mathbb{C}\setminus\{0\}), \tag{1.9}$$

and the class $C_{b,p}(\phi)$ of all functions in $f\in\mathcal{A}_p$ for which

$$1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\phi(z) \quad (z\in\Delta,\ b\in\mathbb{C}\setminus\{0\}). \tag{1.10}$$

Ali et al. [1] also defined and studied the class $R_{b,p}(\phi)$ to be the class of all functions $f\in\mathcal{A}_p$ for which

$$1+\frac{1}{b}\left(\frac{f'(z)}{pz^{p-1}}-1\right)\prec\phi(z) \quad (z\in\Delta,\ b\in\mathbb{C}\setminus\{0\}). \tag{1.11}$$

Note that $S^*_{1,1}(\phi)=S^*(\phi)$ and $C_{1,1}(\phi)=C(\phi)$, the classes introduced and studied by Ma and Minda [2]. The familiar class $S^*(\gamma)$ of *starlike functions* of order γ and the class $C(\gamma)$ of *convex functions* of order γ, $0\le\gamma<1$ are the special case of $S^*_{1,1}(\phi)$ and $C_{1,1}(\phi)$, respectively, when $\phi(z)=(1+(1-2\gamma)z)/(1-z)$.

Owa [3] introduced and studied the class $H_p(A,B,\alpha,\beta)$ of all functions $f\in\mathcal{A}_P$ satisfying

$$(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha}+\beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha}\prec\frac{1+Az}{1+Bz}, \tag{1.12}$$

where $z\in\Delta$, $-1\le B<A\le 1$, $0\le\beta\le 1$, $\alpha\ge 0$. We note that $H_1(A,B,\alpha,\beta)$ is a subclass of Bazilevič functions [4].

Motivated by the classes $H_p(A,B,\alpha,\beta)$ and $R_{b,p}(\phi)$ studied, respectively, by Owa [3] and Ali et al. [1], we now define a class of functions which extends the classes $S^*_{b,p}(\phi)$, $H_p(A,B,\alpha,\beta)$, and $R_{b,p}(\phi)$ in the following.

Definition 1.1. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk Δ onto a region in the right half-plane and is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{A}_p$ is in the class $R_{p,b,\alpha,\beta}(\phi)$ if

$$1 + \frac{1}{b}\left\{(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} - 1\right\} \prec \phi(z) \quad (0 \le \beta \le 1,\ \alpha \ge 0). \tag{1.13}$$

Also, $R_{p,b,\alpha,\beta,g}(\phi)$ is the class of all functions $f \in \mathcal{A}_p$ for which $f * g \in R_{p,b,\alpha,\beta}(\phi)$, where g is a fixed function with positive coefficients.

The class $R_{p,b,\alpha,\beta}(\phi)$ reduces to the following earlier classes.

(1) $R_{p,b,0,1}(\phi) \equiv S^*_{b,p}(\phi)$ introduced and studied by Ali et al. [1].

(2) $R_{p,b,1,1}(\phi) \equiv R_{b,p}(\phi)$ introduced and studied by Ali et al. [1].

(3) $R_{1,1,\alpha,1}(\phi) \equiv B^{\alpha}(\phi)$ introduced and studied by Ravichandran et al. [5].

(4) For $\phi(z) = (1 + Az)/(1 + Bz)$, the class $R_{p,1,\alpha,\beta}(\phi)$ reduces to $H_p(A,B,\alpha,\beta)$ introduced and studied by Owa [3].

(5) For $\phi(z) = (1 + (1 - 2\gamma)z)/(1 - z)$, the class $R_{p,1,\alpha,0}(\phi)$ reduces to

$$\begin{aligned} H_p(1-2\gamma,-1,\alpha,0) &\equiv \mathcal{B}_p(\gamma,\alpha) \\ &= \left\{ f \in \mathcal{A}_p : \operatorname{Re}\left(\frac{f(z)}{z^p}\right)^{\alpha} > \gamma,\ 0 \le \gamma < 1,\ z \in \Delta \right\}. \end{aligned} \tag{1.14}$$

(6) For $\phi(z) = (1 + (1 - 2\gamma)z)/(1 - z)$, the class $R_{p,1,\alpha,1}(\phi)$ reduces to

$$\begin{aligned} H_p(1-2\gamma,-1,\alpha,1) &\equiv \mathcal{C}_p(\gamma,\alpha) \\ &= \left\{ f \in \mathcal{A}_p : \operatorname{Re}\left(\frac{f'(z)(f(z))^{\alpha-1}}{pz^{p-1}}\right) > \gamma,\ 0 \le \gamma < 1,\ z \in \Delta \right\}. \end{aligned} \tag{1.15}$$

(7) $R_{1,1,0,1}(\phi) \equiv S^*(\phi)$ [2].

Very recently, Ali et al. [1] obtained the sharp coefficient inequality for functions in the class $S^*_{b,p}(\phi)$ and many other subclasses $\mathcal{A}_p$.

In the present paper, we prove a sharp coefficient inequality in Theorem 2.1 for the more general class $R_{p,1,\alpha,\beta}(\phi)$. Also we give applications of our results to certain functions defined through Hadamard product. The results obtained in this paper generalize the results obtained by Ali et al. [1], Ma and Minda [2], Ravichandran et al. [5], and Srivastava and Mishra [6].

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \cdots \tag{1.16}$$

in the open unit disk Δ satisfying $|w(z)| < 1$.

To prove our main result, we need the following.

Lemma 1.2 [1]. *If $w \in \Omega$, then*

$$|w_2 - t w_1^2| \le \begin{cases} -t & \text{if } t < -1, \\ 1 & \text{if } -1 \le t \le 1, \\ t & \text{if } t > 1. \end{cases} \tag{1.17}$$

When $t<-1$ or $t>1$, the equality holds if and only if $w(z)=z$ or one of its rotations. If $-1<t<1$, then equality holds if and only if $w(z)=z^2$ or one of its rotations. Equality holds for $t=-1$ if and only if

$$w(z)=z\frac{\lambda+z}{1+\lambda z}\quad(0\le\lambda\le 1) \tag{1.18}$$

or one of its rotations, while for $t=1$, the equality holds if and only if

$$w(z)=-z\frac{\lambda+z}{1+\lambda z}\quad(0\le\lambda\le 1) \tag{1.19}$$

or one of its rotations.

Although the above upper bound is sharp, it can be improved as follows when $-1<t<1$:

$$\begin{aligned}|w_2-tw_1^2|+(t+1)|w_1|^2&\le 1\quad(-1<t\le 0),\\ |w_2-tw_1^2|+(1-t)|w_1|^2&\le 1\quad(0<t<1).\end{aligned} \tag{1.20}$$

Lemma 1.3 [7]. *If $w\in\Omega$, then for any complex number t,*

$$|w_2-tw_1^2|\le\max\{1;|t|\}. \tag{1.21}$$

The result is sharp for the functions $w(z)=z$ or $w(z)=z^2$.

Lemma 1.4 [8]. *If $w\in\Omega$, then for any real numbers q_1 and q_2, the following sharp estimate holds:*

$$|w_3+q_1w_1w_2+q_2w_1^3|\le H(q_1,q_2), \tag{1.22}$$

where

$$H(q_1,q_2)=\begin{cases}1 & \text{for } (q_1,q_2)\in D_1\cup D_2,\\ |q_2| & \text{for } (q_1,q_2)\in\bigcup_{k=3}^{7}D_k,\\ \dfrac{2}{3}(|q_1|+1)\left(\dfrac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{1/2} & \text{for } (q_1,q_2)\in D_8\cup D_9,\\ \dfrac{q_2}{3}\left(\dfrac{q_1^2-4}{q_1^2-4q_2}\right)\left(\dfrac{q_1^2-4}{3(q_2-1)}\right)^{1/2} & \text{for } (q_1,q_2)\in D_{10}\cup D_{11}\setminus\{\pm 2,1\},\\ \dfrac{2}{3}(|q_1|-1)\left(\dfrac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{1/2} & \text{for } (q_1,q_2)\in D_{12}.\end{cases} \tag{1.23}$$

The extremal functions, up to rotations, are of the form

$$\begin{gathered}
w(z)=z^3, \qquad w(z)=z, \qquad w(z)=w_0(z)=\frac{(z[(1-\lambda)\varepsilon_2+\lambda\varepsilon_1]-\varepsilon_1\varepsilon_2 z)}{1-[(1-\lambda)\varepsilon_1+\lambda\varepsilon_2]z},\\
w(z)=w_1(z)=\frac{z(t_1-z)}{1-t_1 z}, \qquad w(z)=w_2(z)=\frac{z(t_2+z)}{1+t_2 z},\\
|\varepsilon_1|=|\varepsilon_2|=1, \quad \varepsilon_1=t_0-e^{-i\theta_0/2}(a\mp b),\ \varepsilon_2=-e^{-i\theta_0/2}(ia\pm b),\\
a=t_0\cos\frac{\theta_0}{2}, \qquad b=\sqrt{1-t_0^2\sin^2\frac{\theta_0}{2}}, \qquad \lambda=\frac{b\pm a}{2b},\\
t_0=\left[\frac{2q_2(q_1^2+2)-3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right]^{1/2}, \qquad t_1=\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{1/2},\\
t_2=\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{1/2}, \qquad \cos\frac{\theta_0}{2}=\frac{q_1}{2}\left[\frac{q_2(q_1^2+8)-2(q_1^2+2)}{2q_2(q_1^2+2)-3q_1^2}\right].
\end{gathered} \tag{1.24}$$

The sets D_k, $k=1,2,\dots,12$, are defined as follows:

$$\begin{aligned}
D_1&=\left\{(q_1,q_2):|q_1|\le\frac{1}{2},\ |q_2|\le 1\right\},\\
D_2&=\left\{(q_1,q_2):\frac{1}{2}\le|q_1|\le 2,\ \frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\le q_2\le 1\right\},\\
D_3&=\left\{(q_1,q_2):|q_1|\le\frac{1}{2},\ q_2\le -1\right\},\\
D_4&=\left\{(q_1,q_2):|q_1|\ge\frac{1}{2},\ q_2\le -\frac{2}{3}(|q_1|+1)\right\},\\
D_5&=\{(q_1,q_2):|q_1|\le 2,\ q_2\ge 1\},\\
D_6&=\left\{(q_1,q_2):2\le|q_1|\le 4,\ q_2\ge\frac{1}{12}(q_1^2+8)\right\},\\
D_7&=\left\{(q_1,q_2):|q_1|\ge 4,\ q_2\ge\frac{2}{3}(|q_1|-1)\right\},\\
D_8&=\left\{(q_1,q_2):\frac{1}{2}\le|q_1|\le 2,\ -\frac{2}{3}(|q_1|+1)\le q_2\le\frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\right\},\\
D_9&=\left\{(q_1,q_2):|q_1|\ge 2,\ -\frac{2}{3}(|q_1|+1)\le q_2\le\frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},\\
D_{10}&=\left\{(q_1,q_2):2\le|q_1|\le 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\le q_2\le\frac{1}{12}(q_1^2+8)\right\},\\
D_{11}&=\left\{(q_1,q_2):|q_1|\ge 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\le q_2\le\frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\},\\
D_{12}&=\left\{(q_1,q_2):|q_1|\ge 4,\ \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\le q_2\le\frac{2}{3}(|q_1|-1)\right\}.
\end{aligned} \tag{1.25}$$

2. Coefficient bounds

By making use of Lemmas 1.2–1.4, we prove the following.

THEOREM 2.1. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, *where* B_n*'s are real with* $B_1 > 0$ *and* $B_2 \geq 0$. *Let* $0 < \beta \leq 1$, $\alpha \geq 0$, $0 \leq \mu \leq 1$, *and*

$$\begin{aligned}
\sigma_1 &:= \frac{(\alpha p+\beta)^2}{2pB_1^2(\alpha p+2\beta)}\left\{2(B_2-B_1) - pB_1^2\frac{(\alpha-1)(\alpha p+2\beta)}{(\alpha+\beta)^2}\right\},\\
\sigma_2 &:= \frac{(\alpha p+\beta)^2}{2pB_1^2(\alpha p+2\beta)}\left\{2(B_2+B_1) - pB_1^2\frac{(\alpha-1)(\alpha p+2\beta)}{(\alpha+\beta)^2}\right\},\\
\sigma_3 &:= \frac{(\alpha p+\beta)^2}{2pB_1^2(\alpha p+2\beta)}\left\{2B_2 - pB_1^2\frac{(\alpha-1)(\alpha p+2\beta)}{(\alpha+\beta)^2}\right\},\\
\Lambda(p,\alpha,\beta,\mu) &:= \frac{(\alpha p+2\beta)(2\mu+\alpha-1)}{2(\alpha p+\beta)^2}.
\end{aligned} \tag{2.1}$$

If $f(z)$ *given by (1.1) belongs to* $R_{p,1,\alpha,\beta}(\phi)$, *then*

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} \dfrac{p}{\alpha p+2\beta}\{B_2 - pB_1^2\Lambda(p,\alpha,\beta,\mu)\} & \text{if } \mu < \sigma_1,\\[2mm] \dfrac{pB_1}{\alpha p+2\beta} & \text{if } \sigma_1 \leq \mu \leq \sigma_2,\\[2mm] -\dfrac{p}{\alpha p+2\beta}\{B_2 - pB_1^2\Lambda(p,\alpha,\beta,\mu)\} & \text{if } \mu > \sigma_2. \end{cases} \tag{2.2}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{1}{2pB_1}\left\{2\left(1-\frac{B_2}{B_1}\right)\frac{(\alpha p+\beta)^2}{\alpha p+2\beta} + (2\mu+\alpha-1)pB_1\right\}|a_{p+1}|^2 \leq \frac{pB_1}{\alpha p+2\beta}. \tag{2.3}$$

If $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$|a_{p+2} - \mu a_{p+1}^2| + \frac{1}{2pB_1}\left\{2\left(1+\frac{B_2}{B_1}\right)\frac{(\alpha p+\beta)^2}{\alpha p+2\beta} - (2\mu+\alpha-1)pB_1\right\}|a_{p+1}|^2 \leq \frac{pB_1}{\alpha p+2\beta}. \tag{2.4}$$

For any complex number μ,

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{pB_1}{\alpha p+2\beta}\max\left\{1, \left|\frac{pB_1}{2}\Lambda(p,\alpha,\beta,\mu) - \frac{B_2}{B_1}\right|\right\}. \tag{2.5}$$

Further,

$$|a_{p+3}| \leq \frac{pB_1}{\alpha p+3\beta}H(q_1,q_2), \tag{2.6}$$

where $H(q_1,q_2)$ is as defined in Lemma 1.4,

$$q_1 := 2\frac{B_2}{B_1} + pB_1\frac{(1-\alpha)(\alpha p+3\beta)}{(\alpha p+\beta)(\alpha p+2\beta)},$$
$$q_2 := \frac{B_3}{B_1} + p^2B_1^2\frac{(\alpha-1)(2\alpha-1)(\alpha p+3\beta)}{6(\alpha p+\beta)^3} + pB_2\frac{(1-\alpha)(\alpha p+3\beta)}{(\alpha p+\beta)(\alpha p+2\beta)}. \tag{2.7}$$

These results are sharp.

Proof. If $f(z) \in R_{p,1,\alpha,\beta}(\phi)$, then there is a Schwarz function

$$w(z) = w_1z + w_2z^2 + \cdots \in \Omega \tag{2.8}$$

such that

$$(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} = \phi(w(z)). \tag{2.9}$$

Since

$$(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha}$$
$$= \begin{cases} 1 + \dfrac{1}{p}(\alpha p+\beta)a_{p+1}z + \dfrac{1}{2p}(\alpha p+2\beta)\{2a_{p+2} + (\alpha-1)a_{p+1}^2\}z^2 \\ + \dfrac{\alpha p+3\beta}{p}\left\{a_{p+3} + (\alpha-1)a_{p+1}a_{p+2} + \dfrac{(\alpha-1)(\alpha-2)}{6}a_{p+1}^3\right\}z^3 + \cdots, \end{cases} \tag{2.10}$$

from (2.9), we have

$$\begin{aligned} a_{p+1} &= \frac{pB_1w_1}{\alpha p+\beta}, \\ a_{p+2} &= \frac{pB_1}{\alpha p+2\beta}\left\{w_2 - w_1^2\left\{pB_1\left(\frac{\alpha-1}{2}\right)\left(\frac{\alpha p+2\beta}{(\alpha p+\beta)^2}\right) - \frac{B_2}{B_1}\right\}\right\}, \\ a_{p+3} &= \frac{pB_1}{\alpha p+3\beta}\{w_3 + q_1w_1w_2 + q_3w_1^3\}, \end{aligned} \tag{2.11}$$

where q_1 and q_2 as defined in (2.7). Therefore, we have

$$a_{p+2} - \mu a_{p+1}^2 = \frac{pB_1}{\alpha p+2\beta}\{w_2 - vw_1^2\}, \tag{2.12}$$

where

$$v := pB_1\Lambda(p,\alpha,\beta,\mu) - \frac{B_2}{B_1}. \tag{2.13}$$

The results (2.2)–(2.5) are established by an application of Lemma 1.2, inequality (2.5) by Lemma 1.3, and (2.6) follows from Lemma 1.4. To show that the bounds in (2.2)–(2.5) are sharp, we define the functions $K_{\phi n}$ $(n = 2,3,\dots)$ by

$$(1-\beta)\left(\frac{K_{\phi n}(z)}{z^p}\right)^{\alpha}+\beta\frac{zK'_{\phi n}(z)}{pf(z)}\left(\frac{K_{\phi n}(z)}{z^p}\right)^{\alpha}=\phi(z^{n-1}),\quad K_{\phi n}(0)=0=\left[K_{\phi n}\right]'(0)-1 \tag{2.14}$$

and the functions F_λ and G_λ $(0 \le \lambda \le 1)$ by

$$\begin{gathered}(1-\beta)\left(\frac{F_\lambda(z)}{z^p}\right)^{\alpha}+\beta\frac{zF_\lambda(z)}{pf(z)}\left(\frac{F_\lambda(z)}{z^p}\right)^{\alpha}=\phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right),\quad F_\lambda(0)=0=F'_\lambda(0)-1,\\ (1-\beta)\left(\frac{G_\lambda(z)}{z^p}\right)^{\alpha}+\beta\frac{zG_\lambda(z)}{pf(z)}\left(\frac{G_\lambda(z)}{z^p}\right)^{\alpha}=\phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right),\quad G_\lambda(0)=0=G'_\lambda(0)-1.\end{gathered} \tag{2.15}$$

Clearly, the functions $K_{\phi n}, F_\lambda, G_\lambda \in R_{p,1,\alpha,\beta}(\phi)$. Also we write $K_\phi := K_{\phi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_λ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is G_λ or one of its rotations. □

Remark 2.2. For $\alpha = 0$ and $\beta = 1$, results (2.2)–(2.6) coincide with the results obtained for the class $S_p^*(\phi)$ by Ali et al. [1].

Remark 2.3. For $\alpha = 0$, $p = 1$ and $\beta = 1$, results (2.2)–(2.6) coincide with the results obtained for the class $S^*(\phi)$ by Ma and Minda [2].

Remark 2.4. For $p = 1$ and $\beta = 1$, results (2.2)–(2.6) coincide with the results obtained for the Bazilevic class $B^\alpha(\phi)$ by Ravichandran et al. [5].

3. Applications to functions defined by convolution

We define $R_{p,b,\alpha,\beta,g}(\phi)$ to be the class of all functions $f \in \mathcal{A}_p$ for which $f * g \in R_{p,b,\alpha,\beta}(\phi)$, where g is a fixed function with positive coefficients and the class $R_{p,b,\alpha,\beta}(\phi)$ is as defined in Definition 1.1. In Theorem 2.1, we obtained the coefficient estimate for the class $R_{p,1,\alpha,\beta}(\phi)$. Now, we obtain the coefficient estimate for the class $R_{p,1,\alpha,\beta,g}(\phi)$.

Theorem 3.1. *Let $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$, where B_n's are real with $B_1 > 0$ and $B_2 \ge 0$. Let $0 < \beta \le 1$, $\alpha \ge 0$, $0 \le \mu \le 1$, and*

$$\begin{aligned}\sigma_1 &:= \frac{g_{p+1}^2}{g_{p+2}}\frac{(\alpha p+\beta)^2}{2pB_1^2(\alpha p+2\beta)}\left\{2(B_2-B_1)-pB_1^2\frac{(\alpha-1)(\alpha p+2\beta)}{(\alpha+\beta)^2}\right\},\\ \sigma_2 &:= \frac{g_{p+1}^2}{g_{p+2}}\frac{(\alpha p+\beta)^2}{2pB_1^2(\alpha p+2\beta)}\left\{2(B_2+B_1)-pB_1^2\frac{(\alpha-1)(\alpha p+2\beta)}{(\alpha+\beta)^2}\right\},\\ \sigma_3 &:= \frac{g_{p+1}^2}{g_{p+2}}\frac{(\alpha p+\beta)^2}{2pB_1^2(\alpha p+2\beta)}\left\{2B_2-pB_1^2\frac{(\alpha-1)(\alpha p+2\beta)}{(\alpha+\beta)^2}\right\},\\ \Lambda_1(p,\alpha,\beta,g,\mu) &:= \frac{(\alpha p+2\beta)\left(2\mu\left((g_{p+2})/(g_{p+1}^2)\right)+\alpha-1\right)}{2(\alpha p+\beta)^2}.\end{aligned} \tag{3.1}$$

If $f(z)$ given by (1.1) belongs to $R_{p,1,\alpha,\beta,g}(\phi)$, then

$$|a_{p+2}-\mu a_{p+1}^2| \le \begin{cases} \dfrac{p}{(\alpha p+2\beta)g_{p+2}}\{B_2-pB_1^2\Lambda_1(p,\alpha,\beta,g,\mu)\} & \text{if } \mu<\sigma_1, \\ \dfrac{pB_1}{(\alpha p+2\beta)g_{p+2}} & \text{if } \sigma_1\le\mu\le\sigma_2, \\ -\dfrac{p}{(\alpha p+2\beta)g_{p+2}}\{B_2-pB_1^2\Lambda_1(p,\alpha,\beta,g,\mu)\} & \text{if } \mu>\sigma_2. \end{cases} \tag{3.2}$$

Further, if $\sigma_1\le\mu\le\sigma_3$, then

$$\begin{aligned} &|a_{p+2}-\mu a_{p+1}^2| + \frac{g_{p+1}^2}{g_{p+2}}\frac{1}{2pB_1}\left\{2\left(1-\frac{B_2}{B_1}\right)\frac{(\alpha p+\beta)^2}{\alpha p+2\beta}+(2\mu+\alpha-1)pB_1\right\}|a_{p+1}|^2 \\ &\quad\le \frac{pB_1}{(\alpha p+2\beta)g_{p+2}}. \end{aligned} \tag{3.3}$$

If $\sigma_3\le\mu\le\sigma_2$, then

$$\begin{aligned} &|a_{p+2}-\mu a_{p+1}^2| + \frac{g_{p+1}^2}{g_{p+2}}\frac{1}{2pB_1}\left\{2\left(1+\frac{B_2}{B_1}\right)\frac{(\alpha p+\beta)^2}{\alpha p+2\beta}-(2\mu+\alpha-1)pB_1\right\}|a_{p+1}|^2 \\ &\quad\le \frac{pB_1}{(\alpha p+2\beta)g_{p+2}}. \end{aligned} \tag{3.4}$$

For any complex number μ,

$$|a_{p+2}-\mu a_{p+1}^2| \le \frac{pB_1}{(\alpha p+2\beta)g_{p+2}}\max\left\{1, \left|\frac{pB_1}{2}\Lambda_1(p,\alpha,\beta,g,\mu)-\frac{B_2}{B_1}\right|\right\}. \tag{3.5}$$

Further,

$$|a_{p+3}| \le \frac{pB_1}{(\alpha p+3\beta)g_{p+3}}H(q_1,q_2), \tag{3.6}$$

where $H(q_1,q_2)$ is as defined in Lemma 1.4,

$$\begin{aligned} q_1 &:= 2\frac{B_2}{B_1}+pB_1\frac{(1-\alpha)(\alpha p+3\beta)}{(\alpha p+\beta)(\alpha p+2\beta)}, \\ q_2 &:= \frac{B_3}{B_1}+p^2B_1^2\frac{(\alpha-1)(2\alpha-1)(\alpha p+3\beta)}{6(\alpha p+\beta)^3}+pB_2\frac{(1-\alpha)(\alpha p+3\beta)}{(\alpha p+\beta)(\alpha p+2\beta)}. \end{aligned} \tag{3.7}$$

These results are sharp.

Proof. If $f(z)\in R_{p,1,\alpha,\beta,g}(\phi)$, then there is a Schwarz function

$$w(z)=w_1z+w_2z^2+\cdots\in\Omega \tag{3.8}$$

such that

$$(1-\beta)\left(\frac{(f*g)(z)}{z^p}\right)^{\alpha}+\beta\frac{z(f*g)'(z)}{p(f*g)(z)}\left(\frac{(f*g)(z)}{z^p}\right)^{\alpha}=\phi(w(z)). \tag{3.9}$$

Hence

$$\begin{aligned}(1-\beta)&\left(\frac{(f*g)(z)}{z^p}\right)^{\alpha}+\beta\frac{z(f*g)'(z)}{p(f*g)(z)}\left(\frac{(f*g)(z)}{z^p}\right)^{\alpha}\\ &=\begin{cases}1+\dfrac{1}{p}(\alpha p+\beta)a_{p+1}g_{p+1}z+\dfrac{1}{2p}(\alpha p+2\beta)\{2a_{p+2}g_{p+2}+(\alpha-1)a_{p+1}^2g_{p+1}^2\}z^2\\ +\dfrac{\alpha p+3\beta}{p}\left\{a_{p+3}g_{p+3}+(\alpha-1)a_{p+1}g_{p+1}a_{p+2}g_{p+2}+\dfrac{(\alpha-1)(\alpha-2)}{6}a_{p+1}^3g_{p+1}^3\right\}z^3+\cdots.\end{cases}\end{aligned} \tag{3.10}$$

The remaining proof of the theorem is similar to the proof of Theorem 2.1 and hence omitted. □

Remark 3.2. For $\alpha=1$ and $\beta=1$, results (3.2)–(3.4) coincide with the results obtained for the class $R_{b,p}(\phi)$ by Ali et al. [1].

Remark 3.3. For $p=1$, $\alpha=0$, and $\beta=1$, results (3.5) coincide with the result for the class $S_b^*(\phi)$ obtained by Ravichandran et al. [9].

Remark 3.4. For $p=1$, $\alpha=1$, $\beta=1$, and $\phi(z)=(1+Az)/(1+Bz)$, $-1\le B<A\le 1$, inequality (3.5) coincides with the result obtained by Dixit and Pal [10].

Remark 3.5. For $p=1$, $\alpha=0$, and $\beta=1$,

$$\begin{gathered}g_2:=\frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)}=\frac{2}{2-\lambda},\\ g_3:=\frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)}=\frac{6}{(2-\lambda)(3-\lambda)},\\ B_1=\frac{8}{\pi^2},\qquad B_2=\frac{16}{3\pi^2},\end{gathered} \tag{3.11}$$

in inequalities (3.2)–(3.4), we get the result obtained by Srivastava and Mishra [6].

THEOREM 3.6. *Let $\phi(z)$ be as in Theorem 2.1. If $f(z)$ given by (1.1) belongs to $R_{p,b,\alpha,\beta,g}(\phi)$, then for any complex number μ, with $B_1>0$, $B_2\ge 0$, $0<\beta\le 1$, $\alpha\ge 0$,*

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\le\frac{p|b|B_1}{(\alpha p+2\beta)g_{p+2}}\max\left\{1,\left|bpB_1\Lambda_2(p,b,\alpha,\mu,g)+\frac{B_2}{B_1}\right|\right\}, \tag{3.12}$$

where

$$\Lambda_2(p,b,\alpha,\beta,\mu,g):=\frac{(\alpha p+2\beta)(2\mu((g_{p+2})/(g_{p+1}^2))+\alpha-1)}{2(\alpha p+\beta)^2}. \tag{3.13}$$

Proof. The proof is similar to the proof of Theorem 2.1 and hence omitted. □

Remark 3.7. For $p = 1$, $\beta = 1$, and $\alpha = 0$, the result in (3.12) coincides with the results obtained by Ravichandran et al. [9].

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