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Research Article

Topological Classification of Conformal Actions on pq-Hyperelliptic Riemann Surfaces

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A compact Riemann surface X of genus $g > 1$ is said to be *p-hyperelliptic* if X admits a conformal involution ρ, for which X/ρ is an orbifold of genus p. If in addition X is q-hyperelliptic, then we say that X is *pq-hyperelliptic*. Here we study conformal actions on pq-hyperelliptic Riemann surfaces with central p- and q-hyperelliptic involutions.

1. Introduction

We say that the finite group G acts on a topological surface X if there exists a monomorphism $\varepsilon : G \to \mathrm{Hom}^+(X)$, where $\mathrm{Hom}^+(X)$ is the group of orientation-preserving homeomorphisms of X. Two actions of finite groups G and G' on X are topologically equivalent if the images of G and G' are conjugate in $\mathrm{Hom}^+(X)$. There are two reasons for the topological classification of finite actions rather than just the groups of homeomorphisms. Firstly, the equivalence classes of group actions are in 1-1 correspondence to conjugacy classes of finite subgroups of the mapping class group and so such a classification gives some information on the structure of this group. Secondly, the enumeration of finite group actions is a principal component of the analysis of singularities of the moduli space of conformal equivalence classes of Riemann surfaces of a given genus since this space is an orbit space of Teichmüller space by a natural action of the mapping class group, see [1].

The classification of conformal actions up to topological conjugacy is a classical problem which up to now was solved for surfaces of genera $g = 2,3$ in [2] (the paper omits one group for the genus 3), $g = 4$ in [3], for elliptic-hyperelliptic surfaces in [4] and for 2-hyperelliptic Riemann surfaces in [5]. Bujalance et al. [9] determined, for each g, the full automorphism groups of a hyperelleptic Riemann surface and Weaver classified their action in [7].

A compact Riemann surface X of genus $g \geq 2$ is said to be *p-hyperelliptic* if X admits a conformal involution ρ, called a *p-hyperelliptic involution*, such that X/ρ is an orbifold of genus p. This notion has been introduced by Farkas and Kra [6], where they also proved that for $g > 4p+1$, the p-hyperelliptic involution is unique and central in the full automorphism group of X. A Riemann surface which is p- and q-hyperelliptic simultaneously is called *pq-hyperelliptic*. In [8] it was shown that for $0 \leq p \leq q$ with $pq \neq 0$, such a surface of genus $g \geq 2$ exists if and only if $2q-1 \leq g \leq 2p+2q+1$. Here we restrict our attention to conformal actions on pq-hyperelliptic Riemann surfaces whose p- and q-hyperelliptic involutions are central in the full automorphism group. In particular, according to [8, Theorem 3.7], this class of surfaces contains all pq-hyperelliptic Riemann surfaces of genera g for $2 \leq p < q < 2p$ and $g > 3q+1$.

For commuting p- and q-hyperelliptic involutions δ, ρ, let k be the genus of $X/\langle\rho,\delta\rangle$. Then $\rho\delta$ is a $(g-p-q+2k)$-hyperelliptic involution and k is in the range $0 \leq k \leq (2p+2q+1-g)/4$. Let $X_k^{p,q}$ denote a Riemann surface with central p- and q-hyperelliptic involutions corresponding given k and let G be an automorphism group of $X_k^{p,q}$. The group $\widetilde{G} = G/\langle\delta,\rho\rangle$ acts on the surface $X_k^{p,q}/\langle\delta,\rho\rangle$ of genus k. Using the known classification of finite group actions on surfaces of low genera, we determine the presentation of $\widetilde{G}$ and next we lift $\widetilde{G}$ to the group acting on the surface $X_k^{p,q}$. The method is similar to that used in [4, 5, 9], however this time it involves many more calculations and the set of topological classes of actions is much bigger. For this reason, we restrict ourselves to $k = 0,1,2$ only. These are the only possible values of k corresponding g in range $2p+2q-10 \leq g \leq 2p+2q+1$. We give the full topological classification of actions on surfaces $X_k^{p,q}$ of such genera except $X_1^{p,q}$ of genus $2p+2q-3$ and $X_2^{p,q}$ of genus $2p+2q-7$ and decide which of them can be chosen to be full. For $k=0$, we enlarge our assumption to $g > 2q-1$. In the general case, the two exceptional surfaces need many more calculations than others and so we omit them. However, for the particular values of g, p, and q it is not difficult to complete the gap.

The main results are presented in Theorems 3.4–3.7 and the supporting tables. In particular, Theorem 3.7 lists the actions on any pq-hyperelliptic Riemann surface of genus g in range $2p+2q-2 \leq g \leq 2p+2q+1$ for $5 \leq p < q < 2p-3$. As an example, we give the actions on 5- , 6-hyperelliptic Riemann surfaces of genus 20, 21, 22, and 23 (see Table 1.1). Every action is determined by the finite group of automorphisms G, the signature of a Fuchsian group Λ and a surface-kernel epimorphism $\theta : \Lambda \to G$ defined by a so-called generating vector which is the sequence of the images of the canonical generators of Λ.

In addition, we give the examples of group actions on Riemann surfaces of genus 2, 3, and 4 with central p- and q-hyperelliptic involutions. Since $2q-1 \leq g$, it follows that $q=1$ for $g=2$ and $q \in \{1,2\}$ for $g \in \{3,4\}$. Thus there are the following surfaces: $X_0^{0,1}$ and $X_0^{1,1}$ of genus 2, $X_0^{0,2}$, $X_0^{1,2}$, $X_1^{1,2}$, $X_1^{2,2}$, $X_0^{0,1}$, $X_0^{1,1}$ of genus 3, and $X_0^{1,1}$, $X_0^{0,2}$, $X_0^{1,2}$, $X_0^{2,2}$, $X_1^{2,2}$ of genus 4 (see Table 1.2).

2. Preliminaries

We will approach the problem using the Riemann uniformization theorem by which each compact Riemann surface X of genus $g \geq 2$ can be represented as the orbit space of the

Table 1.1

Presentation of G	$\sigma(\Lambda)$	Generating vector
	$g = 20$	
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{23}{\ldots},2]$	$(z,\overset{11}{\ldots},z,w,\overset{9}{\ldots},w,zw,zw,zw)$
$\langle x : x^6\rangle \oplus \langle z : z^2\rangle$	$[6,6,2,\overset{7}{\ldots},2]$	$(x,x^{-1},x^3,x^3,x^3,z,z,z,zx^3)$
	$g = 21$	
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{24}{\ldots},2]$	$(z,\overset{12}{\ldots},z,w,\overset{10}{\ldots},w,zw,zw)$
$\langle x : x^2\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{14}{\ldots},2]$	$(xz,x^{-1},z,\overset{6}{\ldots},z,w,\overset{5}{\ldots},w,zw)$
$\langle x : x^4\rangle \oplus \langle z : z^2\rangle$	$[4,4,2,\overset{11}{\ldots},2]$	$(x,x^{-1},x^2,\overset{5}{\ldots},x^2,z,\overset{5}{\ldots},z,x^2z)$
—	—	$(x^3z,x^{-1},x^2,\overset{4}{\ldots},x^2,z,\overset{6}{\ldots},z,zx^2)$
—	—	$(x,x^{-1},z,\overset{6}{\ldots},z,x^2z,\overset{5}{\ldots},zx^2)$
$\langle x,y : x^2,y^4,(xy)^4,(y^2x)^2,((xy)^2x)^2\rangle$	$[2,4,4,2,\overset{5}{\ldots},2]$	$(x,y^3,xy,y^2,y^2,(xy)^2y^2,\overset{3}{\ldots},(xy)^2y^2)$
	$g = 22$	
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{25}{\ldots},2]$	$(z,\overset{13}{\ldots},z,w,\overset{11}{\ldots},w,zw)$
	$g = 23$	
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{26}{\ldots},2]$	$(z,\overset{14}{\ldots},z,w,\overset{12}{\ldots},w)$
$\langle x : x^2\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{15}{\ldots},2]$	$(xz,x^{-1},z,\overset{7}{\ldots},z,w,\overset{6}{\ldots},w)$
$\langle x : x^4\rangle \oplus \langle z : z^2\rangle$	$[4,4,2,\overset{12}{\ldots},2]$	$(x,x^{-1},x^2,\overset{6}{\ldots},x^2,z,\overset{6}{\ldots},z)$
—	—	$(x^3z,x^{-1},x^2,\overset{5}{\ldots},x^2,z,\overset{7}{\ldots},z)$
$\langle x : x^6\rangle \oplus \langle z : z^2\rangle$	$[6,6,2,\overset{8}{\ldots},2]$	$(x,x^{-1},x^3,x^3,z^3,x^3,z,z,z,z)$
$\langle x : x^8\rangle \oplus \langle z : z^2\rangle$	$[8,8,2,\overset{6}{\ldots},2]$	$(x^5z,x^{-1},x^4,x^4,x^4,z,z,z)$
$\langle x : x^{12}\rangle \oplus \langle z : z^2\rangle$	$[12,12,2,2,2,2]$	(x,x^{-1},x^6,x^6,z,z)
$\langle x : x^{24}\rangle \oplus \langle z : z^2\rangle$	$[24,24,2,2]$	$(x^{13}z,x^{-1},x^{12},z)$
$\langle x,y : x^4,y^2,(xy)^2,(x^{-1}y)^2\rangle \oplus \langle z : z^2\rangle$	$[2,4,2,\overset{7}{\ldots},2]$	$((xy)^{-1}z,x^3,y,x^2,x^2,x^2,z,z,z)$
$\langle x,y : x^2,y^2,(xy)^4\rangle \oplus \langle z : z^2\rangle$	$[2,2,4,2,\overset{6}{\ldots},2]$	$(xz,y,xy,(xy)^2,(xy)^2,(xy)^2,z,z,z)$
$\langle x,y : x^2,y^2,(xy)^6\rangle \oplus \langle z : z^2\rangle$	$[2,2,6,2,2,2,2]$	$(x,y,(xy)^{-1},(xy)^3,(xy)^3,z,z)$
$\langle x,y : x^2,y^2,(xy)^{12}\rangle \oplus \langle z : z^2\rangle$	$[2,2,12,2,2]$	$(xz,y(xy)^6,(xy)^{-1},(xy)^6,z)$
$\langle x,y : x^2y^2,y^4,x^{-1}yxy\rangle \oplus \langle z : z^2\rangle$	$[4,4,4,2,\overset{5}{\ldots},2]$	$(xz,(yx)^{-1},y,x^2,x^2,z,z,z)$
$\langle x,y : x^2y^3,y^6,x^{-1}yxy\rangle \oplus \langle z : z^2\rangle$	$[4,4,6,2,2,2]$	$(x^3,(yx)^{-1},y,x^2,z,z)$
$\langle x,y : x^2y^6,y^{12},x^{-1}yxy\rangle \oplus \langle z : z^2\rangle$	$[4,4,12,2]$	$(xz,(yx)^{-1},y,z)$
$\langle x,y : x^2y^2,y^4,(xy)^4\rangle$	$[4,4,4,2,\overset{5}{\ldots},2]$	$(x,y,xy,x^2,x^2,(xy)^2,(xy)^2,(xy)^2)$
$\langle x,y : x^2y^2,y^4,(xy)^6\rangle$	$[4,4,6,2,2,2]$	$(x^3,y,(xy)^{-1},x^2,(xy)^3,(xy)^3)$
$\langle x,y : x^2y^2,y^4,(xy)^{12}\rangle$	$[4,4,12,2]$	$(x,y(xy)^6,(xy)^{-1},(xy)^6)$
$\langle x,y : x^4,x^2(xy)^4,y^4,y^2xy^2x^{-1}\rangle$	$[4,4,8,2,2]$	$(x^3,y^3,(xy)^{-1},x^2,y^2)$

Table 1.2

Presentation of G	$\sigma(\Lambda)$	Generating vector
Actions on the surfaces $X_0^{0,1}$ and $X_0^{1,1}$ of genus 2		
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{5}{\ldots},2]$	(z,z,z,w,zw)
$\langle x : x^6\rangle \oplus \langle z : z^2\rangle$	$[6,6,2]$	(x,x^2z,x^3z)
Actions on the surfaces $X_0^{0,2}$, $X_0^{1,2}$, $X_0^{0,1}$ of genus 3		
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{6}{\ldots},2]$	(z,z,z,z,w,w)
$\langle x : x^2\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{5}{\ldots},2]$	(xw,x,z,z,w)
$\langle x : x^4\rangle \oplus \langle z : z^2\rangle$	$[4,4,2,2]$	(xz,x,z,x^2)
$\langle x : x^4\rangle \oplus \langle z : z^2\rangle$	$[4,4,2,2]$	(x,x^{-1},z,z)
$\langle x : x^8\rangle \oplus \langle z : z^2\rangle$	$[8,8,2]$	$(x,x^{-1}z,z)$
$\langle x,y : x^4,y^2,(xy)^2,(x^{-1}y)^2\rangle \oplus \langle z : z^2\rangle$	$[2,4,2,2]$	$(yx^{-1}z,x,y,z)$
$\langle x,y : x^4,x^2y^2,(xy)^4\rangle$	$[4,4,4]$	$(x,y,(xy)^{-1})$
Actions on the surfaces $X_1^{1,2}$, $X_1^{2,2}$ of genus 3		
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$(1;2,2)$	$(z,1,w,w)$
$\langle x,x^2,y^4,xyx^{-1}y\rangle$	$(1;2)$	(x,xy,y^2)
Actions on the surface $X_0^{1,1}$ of genus 3		
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{6}{\ldots},2]$	(z,z,w,w,zw,zw)
$\langle x : x^2\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{5}{\ldots},2]$	(x,x,z,w,zw)
$\langle x : x^4\rangle \oplus \langle z : z^2\rangle$	$[4,4,2,2]$	(xz,x,z,x^2)
Actions on the surfaces $X_0^{1,1}$, $X_0^{1,2}$ of genus 4		
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{7}{\ldots},2]$	(z,z,z,w,w,w,zw)
$\langle x : x^6\rangle \oplus \langle z : z^2\rangle$	$[3,6,2,2]$	(x^4,x^{-1},z,x^3z)
Actions on the surface $X_1^{2,2}$ of genus 4		
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$(1;2,2,2)$	$(1,1,z,w,zw)$
Actions on the surfaces $X_0^{0,2}$, $X_0^{2,2}$ of genus 4		
$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$[2,\overset{7}{\ldots},2]$	(z,z,z,z,z,w,zw)
$\langle x : x^{10}\rangle \oplus \langle z : z^2\rangle$	$[10,10,2]$	(x,x^4z,x^5z)

hyperbolic plane $\mathcal{H}$ under the action of some Fuchsian surface group Γ. Furthermore, a group of automorphisms of a surface $X = \mathcal{H}/\Gamma$ can be represented as Λ/Γ for another Fuchsian group Λ. The algebraic structure of a Fuchsian group Λ is determined by the signature

$$\sigma(\Lambda) = (\gamma; m_1,\ldots,m_r), \tag{2.1}$$

where γ, m_i are integers satisfying $\gamma \geq 0$, $m_i \geq 2$. In the case $\gamma = 0$, we will write simply $\sigma(\Lambda) = [m_1,\dots,m_r]$ and we will denote a sequence of numbers $m,\overset{n}{\dots},m$ by m^n. The group with signature (2.1) has the presentation given by

$$\begin{gathered}\text{generators}: x_1,\dots,x_r,a_1,b_1,\dots,a_\gamma,b_\gamma,\\ \text{relations}: x_1^{m_1} = \cdots = x_r^{m_r} = x_1\cdots x_r[a_1,b_1]\cdots[a_\gamma,b_\gamma] = 1.\end{gathered} \tag{2.2}$$

Such set of generators is called the *canonical set of generators* and often by abuse of language, the set of *canonical generators*. Geometrically, x_i are elliptic elements which correspond to hyperbolic rotations and the remaining generators are hyperbolic translations. The integers $m_1,m_2,\dots,m_r$ are called the *periods* of Λ and γ is the genus of the orbit space $\mathcal{H}/\Lambda$. Fuchsian groups with signatures $(\gamma;-)$ are called *surface groups* and they are characterized among Fuchsian groups as these ones which are torsion-free.

The group Λ has associated to it a fundamental region whose area $\mu(\Lambda)$, called the *area of the group*, is

$$\mu(\Lambda) = 2\pi\left(2\gamma - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right). \tag{2.3}$$

An abstract group Λ with the presentation (2.2) is isomorphic to a Fuchsian group with the signature (2.1) if and only if the right-hand side of (2.3) is greater than 0; in that case (2.1) is called a *Fuchsian signature.*

If Λ is a subgroup of finite index in a Fuchsian group Λ', then we have the *Riemann-Hurwitz formula*

$$[\Lambda' : \Lambda] = \frac{\mu(\Lambda)}{\mu(\Lambda')}. \tag{2.4}$$

Propostion 2.1. *If Λ is a normal subgroup of Λ' of index N, $\{x_1,\dots,x_r\}$ is the set of canonical elliptic generators of Λ', $\{m_1,\dots,m_r\}$ the set of periods of Λ', and p_i denotes the order of $\Lambda x_i \in \Lambda'/\Lambda$, then the periods in $\sigma(\Lambda)$ are*

$$\frac{m_1}{p_1},\overset{N/p_1}{\dots},\frac{m_1}{p_1},\dots,\frac{m_r}{p_r},\overset{N/p_r}{\dots},\frac{m_r}{p_r}, \tag{2.5}$$

where we omit those values m_i/p_i which are equal to 1. Furthermore, if the orbit genera of $\mathcal{H}/\Lambda'$ and $\mathcal{H}/\Lambda$ are γ and k, respectively, then

$$2k - 2 = N\left(2\gamma - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{p_i}\right)\right). \tag{2.6}$$

Let G be a finite group acting on a Riemann surface X of genus $g > 1$. If the canonical projection $X \to X/G$ is ramified at r points with multiplicities $m_1,\dots,m_r$ and γ is the genus of X/G, then the vector of numbers $(\gamma : m_1,\dots,m_r)$ is called the *branching data* of G on X.

A $(2\gamma+r)$-tuple $(\widetilde{a}_1,\dots,\widetilde{a}_\gamma,\widetilde{b}_1,\dots,\widetilde{b}_\gamma,\widetilde{x}_1,\dots,\widetilde{x}_r)$ of elements of G satisfying the condition:

(1) $\widetilde{a}_1,\dots,\widetilde{a}_\gamma,\widetilde{b}_1,\dots,\widetilde{b}_\gamma,\widetilde{x}_1,\dots,\widetilde{x}_r$ generate G,

(2) $\widetilde{x}_i$ has order m_i for $i=1,\dots,r$,

(3) $\widetilde{x}_1\dots\widetilde{x}_r\prod_{i=1}^{\gamma}[\widetilde{a}_i,\widetilde{b}_i]=1$

is called a *generating* $(\gamma;m_1,\dots,m_r)$*-vector*.

A group G acts on a surface of genus g with the branching data $(\gamma:m_1,\dots,m_r)$ if and only if G has the generating $(\gamma:m_1,\dots,m_r)$-vector and $2g-2=|G|(2\gamma-2+\sum_{i=1}^{r}(1-1/m_i))$. Indeed, if G satisfies these conditions, then we can take a Fuchsian group Λ with the signature (2.1) and define an epimorphism $\theta:\Lambda\to G$ by the assignment $\theta(a_i)=\widetilde{a}_i$, $\theta(b_i)=\widetilde{b}_i$, and $\theta(x_j)=\widetilde{x}_j$. Then $\Gamma=\ker\theta$ is a surface Fuchsian group of orbit genus g and G acts as an automorphism group on a Riemann surface $\mathcal{H}/\Gamma$.

There is 1-1 correspondence between the set of generating vectors of G and the set of epimorphisms $\theta:\Lambda\to G$ with torsion-free kernels. Two epimorphisms $\theta:\Lambda\to G$ and $\theta':\Lambda'\to G'$ define topologically equivalent actions if

$$\varphi\theta=\theta'\psi \tag{2.7}$$

for some isomorphisms $\varphi:G\to G'$ and $\psi:\Lambda\to\Lambda'$ [2]. The relation of the equivalence of actions induces an equivalence relation on generating vectors in the sense that two such vectors are equivalent if they determine the epimorphisms which give rise to equivalent actions.

We will need some pairs of isomorphisms of Fuchsian groups and abstract groups to demonstrate topological equivalence of actions, as in (2.7).

In the case when G is generated by elements x, y and two central involutions ρ_1,ρ_2, for which the assignment $\phi(x)=x\rho_1^k\rho_2^l$, $\phi(y)=y\rho_1^m\rho_2^n$, $\phi(\rho_j)=\rho_j$ defines an isomorphism from G onto some group G', where $j=1,2$ and $k,l,m,n\in\{0,1\}$, we will use the pair $\Phi_{k,l,m,n}=(\mathrm{id}_\Lambda,\phi)$.

When G is generated by the element x and two central involutions ρ_1,ρ_2 for which the assignment $\omega(x)=x^m\rho_1^k\rho_2^l$ and $\omega(\rho_j)=\rho_j$ defines an isomorphism $\omega:G\to G'$, we will use the pair $\Omega_{k,l,m}=(\mathrm{id}_\Lambda,\omega)$.

If $G=\Lambda/\Gamma$ with $\sigma(\Lambda)=(1;m_1,\dots,m_r)$ and for fixed integers $\alpha_1,\alpha_2\in\{0,1\}$, there exists s or w in range $1\le s,w\le r$ such that $\theta(c)=\rho_1^{\alpha_1}\rho_2^{\alpha_2}$ or $\theta(d)=\rho_1^{\alpha_1}\rho_2^{\alpha_2}$, where $\theta:\Lambda\to G$ is a surface-kernel epimorphism, $c=x_1\cdots x_s$ and $d=x_w\cdots x_r$, then by $\Upsilon_{\alpha_1,\alpha_2}$ and by $\Theta_{\alpha_1,\alpha_2}$ we will denote the pairs (v,id_G) and (μ,id_G), respectively, where v and μ are defined by $v(a)=ca$, $v(b)=b$, $v(x_i)=bx_ib^{-1}$ for $i=1,\dots,s$, $v(x_i)=cx_ic^{-1}$ for $i=s+1,\dots,r$ and by $\mu(a)=a$, $\mu(b)=d^{-1}b$, $\mu(x_i)=d^{-1}x_id$ for $i=1,\dots,w-1$, $\mu(x_i)=ax_ia^{-1}$ for $i=w,\dots,r$.

Finally, in the case when $m_k=m_l$ in the signature of Λ, we will use the pair $\Psi_{k,l}=(\psi_{k,l},\mathrm{id}_G)$, where $\psi_{k,l}$ is defined by the assignment $\psi_{k,l}(a_j)=a_j$, $\psi_{k,l}(b_j)=b_j$, and

$$\psi_{k,l}(x_i)=\begin{cases}x_i, & i=1,\dots k-1,l+1,\dots r,\\ x_l, & i=k,\\ x_l^{-1}x_ix_l, & i=k+1,\dots,l-1,\\ (x_{k+1}\dots x_l)^{-1}x_k(x_{k+1}\dots x_l), & i=l.\end{cases} \tag{2.8}$$

The pairs $\Psi_{k,l}$ induce the equivalence of every two generating vectors admitting the same elements up to permutation of involutions ρ_1, ρ_2, ρ_3, what allows us to write the generating vector in the form

$$v = (\theta(a_1),\ldots,\theta(a_\gamma),\theta(b_1),\ldots,\theta(b_\gamma),\theta(x_1),\ldots,\theta(x_{r-u}) \mid \rho_1^{u_1},\rho_2^{u_2},(\rho_1\rho_2)^{u_3}), \qquad (2.9)$$

where $u = u_1 + u_2 + u_3$ and $\rho_i^{u_i}$ denotes the sequence $\rho_i,\overset{u_i}{\ldots},\rho_i$. For better readability, we will separate the central involutions from the other elements of the generating vector by the vertical line. If in addition G is abelian, then any permutation of elements with the same orders of v provides an equivalent vector.

The signatures $\tau = (\gamma; m_1,\ldots,m_r)$ and $\tau' = (\gamma'; m_1',\ldots m_r')$ give rise to the same sets of the equivalence classes of group actions if and only if $\gamma' = \gamma$ and $m_i' = m_{\nu(i)}$ for some permutation ν of the set $\{1,\ldots,r\}$. Let x_i and x_i' be the canonical elliptic generators of Fuchsian groups Λ and Λ' with signatures τ and τ' which correspond to periods m_i and m_i', respectively. Then according to [2, Proposition 2.3], for any isomorphism $\psi : \Lambda \to \Lambda'$, $\psi(x_i)$ is conjugate to x_j' or $x_j'^{-1}$, where $m_j' = m_i$. Thus ψ induces the permutation $\widetilde{\psi}$ of the set $\{1,\ldots,r\}$ such that $m_i = m'_{\widetilde{\psi}(i)}$. In particular, if a pair (ψ,φ) induces the equivalence of actions given by epimorphisms $\theta : \Lambda \to G$ and $\theta' : \Lambda' \to G'$ and G' is abelian, then

$$\varphi\theta(x_i) = \theta'\left(x'^{j}_{\widetilde{\psi}(i)}\right) \qquad (2.10)$$

for $i = 1,\ldots,r$ and $j = 1$ or $j = -1$.

3. The actions of finite groups on pq-hyperelliptic Riemann surfaces

A conformal involution ρ of a Riemann surface X of genus $g > 1$ is called a *p-hyperelliptic involution* if X/ρ has genus p. For simplicity, we will say that ρ is a p-involution. Here we study conformal actions on Riemann surfaces admitting central p- and q-involutions simultaneously for some integers $q \geq p$. In particular, this class of surfaces contains all pq-hyperelliptic Riemann surfaces of genus g for any integers p, q, g in range $2 \leq p < q < 2p$ and $g > 3q + 1$. The product of commuting p- and q-involutions is a t-involution, where the possible values of t are given in the following lemma which is a consequence of [8, Theorem 3.4].

Lemma 3.1. *Let $g \geq 2$ and $q \geq p \geq 0$ be integers such that $2q - 1 < g \leq 2p + 2q + 1$. Then there exists a Riemann surface of genus g admitting commuting p- and q-involutions whose product is a t-involution if and only if $t = g - p - q + 2k$ for some integer k in range $0 \leq k \leq (2p + 2q + 1 - g)/4$.*

Let $X_k^{p,q}$ denote a Riemann surface of genus g with central p- and q-involutions whose product is a $(g - p - q + 2k)$-involution for some fixed k in the range $0 \leq k \leq (2p + 2q + 1 - g)/4$. The group $Z_2 \oplus Z_2$ generated by the p- and q-involutions of $X_k^{p,q}$ can be represented as Δ/Γ for some Fuchsian group Δ, which by Proposition 2.1 has the signature $(k; 2,\overset{g+3-4k}{\ldots},2)$. Thus the dimension d of the corresponding locus in the moduli space is $6(k-1) + 2(g + 3 - 4k) = 2g - 2k$. So by the inequality $g \geq 2q - 1$ and the restrictions on k, we obtain the following lemma.

Lemma 3.2. *For $q \geq p$, the pq-hyperelliptic locus in the moduli space corresponding to classes of surfaces admitting central p- and q-involutions is a finite union of manifolds of dimensions ranging between $3(p-1)$ and $2g$.*

Given an integer $g \geq 2$, a group G is said to be a *$g-(p,q,k)$-hyperelliptic subgroup* if there exists a surface $X_k^{p,q}$ of genus g such that $G \subseteq \mathrm{Aut}(X_k^{p,q})$ and the p- and q-involutions belong to G. In such a case $X_k^{p,q} = \mathcal{H}/\Gamma$ for some Fuchsian surface group of the orbit genus g and $G = \Lambda/\Gamma$ for some Fuchsian group Λ, say with the signature $\tau = (\gamma; m_1, \ldots, m_r)$, containing Δ as a normal subgroup. Assume that $\widetilde{G} = \Lambda/\Delta$ has order N. Let θ and π be the canonical epimorphisms from Λ onto G and G onto $\widetilde{G}$ and $\widetilde{\theta} = \pi\theta$. Let p_i be the order of $\widetilde{\theta}(x_i)$. Then by Proposition 2.1, $m_i/p_i = 2$ or $m_i/p_i = 1$ and (2.6) is satisfied.

The group G admits central involutions ρ_1, ρ_2, and $\rho_3 = \rho_1\rho_2$ such that $\theta^{-1}(\rho_j)$ is a Fuchsian group with the signature $(\mu(j); [2, \overset{2g+2-4\mu(j)}{\ldots}, 2])$ for some assignment $\mu : \{1,2,3\} \to \{p,q,t\}$. Since Γ is a surface group, it follows that $\theta(x_i)$ has order m_i. Furthermore, $\theta(x_i)^{p_i} = \rho_1^{r_i}\rho_2^{s_i}$ for some $r_i, s_i \in \{0,1\}$. So applying Proposition 2.1 for Λ and its normal subgroup $\theta^{-1}(\langle\rho_j\rangle)$, we calculate that θ maps $(2g+2-4\mu(j))/2N - \sum_{p_i \neq 1} \varepsilon_i(\rho_j)/p_i$ of the elliptic generators corresponding to $p_i = 1$ onto ρ_j, where $\varepsilon_i(\rho_j)$ is 1 or 0 according as ρ_j is or is not equal to $\theta(x_i)^{p_i}$. Thus we obtain the following proposition.

Propostion 3.3. *Let G be an abstract group of order $4N$. Let $p_1, \ldots, p_r$ and γ be nonnegative integers satisfying the equation*

$$2k - 2 + N(2 - 2\gamma) = N \sum_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) \tag{3.1}$$

for some fixed k in range $0 \leq k \leq (2p + 2q + 1 - g)/4$ and let $\tau = (\gamma; m_1, \ldots, m_r)$, where $m_i = 2$ for $p_i = 1$ and $m_i = \xi_i p_i$ for $p_i \neq 1$ with $\xi_i = 1$ or $\xi_i = 2$. Then G is a $g-(p,q,k)$-hyperelliptic subgroup if and only if it admits central involutions ρ_1, ρ_2, $\rho_3 = \rho_1\rho_2$ and there exists a Fuchsian group Λ with the signature τ and an epimorphism $\theta : \Lambda \to G$ satisfying the conditions

(i) *$\theta(x_i)$ has order m_i,*
(ii) $\theta(x_1), \ldots, \theta(x_r) \prod_{i=1}^{\gamma} [\theta(a_i), \theta(b_i)] = 1$,
(iii) *$\theta(x_i)^{p_i} = \rho_1^{r_i}\rho_2^{s_i}$ for some $r_i, s_i \in \{0,1\}$,*
(iv) *θ maps exactly*

$$u_j = \frac{2g + 2 - 4\mu(j)}{2N} - \sum_{p_i \neq 1} \frac{\varepsilon_i(\rho_j)}{p_i} \tag{3.2}$$

of the elliptic generators of Λ corresponding to $p_i = 1$ onto ρ_j for some assignment $\mu : \{1,2,3\} \to \{p,q,t\}$, where $\varepsilon_i(\rho_j)$ is 1 or 0 according as ρ_j is or is not equal to $\theta(x_i)^{p_i}$.

We describe $g-(p,q,k)$-subgroups corresponding to g in range $2p + 2q - 10 \leq g \leq 2p + 2q + 1$, where according to Lemma 3.1, k takes one of the values 0, 1, 2. In order to find the complete list of such groups acting on a surface of given genus, first of all we determine all the possible signatures of Λ, up to permutation of periods, however any such permutation gives rise to the same sets of topological types of actions. For, we find

Table 3.1

g	F_p	F_q	F_{g-p-q}	$F_{g-p-q+2}$	$F_{g-p-q+4}$
$g_1 = 2p+2q+1$	$4q+4$	$4p+4$	0	—	—
$g_2 = 2p+2q$	$4q+2$	$4p+2$	2	—	—
$g_3 = 2p+2q-1$	$4q$	$4p$	4	—	—
$g_4 = 2p+2q-2$	$4q-2$	$4p-2$	6	—	—
$g_5 = 2p+2q-3$	$4q-4$	$4p-4$	8	0	—
$g_6 = 2p+2q-4$	$4q-6$	$4p-6$	10	2	—
$g_7 = 2p+2q-5$	$4q-8$	$4p-8$	12	4	—
$g_8 = 2p+2q-6$	$4q-10$	$4p-10$	14	6	—
$g_9 = 2p+2q-7$	$4q-12$	$4p-12$	16	8	0
$g_{10} = 2p+2q-8$	$4q-14$	$4p-14$	18	10	2
$g_{11} = 2p+2q-9$	$4q-16$	$4p-16$	20	12	4
$g_{12} = 2p+2q-10$	$4q-18$	$4p-18$	22	14	6

all the possible values of N, p_i, and γ satisfying (3.1) and next we take $m_i = 2$ for $p_i = 1$ and $m_i = \xi_i p_i$ for $p_i \neq 1$, where $\xi_i = 1$ or $\xi_i = 2$. Such a signature will induce an action of a $g-(p,q,k)$-subgroup G of order $4N$ with central involutions ρ_1, ρ_2, and $\rho_3 = \rho_1\rho_2$ if and only if there exists an epimorphism $\theta : \Lambda \to G$ satisfying the conditions of Proposition 3.3. In particular, for $\xi_i = 2$, $\theta(x_i)^{p_i} = \rho_1^{r_i}\rho_2^{s_i}$ for some $r_i, s_i \in \{0,1\}$ such that $r_i s_i \neq 0$ and θ maps u_j of elliptic generators with $p_i = 1$ onto ρ_j, where u_j are given by (3.2). So we examine all the possible values of r_i, s_i and choose those for which the numbers u_j are integers for some $0 \leq p \leq q$ with $pq \neq 0$. In this way, we obtain the connection between the number of periods $m_i = 2$ corresponding to $p_i = 1$ and the values of p and q. To simplify the calculations we list in Table 3.1 the values of F_p, F_q, and F_t corresponding the considered surfaces, where $F_{\mu(j)} = 2g + 2 - 4\mu(j)$ denotes the number of fixed points of $\mu(j)$-involution for the assignment $\mu : \{1,2,3\} \to \{p,q,t\}$.

Having all candidates for the signature of Λ, we determine the presentations of the corresponding groups $\widetilde{G} = G/\langle \rho_1, \rho_2 \rangle$ by inspecting groups of automorphisms of a genus k surface and choosing those of them which admit generators of orders p_i not leading to a contrary with the values of r_i and s_i (some p_i may be equal to 1). The generating vector of $\widetilde{G}$ given in Tables 3.2, 3.5, and 3.9 according as $k = 2, 1$ or 0 determines, up to topological equivalence, how an epimorphism $\widetilde{\theta} : \Lambda \mapsto \widetilde{G}$ maps the canonical generators of Λ, except the elliptic elements corresponding to $p_i = 1$ which are mapped onto 1. A generating vector of G can be written as

$$\left(e_1\rho_1^{\alpha_1}\rho_2^{\beta_1},\dots,e_\gamma\rho_1^{\alpha_\gamma}\rho_2^{\beta_\gamma},f_1\rho_1^{o_1}\rho_2^{\varsigma_2},\dots,f_\gamma\rho_1^{o_\gamma}\rho_2^{\varsigma_\gamma},g_1\rho_1^{k_1}\rho_2^{l_1},\dots,g_m\rho_1^{k_m}\rho_2^{l_m} \mid \rho_1^{u_1},\rho_2^{u_2},\rho_3^{u_3}\right), \tag{3.3}$$

where $m = r - u_1 - u_1 - u_3$, $\pi(e_i) = \widetilde{\theta}(a_i)$, $\pi(f_i) = \widetilde{\theta}(b_i)$, $\pi(g_i) = \widetilde{\theta}(x_i)$, $k_i, l_i, \alpha_i, \beta_i, o_i, \varsigma_i \in \{0,1\}$ and $\rho_i^{u_i}$ denotes $\rho_i, \overset{u_i}{\dots}, \rho_i$. In order to determine the set of relations in the presentation of G, we assume that any relation

$$R(\widetilde{\theta}(a_1),\dots,\widetilde{\theta}(b_\gamma),\widetilde{\theta}(x_1),\dots,\widetilde{\theta}(x_r)) = 1 \tag{3.4}$$

Table 3.2

Case	Presentation of $\widetilde{G}$	Data	N	Generating vector
2.1	$Z_2 = \langle \widetilde{x} : \widetilde{x}^2 \rangle$	$(1:2^2)$	2	$(1,1,\widetilde{x},\widetilde{x})$
2.2	$Z_2 = \langle \widetilde{x} : \widetilde{x}^2 \rangle$	(2^6)	2	$(\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x})$
2.3	$Z_3 = \langle \widetilde{x} : \widetilde{x}^3 \rangle$	(3^4)	3	$(\widetilde{x},\widetilde{x},\widetilde{x}^{-1},\widetilde{x}^{-1})$
2.4	$Z_4 = \langle \widetilde{x} : \widetilde{x}^4 \rangle$	$(2^2,4^2)$	4	$(\widetilde{x}^2,\widetilde{x}^2,\widetilde{x},\widetilde{x}^{-1})$
2.5	$Z_2 \oplus Z_2 = \langle \widetilde{x},\widetilde{y} : \widetilde{x}^2,\widetilde{y}^2,[\widetilde{x},\widetilde{y}] \rangle$	(2^5)	4	$(\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{y},\widetilde{x}\widetilde{y})$
2.6	$Z_5 = \langle \widetilde{x} : \widetilde{x}^5 \rangle$	(5^3)	5	$(\widetilde{x},\widetilde{x},\widetilde{x}^3)$
2.7	$Z_6 = \langle \widetilde{x} : \widetilde{x}^6 \rangle$	$(3,6^2)$	6	$(\widetilde{x}^4,\widetilde{x},\widetilde{x})$
2.8	$Z_6 = \langle \widetilde{x} : \widetilde{x}^6 \rangle$	$(2^2,3^2)$	6	$(\widetilde{x}^3,\widetilde{x}^3,\widetilde{x}^2,\widetilde{x}^4)$
2.9	$D_3 = \langle \widetilde{x},\widetilde{y} : \widetilde{x}^2,\widetilde{y}^3,\widetilde{x}\widetilde{y}\widetilde{x}^{-1}\widetilde{y} \rangle$	$(2^2,3^2)$	6	$(\widetilde{x},\widetilde{x},\widetilde{y},\widetilde{y}^{-1})$
2.10	$Z_8 = \langle \widetilde{x} : \widetilde{x}^8 \rangle$	$(2,8^2)$	8	$(\widetilde{x}^4,\widetilde{x}^3,\widetilde{x})$
2.11	$\widetilde{D}_2 = \langle \widetilde{x},\widetilde{y} : \widetilde{x}^4,\widetilde{y}^4,\widetilde{x}^2\widetilde{y}^2,\widetilde{x}\widetilde{y}\widetilde{x}^{-1}\widetilde{y} \rangle$	(4^3)	8	$(\widetilde{x},\widetilde{y},\widetilde{y}\widetilde{x})$
2.12	$D_4 = \langle \widetilde{x},\widetilde{y} : \widetilde{x}^2,\widetilde{y}^4,\widetilde{x}\widetilde{y}\widetilde{x}^{-1}\widetilde{y} \rangle$	$(2^3,4)$	8	$(\widetilde{x},\widetilde{x}\widetilde{y},\widetilde{y}^2,\widetilde{y})$
2.13	$Z_{10} = \langle \widetilde{x} : \widetilde{x}^{10} \rangle$	$(2,5,10)$	10	$(\widetilde{x}^5,\widetilde{x}^4,\widetilde{x})$
2.14	$Z_2 \oplus Z_6 = \langle \widetilde{x},\widetilde{y} : \widetilde{x}^2,\widetilde{y}^6,[\widetilde{x},\widetilde{y}] \rangle$	$(2,6^2)$	12	$(\widetilde{x},\widetilde{x}\widetilde{y},\widetilde{y}^{-1})$
2.15	$D_{4,3,-1} = \langle \widetilde{x},\widetilde{y} : \widetilde{x}^4,\widetilde{y}^3,\widetilde{x}\widetilde{y}\widetilde{x}^{-1}\widetilde{y} \rangle$	$(3,4^2)$	12	$(\widetilde{y},(\widetilde{x}\widetilde{y})^{-1},\widetilde{x})$
2.16	$D_6 = \langle \widetilde{x},\widetilde{y} : \widetilde{x}^2,\widetilde{y}^6,\widetilde{x}\widetilde{y}\widetilde{x}^{-1}\widetilde{y} \rangle$	$(2^3,3)$	12	$(\widetilde{x},\widetilde{x}\widetilde{y},\widetilde{y}^3,\widetilde{y}^2)$
2.17	$D_{2,8,3} = \langle \widetilde{x},\widetilde{y} : \widetilde{x}^2,\widetilde{y}^8,\widetilde{x}\widetilde{y}\widetilde{x}^{-1}\widetilde{y}^5$	$(2,4,8)$	16	$(\widetilde{x},(\widetilde{y}\widetilde{x})^{-1},\widetilde{y})$
2.18	$\langle \widetilde{x},\widetilde{y},\widetilde{z},\widetilde{w} : \widetilde{x}^2,\widetilde{y}^2,\widetilde{z}^2,\widetilde{w}^3,[\widetilde{y},\widetilde{z}],[\widetilde{y},\widetilde{w}],$ $[\widetilde{z},\widetilde{w}],[\widetilde{x},\widetilde{y}],\widetilde{x}\widetilde{w}\widetilde{x}^{-1}\widetilde{w},\widetilde{x}\widetilde{z}\widetilde{x}^{-1}\widetilde{y}^{-1}\widetilde{z}^{-1} \rangle$	$(2,4,6)$	24	$(\widetilde{x},(\widetilde{z}\widetilde{w}\widetilde{x})^{-1},\widetilde{z}\widetilde{w})$
2.19	$\mathrm{SL}_2(3) = \langle \widetilde{x},\widetilde{y} : \widetilde{x}^3,\widetilde{y}^4,(\widetilde{x}\widetilde{y})^3,\widetilde{x}\widetilde{y}^2\widetilde{x}^{-1}\widetilde{y}^2 \rangle$	$(3^2,4)$	24	$(\widetilde{x},(\widetilde{y}\widetilde{x})^{-1},\widetilde{y})$
2.20	$\mathrm{GL}_2(3) = \langle \widetilde{x},\widetilde{y} : \widetilde{x}^2,\widetilde{y}^3,(\widetilde{x}\widetilde{y})^8,((\widetilde{x}\widetilde{y})^4x)^2 \rangle$	$(2,3,8)$	48	$(\widetilde{x},\widetilde{y},(\widetilde{x}\widetilde{y})^{-1})$

in the presentation of $\widetilde{G}$ involving $\widetilde{\theta}(a_1),\dots,\widetilde{\theta}(b_\gamma),\widetilde{\theta}(x_1),\dots,\widetilde{\theta}(x_r)$ induces the relation

$$R(\theta(a_1),\dots,\theta(b_\gamma),\theta(x_1),\dots,\theta(x_r)) = \rho_1^{\alpha}\rho_2^{\beta} \tag{3.5}$$

for some $\alpha,\beta \in \{0,1\}$. In most cases, the parameters α, β are determined by the values of r_i, s_i. In the other case, we must decide if two presentations corresponding to different values of such parameters provide equivalent actions. Sometimes it does lead to nonequivalent actions and so a lifting of the given group $\widetilde{G}$ may not be unique. Even more, the group $\widetilde{G}$ sometimes induces actions on pq-hyperelliptic surfaces with different values of p and q as we can observe for genus 3 actions.

Finally, we check if the determined set of generators and relations presents a group of the required order $4N$ and if the generating vectors of such a group corresponding to different values of parameters k_i, l_i, α_m, β_m, o_m, ς_m are topologically equivalent.

Following the program outlined above we classify the finite group actions on surfaces $X_k^{p,q}$ of genera g in range $2p+2q-10 \le g \le 2p+2q+1$ except X_1 of genus g_5 and X_2 of genus g_9 since the two cases require lifting all groups of automorphisms of a genus 1 or a genus 2 surface, respectively, and the corresponding sets of the topological classes of actions are very big. Additionally, for $k = 0$, we extend the assumption to $g \geq 2q - 1$. The results can be applied for any pq-hyperelliptic Riemann surface of genus $g > 3q + 1$ with $p < q$.

3.1. Classification of conformal actions on a surface $X_2^{p,q}$

THEOREM 3.4. *The topological type of the action on a surface $X_2^{p,q}$ of genus g_{10}, g_{11}, or g_{12} is determined by the group of automorphisms G, the signature of Λ, and the generating vector listed in Tables 3.3 and 3.4, where ν denotes a permutation of the set $\{p,q\}$.*

Proof. Let G be an automorphism group of a surface $X_2^{p,q}$ of genus g_{10}, g_{11}, or g_{12} and let ρ_1, ρ_2, and $\rho_3 = \rho_1\rho_2$ denote the central involutions of G. According to the classification of finite group actions on a genus 2 surface given up to topological equivalence by Broughton [2] and the solutions of (3.1) for $k = 2$, the corresponding group $\widetilde{G}$ is isomorphic to one of the groups listed in Table 3.2. We exclude the cases 2.4, 2.9–2.12, 2.15–2.20, where the numbers u_i defined by (3.2) cannot be integers for any values of p and q. For example, let us consider the case 2.15. By inspecting Table 3.1, we conclude that u_j can be integers only for a surface of genus g_{12} if there is exactly one t-involution among the elements $\theta(x_i)^{p_i}$ with $p_i \neq 1$ corresponding to $p_i = 4$. However, this condition cannot be satisfied because $p_i = 4$ for $i = 2,3$, and $\theta(x_2)^{p_2} = \theta(x_3)^{p_3}$. Indeed, since $\widetilde{G} \cong \langle \widetilde{x}, \widetilde{y} : \widetilde{x}^4, \widetilde{y}^3, \widetilde{x}\widetilde{y}\widetilde{x}^{-1}\widetilde{y}\rangle$ and the epimorphism $\widetilde{\theta}$ is defined by $\widetilde{\theta}(x_1) = \widetilde{y}$, $\widetilde{\theta}(x_2) = (\widetilde{x}\widetilde{y})^{-1}$, $\widetilde{\theta}(x_3) = \widetilde{x}$, and $\widetilde{\theta}(x_j) = 1$ for $j = 4,\dots,r$, it follows that any generating vector of G has a form $v = (y\rho_1^{k_1}\rho_2^{l_1}, (xy)^{-1}\rho_1^{k_2}\rho_2^{l_2}, x\rho_1^{k_3}\rho_2^{l_3} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3})$ for x and y belonging to $\pi^{-1}(x)$ and $\pi^{-1}(\widetilde{y})$, respectively, and $k_i, l_i \in \{0,1\}$, which implies that $\theta(x_2)^{p_2} = x^4 = \theta(x_3)^{p_3}$.

Let us consider the remaining cases from Table 3.2. First suppose that $\widetilde{G}$ is isomorphic to $Z_n = \langle\widetilde{x}\rangle$. Then G is generated by element $x \in \pi^{-1}(\widetilde{x})$ and two central involutions ρ_1, ρ_2. Since $x^n = \rho_1^{\delta_1}\rho_2^{\delta_2}$ for some $\delta_1, \delta_2 \in \{0,1\}$, it follows that G is isomorphic to $Z_n \oplus Z_2 \oplus Z_2$ if $\delta_1 = \delta_2 = 0$ or to $Z_{2n} \oplus Z_2$ otherwise.

Cases 2.1, 2.2. Here $n = 2$ and a generating vector of G has a form

$$v_{2.1} = (\rho_1^{\alpha_1}\rho_2^{\alpha_2}, \rho_1^{\beta_1}\rho_2^{\beta_2}, x\rho_1^{k_1}\rho_2^{l_1}, x\rho_1^{k_2}\rho_2^{l_2} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3}) \tag{3.6}$$

or

$$v_{2.2} = (x\rho_1^{k_1}\rho_2^{l_1}, x\rho_1^{k_2}\rho_2^{l_2}, x\rho_1^{k_3}\rho_2^{l_3}, x\rho_1^{k_4}\rho_2^{l_4}, x\rho_1^{k_5}\rho_2^{l_5}, x\rho_1^{k_6}\rho_2^{l_6} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3}), \tag{3.7}$$

respectively. Using the pairs $\Upsilon_{\alpha_1,\alpha_2}$ and Θ_{β_1,β_2} if necessary we can assume that $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$. Since $\theta(x_i)^2 = x^2 = \rho_1^{\delta_1}\rho_2^{\delta_2}$ for $p_i \neq 1$, it follows that Λ has the signature $(1; 2, 2, 2, \overset{(g-5)/2}{\dots}, 2)$ in the case 2.1 or $(0; 2, 2, 2, 2, 2, 2, 2, \overset{(g-5)/2}{\dots}, 2)$ in the case 2.2 and consequently $G = Z_2 \oplus Z_2 \oplus Z_2$ or Λ has the signature $(1; 4, 4, 2, \overset{(g-7)/2}{\dots}, 2)$ or $(0; 4, 4, 4, 4, 4, 4, 2, \overset{(g-11)/2}{\dots}, 2)$, respectively, and $G = Z_4 \oplus Z_2$. Now suppose that a pair (ψ, φ) induces the equivalence of two vectors corresponding to k_i, l_i and k_i', l_i', respectively. Clearly $\varphi(\rho_i) = \rho_i$ and $\varphi(x) = x\rho_1^a\rho_2^b$ for some $a, b \in \{0,1\}$. Thus by (2.10), we have $k_i = k'_{\widetilde{\psi}(i)} + a$ and $l_i = l'_{\widetilde{\psi}(i)} + b$ for $i = 1,\dots,r-u$, where $u = u_1 + u_2 + u_3$. Thus either $\sum_{i=1}^{r-u} k_i = \sum_{i=1}^{r-u} k_i'$ if

Table 3.3

Case	$\sigma(\Lambda)$	Presentation of G	g
2.1.a	$(1;2,2,2^{(g-5)/2})$	$\langle x:x^2\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{11}
2.1.b	$(1;4^2,2^{(g-7)/2})$	$\langle x:x^4\rangle\oplus\langle z:z^2\rangle$	g_{11}
2.2.a	$(2^6,2^{(g-5)/2})$	$\langle x:x^2\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{11}
2.2.b	$(4^6,2^{(g-11)/2})$	$\langle x:x^4\rangle\oplus\langle z:z^2\rangle$	g_{11}
2.3.a	$(3^4,2^{(g-5)/3})$	$\langle x:x^3\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{12}
2.3.b	$(6,3^3,2^{(g-6)/3})$	$\langle x:x^3\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{10}, g_{12}
2.3.c	$(6^2,3^2,2^{(g-7)/3})$	$\langle x:x^3\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{10}, g_{11}, g_{12}
2.3.d	$(6^3,3,2^{(g-8)/3})$	$\langle x:x^3\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{10}, g_{11}, g_{12}
2.3.e	$(6^4,2^{(g-9)/3})$	$\langle x:x^3\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{10}, g_{11}, g_{12}
2.5.a	$(2^4,4,2^{(g-7)/4})$	$\langle x,y,z,w:x^2,y^2,z^2,w^2,[z,x],$ $[z,y],[w,x],[w,y],[x,y]zw\rangle$	g_{11}
2.5.b	$(2^3,4^2,2^{(g-9)/4})$	$\langle x,y,z:x^2,y^4,z^2,[x,z],[y,z],[x,y]z\rangle$	g_{11}
2.5.c	$(4^4,2,2^{(g-13)/4})$	$\langle x,y:x^4,y^4,[x,y]x^2y^2,[x^2,y]\rangle$	g_{11}
2.5.d	$(4^5,2^{(g-15)/4})$	$\langle x:x^4\rangle\oplus\langle y:y^4\rangle$	g_{11}
2.5.d'	$(4^5,2^{(g-15)/4})$	$\langle x,y:x^4,y^4,[x,y]y^2\rangle$	g_{11}
2.6.a	$(10,5^2,2^{(g-6)/5})$	$\langle x:x^5\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{10}
2.6.b	$(10^2,5,2^{(g-7)/5})$	$\langle x:x^5\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{10}, g_{11}
2.6.c	$(10^3,2^{(g-8)/5})$	$\langle x:x^5\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{10}, g_{11}, g_{12}
2.7.a	$(6^3,2^{(g-7)/6})$	$\langle x:x^6\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{11}
2.7.b	$(3,12^2,2^{(g-7)/6})$	$\langle x:x^{12}\rangle\oplus\langle z:z^2\rangle$	g_{11}
2.7.c	$(6,12^2,2^{(g-9)/6})$	$\langle x:x^{12}\rangle\oplus\langle z:z^2\rangle$	g_{11}
2.8.a	$(2^2,6,3,2^{(g-7)/6})$	$\langle x:x^6\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{11}
2.8.b	$(2^2,6^2,2^{(g-9)/6})$	$\langle x:x^6\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{11}
2.8.c	$(4^2,3,6,2^{(g-13)/6})$	$\langle x:x^{12}\rangle\oplus\langle z:z^2\rangle$	g_{11}
2.8.d	$(4^2,6^2,2^{(g-15)/6})$	$\langle x:x^{12}\rangle\oplus\langle z:z^2\rangle$	g_{11}
2.13.a	$(2,10^2,2^{(g-7)/10})$	$\langle x:x^{10}\rangle\oplus\langle z:z^2\rangle\oplus\langle w:w^2\rangle$	g_{11}
2.13.b	$(4,10,20,2^{(g-14)/10})$	$\langle x:x^{20}\rangle\oplus\langle z:z^2\rangle$	g_{11}
2.14.a	$(2,12,6,2^{(g-7)/12})$	$\langle x,y,z,w:x^2,y^6,z^2,w^2,[x,z],[y,z],$ $[x,w],[y,w],[x,y]zw\rangle$	g_{11}
2.14.b	$(2,12^2,2^{(g-9)/12})$	$\langle x,y,z:x^2,y^{12},z^2,[x,z],[y,z],[x,y]z\rangle$	g_{11}
2.14.c	$(4,12,6,2^{(g-13)/12})$	$\langle x,y,z:x^4,y^6,z^2,[x,z],[y,z],[x,y]z\rangle$	g_{11}
2.14.d	$(4,12^2,2^{(g-15)/12})$	$\langle x:x^4\rangle\oplus\langle y:y^{12}\rangle$	g_{11}
2.14.e	$(4,12^2,2^{(g-15)/12})$	$\langle x,y:x^4,y^{12},[x,y]y^6\rangle$	g_{11}

$a=0$ or $\sum_{i=1}^{r-u}k_i=(r-u)-\sum_{i=1}^{r-u}k'_i$ otherwise, and similarly $\sum_{i=1}^{r-u}l_i=\sum_{i=1}^{r-u}l'_i$ or $\sum_{i=1}^{r-u}l_i=(r-u)-\sum_{i=1}^{r-u}l'_i$ according to $b=0$ or $b=1$. So any two vectors listed in Table 3.4 are not

Table 3.4

Case	g	Generating vector	Conditions
2.1.a	g_{11}	$(h,1,x,xz^{\nu(p)-3}w^{\nu(q)-3} \mid z^{\nu(p)-4},w^{\nu(q)-4},zw)$, $h=z$ if $p=q=4$; $h=1$ otherwise	None
2.1.b	g_{11}	$(h,1,x,x^{-1+2\nu(q)}z^{\nu(p)-3} \mid z^{\nu(p)-4},(x^2)^{\nu(q)-5},zx^2)$, $h\in\langle z\rangle$ if $p=4$ and $q=5$; $h=1$ otherwise	None
—	—	$(h,1,x,x^{-1+2\nu(q)}z^{\nu(p)+\nu(q)} \mid (x^2z)^{\nu(q)-4},z^{\nu(p)-4})$, $h=z$ if $p=q=4$; $h=1$ otherwise	None
2.2.a	g_{11}	$(x,x,x,xh,xh,xz^{\nu(p)-3}w^{\nu(q)-3} \mid z^{\nu(p)-4},w^{\nu(q)-4},zw)$; $h\in\langle z,w\rangle$	None
2.2.b	g_{11}	$(x,x,x,xh,xh,x^{-1+2\nu(q)}z^{\nu(p)-3} \mid z^{\nu(p)-4},(x^2)^{\nu(q)-7},x^2z)$; $h\in\langle x^2,z\rangle$	None
2.3.a	g_{12}	$(x,x,x^{-1},x^{-1} \mid z^{2\nu(p)/3-3},w^{2\nu(q)/3-3},zw)$	$p\equiv 0(3)$, $q\equiv 0(3)$
2.3.b	g_{12}	$(xz,x,x^{-1},x^{-1} \mid z^{(2\nu(p)-10)/3},w^{2\nu(q)/3-3},zw)$	$\nu(p)\equiv 2(3)$, $\nu(q)\equiv 0(3)$
2.3.b	g_{10}	$(xzw,x,x^{-1},x^{-1} \mid z^{(2\nu(p)-7)/3},w^{(2\nu(q)-7)/3})$	$p\equiv 2(3)$, $q\equiv 2(3)$
2.3.c	g_{10}	$(xzw,xz,x^{-1},x^{-1} \mid z^{(2\nu(p)-8)/3},w^{(2\nu(q)-7)/3})$	$\nu(p)\equiv 1(3)$, $\nu(q)\equiv 2(3)$
2.3.c	g_{11}	$(xzw,xzw,x^{-1},x^{-1} \mid z^{(2\nu(p)-8)/3},w^{(2\nu(q)-8)/3})$	$p\equiv 1(3)$, $q\equiv 1(3)$
2.3.c	g_{12}	$(xz,xz,x^{-1},x^{-1} \mid z^{(2\nu(p)-11)/3},w^{2\nu(q)/3-3},zw)$	$\nu(p)\equiv 1(3)$, $\nu(q)\equiv 0(3)$
—	—	$(xz,xw,x^{-1},x^{-1} \mid z^{(2\nu(p)-10)/3},w^{(2\nu(q)-10)/3},zw)$	$p\equiv 2(3)$, $q\equiv 2(3)$
2.3.d	g_{10}	$(xzw,xz,x^{-1}w,x^{-1} \mid z^{(2\nu(p)-8)/3},w^{(2\nu(q)-8)/3})$	$p\equiv 1(3)$, $q\equiv 1(3)$
—	—	$(xzw,xz,x^{-1}z,x^{-1} \mid z^{2\nu(p)/3-3},w^{(2\nu(q)-7)/3})$	$\nu(p)\equiv 0(3)$, $\nu(q)\equiv 2(3)$
2.3.d	g_{11}	$(xzw,xzw,x^{-1}z,x^{-1} \mid z^{2\nu(p)/3-3},w^{(2\nu(q)-8)/3})$	$\nu(p)\equiv 0(3)$, $\nu(q)\equiv 1(3)$
2.3.d	g_{12}	$(xzw,xzw,x^{-1}zw,x^{-1} \mid z^{2\nu(p)/3-3},w^{2\nu(q)/3-3})$	$p\equiv 0(3)$, $q\equiv 0(3)$

Table 3.4. Continued.

Case	g	Generating vector	Conditions
—	—	$(xz, xz, x^{-1}z, x^{-1} \mid z^{2\nu(p)/3-4}, w^{2\nu(q)/3-3}, zw)$	$p \equiv 0(3)$, $q \equiv 0(3)$
—	—	$(xz, xz, x^{-1}w, x^{-1} \mid z^{(2\nu(p)-11)/3}, w^{(2\nu(q)-10)/3}, zw)$	$\nu(p) \equiv 1(3)$, $\nu(q) \equiv 2(3)$
2.3.e	g_{10}	$(xzw, xz, x^{-1}z, x^{-1}w \mid z^{2\nu(p)/3-3}, w^{(2\nu(q)-8)/3})$	$\nu(p) \equiv 0(3)$, $\nu(q) \equiv 1(3)$
—	—	$(xzw, xz, x^{-1}z, x^{-1}z \mid z^{(2\nu(p)-10)/3}, w^{(2\nu(q)-7)/3})$	$p \equiv 2(3)$, $q \equiv 2(3)$
2.3.e	g_{11}	$(xzw, xzw, x^{-1}z, x^{-1}w \mid z^{2\nu(p)/3-3}, w^{2\nu(q)/3-3})$	$p \equiv 0(3)$, $q \equiv 0(3)$
—	—	$(xzw, xzw, x^{-1}z, x^{-1}z \mid z^{(2\nu(p)-10)/3}, w^{(2\nu(q)-8)/3})$	$\nu(p) \equiv 2(3)$, $\nu(q) \equiv 1(3)$
2.3.e	g_{12}	$(xz, xz, x^{-1}z, x^{-1}z \mid z^{(2\nu(p)-13)/3}, w^{2\nu(q)/3-3}, zw)$	$\nu(p) \equiv 2(3)$, $\nu(q) \equiv 0(3)$
—	—	$(xzw, xzw, x^{-1}zw, x^{-1}z \mid z^{(2\nu(p)-10)/3}, w^{2\nu(q)/3-3})$	$\nu(p) \equiv 2(3)$, $\nu(q) \equiv 0(3)$
—	—	$(xz, xz, x^{-1}z, x^{-1}w \mid z^{2\nu(p)/3-4}, w^{(2\nu(q)-10)/3}, zw)$	$\nu(p) \equiv 0(3)$, $\nu(q) \equiv 2(3)$
—	—	$(xz, xz, x^{-1}w, x^{-1}w \mid z^{(2\nu(p)-11)/3}, w^{(2\nu(q)-11)/3}, zw)$	$p \equiv 1(3)$, $q \equiv 1(3)$
2.5.a	g_{11}	$(xh, x, x, y, yxz^{\nu(p)/2-2}w^{\nu(q)/2-2}h \mid z^{\nu(p)/2-2}, w^{\nu(q)/2-2})$, $h \in \langle z \rangle$	$p \equiv 0(2)$, $q \equiv 0(2)$
2.5.b	g_{11}	$(xh, x, x, y, y^{p-2}xz^{\nu(q)/2}h \mid z^{\nu(q)/2-2}, (y^2)^{(\nu(p)-5)/2})$, $h \in \langle y^2 \rangle$	$\nu(p) \equiv 1(2)$, $\nu(q) \equiv 0(2)$
2.5.c	g_{11}	$(xh, x, x, y, y^{\nu(q)-1}x^{\nu(p)+\nu(q)-2}h \mid (x^2)^{(\nu(p)-7)/2}, (x^2y^2)^{\nu(q)/2-2})$, $h \in \langle x^2 \rangle$	$\nu(p) \equiv 1(2)$, $\nu(q) \equiv 0(2)$
2.5.d	g_{11}	$(xh, x, x, y, y^{\nu(q)-2}x^{\nu(p)-2}h \mid (x^2)^{(\nu(p)-7)/2}, (y^2)^{(\nu(q)-5)/2})$, $h \in \langle x^2, y^2 \rangle$	$\nu(p) \equiv 1(2)$, $\nu(q) \equiv 1(2)$
2.5.d’	g_{11}	$(xh, x, x, y, y^{\nu(q)-1}x^{\nu(p)+\nu(q)-2}h \mid (x^2)^{\nu(p)/2-4}, (x^2y^2)^{\nu(q)/2-2})$, $h \in \langle x^2 \rangle$	$\nu(p) \equiv 0(2)$, $\nu(q) \equiv 0(2)$
2.6.a	g_{10}	$(xzw, x, x^3 \mid z^{(2\nu(p)-7)/5}, w^{(2\nu(q)-7)/5})$	$p \equiv 1(5)$, $q \equiv 1(5)$

Table 3.4. Continued.

Case	g	Generating vector	Conditions
2.6.b	g_{10}	$(xzw, xz, x^3 \mid z^{(2\nu(p)-8)/5}, w^{(2\nu(q)-7)/5})$	$\nu(p) \equiv 4(5)$, $\nu(q) \equiv 1(5)$
2.6.b	g_{11}	$(xzw, xzw, x^3 \mid z^{(2\nu(p)-8)/5}, w^{(2\nu(q)-8)/5})$	$p \equiv 4(5)$, $q \equiv 4(5)$
2.6.c	g_{10}	$(xzw, xz, x^3 z \mid z^{(2\nu(p)-9)/5}, w^{(2\nu(q)-7)/5})$	$\nu(p) \equiv 2(5)$, $\nu(q) \equiv 1(5)$
—	—	$(xzw, xz, x^3 w \mid z^{(2\nu(p)-8)/5}, w^{(2\nu(q)-8)/5})$	$\nu(p) \equiv 4(5)$, $\nu(q) \equiv 4(5)$
2.6.c	g_{11}	$(xzw, xzw, x^3 z \mid z^{(2\nu(p)-9)/5}, w^{(2\nu(q)-8)/5})$	$\nu(p) \equiv 2(5)$, $\nu(q) \equiv 4(5)$
2.6.c	g_{12}	$(xzw, xzw, x^3 zw \mid z^{(2\nu(p)-9)/5}, w^{(2\nu(q)-9)/5})$	$p \equiv 2(5)$, $q \equiv 2(5)$
2.7.a	g_{11}	$(x^4 zw, x, xz^{(\nu(q)-1)/3} w^{(\nu(p)-1)/3} \mid z^{(\nu(q)-4)/3}, w^{(\nu(p)-4)/3})$	$p \equiv 1(3)$, $q \equiv 1(3)$
2.7.b	g_{11}	$(x^4, x, x^{2\nu(q)-1} z^{(\nu(q)+\mu(p)-8)/3} \mid z^{(\nu(p)-4)/3}, (zx^6)^{(\nu(q)-4)/3})$	$p \equiv 1(3)$, $q \equiv 1(3)$
2.7.c	g_{11}	$(x^{10} z, x, x^{2\nu(q)-9} z^{(\nu(p)-1)/3} \mid z^{(\nu(p)-4)/3}, (x^6)^{(\nu(q)-5)/3})$	$\nu(p) \equiv 1(3)$, $\nu(q) \equiv 2(3)$
—	—	$(x^{10} z, x, x^{2\nu(q)-9} z^{(\nu(p)+\nu(q)-6)/3} \mid z^{(\nu(p)-4)/3}, (x^6 z)^{(\nu(q)-5)/3})$	$\nu(p) \equiv 1(3)$, $\nu(q) \equiv 2(3)$
2.8.a	g_{11}	$(x^3 z^{(\nu(p)-1)/3} w^{(\nu(q)-1)/3}, x^3, x^2 zw, x^4 \mid z^{(\nu(p)-4)/3}, w^{(\nu(q)-4)/3})$	$p \equiv 1(3)$, $q \equiv 1(3)$
2.8.b	g_{11}	$(x^3 z^{(\nu(p)-5)/3} w^{(\nu(q)-1)/3}, x^3, x^2 zw, x^4 z \mid z^{(\nu(p)-5)/3}, w^{(\nu(q)-4)/3})$	$\nu(p) \equiv 2(3)$, $\nu(q) \equiv 1(3)$
2.8.c	g_{11}	$(x^{2\nu(p)+2\nu(q)-1} z^{(\nu(q)-1)/3}, x^3, x^8, x^4 z \mid (x^6 z)^{(\nu(q)-4)/3}, (x^6)^{(\nu(p)-7)/3})$	$\nu(p) \equiv 1(3)$, $\nu(q) \equiv 1(3)$
2.8.d	g_{11}	$(x^{2\nu(p)+2\nu(q)-21} z^{(\nu(p)-1)/3}, x^3, x^2, x^4 z \mid (x^6 z)^{(\nu(p)-4)/3}, (x^6)^{(\nu(q)-8)/3})$	$\nu(p) \equiv 1(3)$, $\nu(q) \equiv 2(3)$
2.13.a	g_{11}	$(x^5, x^4 zw, xz^{(\nu(p)+1)/5} w^{(\nu(q)+1)/5} \mid z^{(\nu(p)-4)/5}, w^{(\nu(q)-4)/5})$	$\nu(p) \equiv 4(5)$, $\nu(q) \equiv 4(5)$
2.13.b	g_{11}	$(x^5, x^{14} z, x^{2\nu(q)-13} z^{(\nu(p)+1)/5} \mid z^{(\nu(p)-4)/5}, (x^{10})^{(\nu(q)-7)/5})$	$\nu(p) \equiv 4(5)$, $\nu(q) \equiv 2(5)$

Table 3.4. Continued.

Case	g	Generating vector	Conditions
2.14.a	g_{11}	$(x,xy,y^{-1}z^{(\nu(p)-4)/6}w^{(\nu(q)-4)/6} \mid z^{(\nu(p)-4)/6},w^{(\nu(q)-4)/6})$	$p\equiv 4(6)$, $q\equiv 4(6)$
2.14.b	g_{11}	$(x,xy,y^{\nu(p)-6}z^{(\nu(q)-4)/6} \mid z^{(\nu(q)-4)/6},(y^6)^{(\nu(p)-5)/6})$	$\nu(p)\equiv 5(6)$, $\nu(q)\equiv 4(6)$
2.14.c	g_{11}	$(x,xy,y^{-1}x^{(\nu(p)-1)/3}z^{(\nu(q)-4)/6} \mid (x^2)^{(\nu(p)-7)/6},z^{(\nu(q)-4)/6})$	$\nu(p)\equiv 1(6)$, $\nu(q)\equiv 4(6)$
2.14.d	g_{11}	$(x,xy,y^{\nu(q)-6}x^{(\nu(p)-1)/3} \mid (x^2)^{(\nu(p)-7)/6},(y^6)^{(\nu(q)-5)/6})$	$\nu(p)\equiv 1(6)$, $\nu(q)\equiv 5(6)$
2.14.e	g_{11}	$(x,xy,y^{\nu(q)-5}x^{(\nu(p)+\nu(q))/3-2} \mid (x^2)^{(\nu(p)-8)/6},(x^2y^6)^{(\nu(q)-4)/6})$	$\nu(p)\equiv 2(6)$, $\nu(q)\equiv 4(6)$

equivalent. Furthermore, composing the pairs $\Psi_{k,l}$ and $\Omega_{k,l,1}$ we can show that any generating vector of G is equivalent to one of them.

Case 2.3. Let $v=(x\rho_1^{k_1}\rho_2^{l_1},x\rho_1^{k_2}\rho_2^{l_2},x^{-1}\rho_1^{k_3}\rho_2^{l_3},x^{-1}\rho_1^{k_4}\rho_2^{l_4} \mid \rho_1^{u_1},\rho_2^{u_2},\rho_3^{u_3})$ and $n=3$. Then $x^3=\rho_1^{k_i+r_i}\rho_2^{l_i+s_i}$ for $i=1,2,3,4$. Using pair $\Omega_{\delta_1,\delta_2,1}$ if necessary, we can assume that $x^3=1$, $\rho_1^{k_i}\rho_2^{l_i}=\rho_1^{r_i}\rho_2^{s_i}$, and G is isomorphic to $Z_3\oplus Z_2\oplus Z_2$.

The cases 2.6, 2.7, 2.8, 2.13 are similar and so we omit them.

Next assume that $\widetilde{G}$ is isomorphic to $Z_2\oplus Z_n=\langle\tilde{x}\rangle\oplus\langle\tilde{y}\rangle$ and let x and y belong to $\pi^{-1}(\tilde{x})$ and $\pi^{-1}(\tilde{y})$, respectively. Then G is generated by elements x, y and two central involutions ρ_1, ρ_2 which satisfy the relations $x^2=\rho_1^{\delta_1}\rho_2^{\delta_2}$, $y^n=\rho_1^{\epsilon_1}\rho_2^{\epsilon_2}$, $[x,y]=\rho_1^{\gamma_1}\rho_2^{\gamma_2}$ for some $\epsilon_1,\varepsilon_2,\delta_1,\delta_2,\gamma_1,\gamma_2\in\{0,1\}$. Clearly for any values of these parameters, the group G has order $8n$.

Case 2.5. Let us assume that $v=(x\rho_1^{k_1}\rho_2^{l_1},x\rho_1^{k_2}\rho_2^{l_2},x\rho_1^{k_3}\rho_2^{l_3},y\rho_1^{k_4}\rho_2^{l_4},yx\rho_1^{k_5}\rho_2^{l_5} \mid \rho_1^{u_1}\rho_2^{u_2},\rho_3^{u_3})$ and $n=2$. Then $\rho_1^{\delta_1}\rho_2^{\delta_2}=x^2=\rho_1^{r_i}\rho_2^{s_i}$ for $i=1,2,3$ and so $m_1=m_2=m_3$. Furthermore, $\rho_1^{\epsilon_1}\rho_2^{\epsilon_2}=y^2=\rho_1^{k_4}\rho_2^{l_4}$ and $\rho_1^{r_5}\rho_2^{s_5}=(yx)^2=x^2y^2\rho_1^{\gamma_1}\rho_2^{\gamma_2}$ imply that $\rho_1^{\gamma_1}\rho_2^{\gamma_2}=\rho_1^{r_1+r_4+r_5}\rho_2^{s_1+s_4+s_5}$. Using the pairs $\Phi_{k,l,m,n}$ and $\Psi_{k,l}$ we can prove that v is equivalent to

$$v_{\alpha,\beta}=\left(x\rho_1^{\alpha}\rho_2^{\beta},x,x,y,y^{-1}x\rho_1^{u_1+u_3+\alpha}\rho_2^{u_2+u_3+\beta} \mid \rho_1^{u_1},\rho_2^{u_2},\rho_3^{u_3}\right)\quad\text{for some }\alpha,\beta\in\{0,1\}. \tag{3.8}$$

Two such vectors $v_{\alpha,\beta}$ and $v_{\alpha',\beta'}$ are equivalent only if $\rho_1^{\alpha'}\rho_2^{\beta'}=\rho_1^{\alpha+\gamma_1}\rho_2^{\beta+\gamma_2}$. Then a pair (ψ,φ) of automorphisms of Λ and G induces their equivalence, for ψ and φ defined by the assignments $\psi(x_1)=x_1^{-1}$, $\psi(x_2)=x_4^{-1}x_3^{-1}x_4$, $\psi(x_3)=x_4^{-1}x_2^{-1}x_4$, $\psi(x_4)=x_4^{-1}$, $\psi(x_5)=x_1^{-1}x_5^{-1}x_1$, $\psi(x_6)=x_1^{-1}x_5x_r^{-1}x_5^{-1}x_1,\dots,\psi(x_r)=x_1^{-1}x_5x_6^{-1}x_5^{-1}x_1$ and by $\varphi(x)=x^{-1}\rho_1^{\gamma_1}\rho_2^{\gamma_2}$, $\varphi(y)=y^{-1}$, $\varphi(\rho_j)=\rho_j$, $j=1,2$, respectively.

By inspecting Table 3.1 we check that the numbers u_j are integers only for a surface of genus g_{11} if exactly one of elements $\theta(x_i)^{p_i}$ with $p_i\neq 1$ is a t-involution. It is possible

Table 3.5

Case	Presentation of $\widetilde{G}$	Bran. data	N	Generating vector
1.1	$\langle \widetilde{x},\widetilde{y},\widetilde{c} \mid \widetilde{y}^n,\widetilde{c}^4,[\widetilde{x},\widetilde{y}],\widetilde{c}\widetilde{x}\widetilde{c}^{-1}\widetilde{y},\widetilde{c}\widetilde{y}\widetilde{c}^{-1}\widetilde{x}^{-1},\widetilde{x}^m\widetilde{y}^{-km}\rangle$ $\cong (Z_m \oplus Z_n) \ltimes Z_4$; m/n, $k^2 \equiv -1(n/m)$	$(2,4^2)$	$4mn$	$(\widetilde{c}^{-2}\widetilde{x},\widetilde{c},\widetilde{y}^{-1}\widetilde{c})$
1.2	$\langle \widetilde{x},\widetilde{y},\widetilde{c} \mid \widetilde{y}^n,\widetilde{c}^6,[\widetilde{x},\widetilde{y}],\widetilde{c}\widetilde{x}\widetilde{c}^{-1}\widetilde{y}^{-1},\widetilde{c}\widetilde{y}\widetilde{c}^{-1}\widetilde{y}^{-1}\widetilde{x},\widetilde{x}^m\widetilde{y}^{-km}\rangle$ $\cong (Z_m \oplus Z_n) \ltimes Z_6$; m/n, $k^2-k+1 \equiv 0(n/m)$	$(2,3,6)$	$6mn$	$(\widetilde{c}^3\widetilde{x},\widetilde{c}^2\widetilde{y},\widetilde{c})$
1.3	$\langle \widetilde{x},\widetilde{y},\widetilde{c} \mid \widetilde{c}^3,\widetilde{y}^n,\widetilde{c}\widetilde{x}\widetilde{c}^{-1}\widetilde{y}^{-1}\widetilde{x},\widetilde{c}\widetilde{y}\widetilde{c}^{-1}\widetilde{x},[\widetilde{x},\widetilde{y}],\widetilde{x}^m\widetilde{y}^{-mk}\rangle$ $\cong (Z_m \oplus Z_n) \ltimes Z_3$; m/n, $k^2-k+1 \equiv 0(n/m)$	(3^3)	$3mn$	$(\widetilde{c},\widetilde{c}^{-2}\widetilde{x},\widetilde{x}^{-1}\widetilde{c})$
1.4	$\langle \widetilde{x},\widetilde{y},\widetilde{c} \mid \widetilde{y}^n,\widetilde{c}^2,[\widetilde{x},\widetilde{y}],\widetilde{c}\widetilde{x}\widetilde{c}^{-1}\widetilde{x},\widetilde{c}\widetilde{y}\widetilde{c}^{-1}\widetilde{y},\widetilde{x}^m\widetilde{y}^{-k}\rangle$ $\cong (Z_M \oplus Z_N) \ltimes Z_2$; $M=\gcd(m,n,k)$, $N=mn/M$	(2^4)	$2MN$	$(\widetilde{c},\widetilde{c}^{-1}\widetilde{x},\widetilde{y}\widetilde{c}^{-1}, \widetilde{c}\widetilde{y}^{-1}\widetilde{x}^{-1})$
1.5	$\langle \widetilde{x},\widetilde{y} : \widetilde{y}^n,[\widetilde{x},\widetilde{y}],\widetilde{x}^m=\widetilde{y}^k\rangle \cong Z_M \oplus Z_N$; $M=\gcd(m,n,k)$, $N=mn/M$	$(1:-)$	MN	$(\widetilde{x},\widetilde{y})$

only for $i=4$ or $i=5$ and the both possibilities provide equivalent actions. For example assume that Λ has the signature $[4^5,2^{(g-15)/4}]$, the remaining cases can be proved in a similar way. If apart from the only t-involution, there are p- and q-involutions among the elements $\theta(x_i)^{p_i}$ with $p_i \neq 1$, then $G \cong \langle x : x^4\rangle \oplus \langle y : y^4\rangle$ and the generating vector of G is equivalent to (3.8), where $\rho_1 = x^2$ and $\rho_2 = y^2$ or $\rho_2 = y^2x^2$ according to $\theta(x_5)^{p_5}$ or $\theta(x_4)^{p_4}$ is the t-involution. However, the pair $(\psi_{4,5},\eta)$ induces the equivalence of two such vectors, where η is an automorphism of G defined by $\eta(x)=x$, $\eta(y)=xy^{-1}\rho_1^{u_1+\alpha}\rho_2^{\beta+u_2}$, and $\eta(\rho_j)=\rho_j$. Next assume that exactly one of elements $\theta(x_i)^{p_i}$ with $p_i \neq 1$ is the t-involution and the remaining ones are the same p- or q-involution. If $\theta(x_4)^{p_4}$ is the t-involution, then G is isomorphic to the group $\langle x,y : x^4,y^4,[x,y]y^2\rangle$ for which $\rho_1 = x^2$ and $\rho_2 = x^2y^2$ in (3.8) while if $\theta(x_5)^{p_5}$ is the t-involution, then G is isomorphic to the group $\langle x,y,z : x^4,x^2y^2,z^2,[x,z],[y,z],[x,y]z\rangle$, where $\rho_1 = x^2$ and $\rho_2 = x^2z$. The pair $(\psi_{4,5},\eta)$ induces the equivalence of these two actions once again.

Case 2.14. Here $n=6$ and $v=(x\rho_1^{k_1}\rho_2^{l_1},x^{-1}y\rho_1^{k_2}\rho_2^{l_2},y^{-1}\rho_1^{k_3}\rho_2^{l_3} \mid \rho_1^{u_1},\rho_2^{u_2},\rho_3^{u_3})$. Thus $\rho_1^{\delta_1}\rho_2^{\delta_2} = \rho_1^{r_1}\rho_2^{s_1}$, $\rho_1^{\epsilon_1}\rho_2^{\epsilon_2} = \rho_1^{r_3}\rho_2^{s_3}$. Since $\rho_1^{r_2}\rho_2^{s_2} = (xy)^6 = (x^2y^2\rho_1^{\gamma_1}\rho_2^{\gamma_2})^3 = \rho_1^{r_1+r_3+\gamma_1}\rho_2^{s_1+s_3+\gamma_2}$, it follows that $\gamma_1 \equiv r_1+r_2+r_3(2)$ and $\gamma_2 \equiv s_1+s_2+s_3(2)$. Using the pairs $\Phi_{k,l,m,n}$ we can show that v is equivalent to $(x,x^{-1}y,y^{-1}\rho_1^{u_1+u_3}\rho_2^{u_2+u_3} \mid \rho_1^{u_1},\rho_2^{u_2},\rho_3^{u_3})$. The numbers u_j are integers only for a surface of genus g_{11} if one of elements $\theta(x_2)^{p_2}$ or $\theta(x_3)^{p_3}$ is the only t-involution among the elements $\theta(x_i)^{p_i}$ with $p_i \neq 1$. As in the case 2.5, we can show that both possibilities provide the equivalent actions, however this time the automorphism $\psi_{2,3}$ of the group Λ is involved. □

3.2. Classification of conformal actions on a surface $X_1^{p,q}$. Let G be an automorphism group of a surface $X_1^{p,q}$ of genus g in range $2p+2q-10 \le g \le 2p+2q-4$. By analyzing the all possible solutions of (3.1) for $k=1$ and the classification of finite group actions

Table 3.6

H	Presentation of H	ρ_1	ρ_2
H_1	$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 = \langle x\rangle \oplus \langle y\rangle \oplus \langle z\rangle \oplus \langle w\rangle$	z	w
H_2	$Z_4 \oplus Z_4 = \langle x\rangle \oplus \langle y\rangle$	x^2	y^2
H_3	$Z_4 \oplus Z_2 \oplus Z_2 = \langle x\rangle \oplus \langle y\rangle \oplus \langle z\rangle$	z	x^2
H_4	$\langle x, y : x^4, y^4, [x, y]x^2y^2, [x^2, y]\rangle$	x^2y^2	x^2
H_5	$\langle x^4, y^4, [x, y]y^2\rangle$	y^2	x^2
H_6	$\langle x, y : x^2, y^4, [x, y]y^2\rangle \oplus \langle z : z^2\rangle$	y^2	z
H_7	$\langle x, y : x^4, x^2y^2, [x, y]y^2\rangle \oplus \langle z : z^2\rangle$	y^2	z
H_8	$\langle x, y, z : x^4, x^2y^2, z^2, [x, y]z, [z, x], [y, z]\rangle$	z	x^2
H_9	$\langle x, y, z : x^2, y^2, z^2, [x, z], [y, z], [x, y]z\rangle$	z	w
$H_{1,1}$	$Z_2 \oplus Z_2 = \langle z\rangle \oplus \langle w\rangle$	z	w
$H_{1,n}$	$Z_n \oplus Z_2 \oplus Z_2 = \langle y\rangle \oplus \langle z\rangle \oplus \langle w\rangle$	z	w
$H_{1,2\times n}$	$Z_{2n} \oplus Z_2 = \langle y\rangle \oplus \langle z\rangle$	y^n	z

Table 3.7

1.5	σ (Λ)	G	g	Conditions
a	$(1; 2^{g-1})$	$H_{1,1}$	$g \geq g_5$	None
b.1	$(1; 2^{(g-1)/2})$	$H_{1,2}$	g_7, g_9, g_{11}	$p \equiv 1(2)$ and $q \equiv 1(2)$
b.2	$(1; 2^{(g-1)/2})$	$H_{1,2\times 2}$	g_7, g_9, g_{11}	$p \equiv 1(2)$ and $q \equiv 1(2)$
c	$(1; 2^{(g-1)/3})$	$H_{1,3}$	g_8, g_{11}	$p \equiv 1(3)$ and $q \equiv 1(3)$
d.1	$(1; 2^{(g-1)/4})$	H_1	g_9	$p \equiv 1(4)$ and $q \equiv 1(4)$
d.2	$(1; 2^{(g-1)/4})$	H_2	g_9	$p \equiv 1(4)$ and $q \equiv 1(4)$
d.3	$(1; 2^{(g-1)/4})$	H_3	g_9	$p \equiv 1(4)$ and $q \equiv 1(4)$
d.4	$(1; 2^{(g-1)/4})$	H_4	g_9	$pq \equiv 1(2)$, $p \equiv 3(4)$ or $q \equiv 3(4)$
d.5	$(1; 2^{(g-1)/4})$	H_5	g_9	$pq \equiv 1(2)$, $p \equiv 3(4)$ or $q \equiv 3(4)$
d.6	$(1; 2^{(g-1)/4})$	H_6	g_9	$pq \equiv 1(2)$, $p \equiv 3(4)$ or $q \equiv 3(4)$
d.7	$(1; 2^{(g-1)/4})$	H_7	g_9	$pq \equiv 1(2)$, $p \equiv 3(4)$ or $q \equiv 3(4)$
d.8	$(1; 2^{(g-1)/4})$	$H_{1,4}$	g_9	$p \equiv 1(4)$ and $q \equiv 1(4)$
d.9	$(1; 2^{(g-1)/4})$	$H_{1,2\times 4}$	g_9	$p \equiv 1(4)$ and $q \equiv 1(4)$
e	$(1; 2^{(g-1)/5})$	$H_{1,5}$	g_{10}	$p \equiv 1(5)$ and $q \equiv 1(5)$
f.1	$(1; 2^{(g-1)/6})$	$H_{1,6}$	g_{11}	$p \equiv 1(6)$ and $q \equiv 1(6)$
f.2	$(1; 2^{(g-1)/6})$	$H_{1,2\times 6}$	g_{11}	$p \equiv 1(6)$ and $q \equiv 1(6)$
g	$(1; 2^{(g-1)/7})$	$H_{1,7}$	g_{12}	$p \equiv 1(7)$ and $q \equiv 1(7)$

on a genus one surface given in [4], we conclude that the corresponding group $\widetilde{G}$ has one of the presentations listed in Table 3.5. According to these cases we will refer to an action of the group G as to (1.1), (1.2), (1.3), (1.4), or (1.5) action, respectively. In order to state our results we need some groups listed in Table 3.6. Let us notice that there are only two pairs of isomorphic groups among them, namely, (H_8, H_5) and (H_9, H_6).

Theorem 3.5. *The topological type of the* (1.5) *action on a surface* $X_1^{p,q}$ *of genus* g *in range* $2p+2q-4 \le g \le 2p+2q-10$ *is determined by the group of automorphisms* G, *the signature of* Λ *given in Table 3.7 and the generating vector* v, *where*

$$\begin{aligned} v &= \left(h, 1 \mid \rho_1^{F_{\nu(p)}/2}, \rho_2^{F_{\nu(q)}/2}, \rho_3^{F_{\nu(t)}/2}\right) \quad \text{in the case (a)}, \\ v &= \left(x, y \mid \rho_1^{F_{\nu(p)}/8}, \rho_2^{F_{\nu(q)}/8}, \rho_3^{F_{\nu(t)}/8}\right) \quad \text{in (d.1)–(d.7)}, \\ v &= \left(h, y \mid \rho_1^{F_{\nu(p)}/2n}, \rho_2^{F_{\nu(q)}/2n}, \rho_3^{F_{\nu(t)}/2n}\right) \quad \text{in the remaining cases} \end{aligned} \tag{3.9}$$

for some permutation ν *of the set* $\{p,q,t\}$, *where* h *denotes the involution* ρ_j *with fixed points if two of* F_p, F_q, F_t *are equal zero and* $h = 1$ *otherwise.*

Proof. In this case, the group Λ has the signature $\sigma(\Lambda) = (1;2,\overset{r}{\ldots},2)$. Let $x_1,\ldots,x_r,a_1,b_1$ be its canonical generators and let x and y denote $\theta(a_1)$ and $\theta(b_1)$, respectively. Then $\widetilde{G} \cong Z_M \oplus Z_N = \langle \pi(x)\rangle \oplus \langle \pi(y)\rangle$ for some positive integers N, M, where M divides N. The generating vector of G has the form $(x\rho_1^{\alpha_1}\rho_2^{\alpha_2}, y\rho_1^{\beta_1}\rho_2^{\beta_2} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3})$, where by (3.2), $u_1 = F_{\nu(p)}/2MN$, $u_2 = F_{\nu(q)}/2MN$, and $u_3 = F_{\nu(t)}/2MN$ for some permutation ν of the set $\{p,q,t\}$. Using pairs $\Upsilon_{\alpha_1,\alpha_2}$ and Θ_{β_1,β_2} if necessary, we can assume that $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$.

The elements x, y, ρ_1, ρ_2 generate G and satisfy the relations $[x,y] = \rho_1^{u_1+u_3}\rho_2^{u_2+u_3}$, $x^M = \rho_1^{\delta_1}\rho_2^{\delta_2}$, $y^N = \rho_1^{\epsilon_1}\rho_2^{\epsilon_2}$ for some integers $\delta_1,\delta_2,\epsilon_1,\epsilon_2 \in \{0,1\}$. If $M \equiv 1(2)$, then by the relation $yxy^{-1} = x\rho_1^{u_1+u_3}\rho_2^{u_2+u_3}$, we obtain $x^M\rho_1^{u_1+u_3}\rho_2^{u_2+u_3} = x^M$ which implies $u_1+u_3 \equiv 0(2)$ and $u_2+u_3 \equiv 0(2)$ and consequently $p \equiv 1(N)$ and $q \equiv 1(N)$. Furthermore, exchanging the identity of G for φ' in the pair $\Upsilon_{\delta_1,\delta_2}$, where φ' is defined by the assignment $\varphi'(x) = x\rho_1^{\delta_1}\rho_2^{\delta_2}$, $\varphi'(y) = y$, $\varphi'(\rho_j) = \rho_j$, we obtain the pair $\Upsilon'_{\delta_1,\delta_2}$ which induces the equivalence of the actions of G and a group corresponding to $\delta_1 = \delta_2 = 0$. In the similar way, we can prove that for odd N, $\epsilon_1 = \epsilon_2 = 0$, $u_1+u_3 \equiv 0(2)$, and $u_2+u_3 \equiv 0(2)$.

By inspecting Table 3.1 and the formula (3.2) for g in range $g_{12} \le g \le g_6$, we conclude that the numbers u_j are integers if and only if $M = 1$ and N divides $F_t/2$, except $g = g_9$, where the additional case $M = N = 2$ is possible.

If $M = 1$, then $G \cong Z_N \oplus Z_2 \oplus Z_2 = \langle y\rangle \oplus \langle \rho_1\rangle \oplus \langle \rho_2\rangle$ and $v = (1, y \mid \rho_1^{u_1}, \rho_2^{u_2}, (\rho_1\rho_2)^{u_3})$ or $G \cong Z_{2N} \oplus Z_2 = \langle y\rangle \oplus \langle \rho_1\rangle$ and $v = (1, y \mid \rho_1^{u_1}, (y^N)^{u_2}, (y^N\rho_1)^{u_3})$. These two actions are equivalent if and only if N is odd.

Now assume that $g = g_9$ and $M = N = 2$. For $i = 1,2$, let us define the automorphisms $\psi_i : \Lambda \to \Lambda$ and $\varphi_i : G \to G$ by the assignments $\psi_1(a) = ab$, $\psi_1(b) = b$, $\psi_1(x_j) = x_j$ for $j = 1,\ldots,r$ and $\varphi_1(x) = xy$, $\varphi_1(y) = y$, $\varphi_1(\rho_j) = \rho_j$ and by $\psi_2(a) = b$, $\psi_2(b) = a$, $\psi_2(x_1) = x_r^{-1},\ldots,\psi_2(x_r) = x_1^{-1}$ and $\varphi_2(x) = y$, $\varphi_2(y) = x$, $\varphi_2(\rho_j) = \rho_j$, respectively. Then using the pairs (ψ_i,φ_i) and $\Psi_{k,l}$ we can prove that the action of G is equivalent to one of d.1–d.7 in Table 3.7. ◻

Theorem 3.6. *For $i = 1,2,3,4$, the (1.i)-action on a surface $X_1^{p,q}$ of genus g in range $2p+2q-10 \le g \le 2p+2q-4$ is possible only for $g = 2p+2q-7$. The topological type of Cases 1.1, 1.2, 1.3, and 1.4-action on such a surface is determined by the group of automorphisms G, the signature of given in Tables 3.8, 3.9, 3.10, 3.11, respectively, and the generating vector v, where*

$$v = \begin{cases} (c^{-2}x, c, y^{-1}c\rho_1^{u_1+u_3}\rho_2^{u_2+u_3} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3}), & i = 1, \\ (c^3x, c^2y, c\rho_1^{u_1+u_3}\rho_2^{u_2+u_3} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3}), & i = 2, \\ (c, c^{-2}x, x^{-1}c\rho_1^{u_1+u_3}\rho_2^{u_2+u_3} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3}), & i = 3, \\ (c, c^{-1}x, yc^{-1}, cy^{-1}x^{-1}\rho_1^{u_1+u_3}\rho_2^{u_2+u_3} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3}), & i = 4, \end{cases} \tag{3.10}$$

and v denotes a permutation of the set $\{p,q\}$.

Proof. By inspecting Table 3.1 and the formula (3.2) for g in range $g_{12} \le g \le g_6$, we conclude that the actions (1.1)–(1.4) are possible only on a surface of genus g_9 if both elements $\widetilde{x}$ and $\widetilde{y}$ in the presentation of $\widetilde{G}$ have the orders 2 and exactly one of elements $\theta(x_i)^{p_i}$ is the t-involution for some i with $2|\widetilde{G}|/p_i = 8$.

Case 1.1. G is generated by the elements $c = \theta(x_2)$, $x = \theta(x_2)^2\theta(x_1)$, $y = \theta(x_2)\theta(x_1)\theta(x_2)$ and two central involutions ρ_1, ρ_2 which satisfy the relations $y^2 = \rho_1^{\epsilon_1}\rho_2^{\epsilon_2}$, $c^4 = \rho_1^{r_2}\rho_2^{s_2}$, $cxc^{-1} = y^{-1}\rho_1^{r_1+r_2}\rho_2^{s_1+s_2}$, $cyc^{-1} = x$, $[x,y] = \rho_1^{r_2+r_3}\rho_2^{s_2+s_3}$ for some $\epsilon_1, \epsilon_2 \in \{0,1\}$. Two groups corresponding to different values of parameters ϵ_1, ϵ_2 have different numbers of elements whose squares are equal to $\rho_1^{\epsilon_1}\rho_2^{\epsilon_2}$ and therefore their actions are not equivalent. Let us notice that x, y, ρ_1, and ρ_2 generate a subgroup of G isomorphic to H_i for some $i = 7,8,9$ and $G \simeq H_i \ltimes \langle c\rangle$ if $\rho_1^{r_2}\rho_2^{s_2} = 1$ or $G \simeq (H_i \ltimes \langle c\rangle)/\langle c^4\rho_1^{r_2}\rho_2^{s_2}\rangle$ otherwise.

The generating vector of G has the form $(c^{-2}x, c, y^{-1}c\rho_1^{u_1+u_3}\rho_2^{u_2+u_3} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3})$. The numbers u_j are integers if and only if exactly one of elements $\theta(x_2)^4$ or $\theta(x_3)^4$ is the only t-involution among the elements $\theta(x_i)^{p_i}$. The pair $(\psi_{2,3}, \varphi)$ induces the equivalence of two actions corresponding to both possibilities, where $\psi_{2,3}$ is given by (2.8) and φ is defined by $\varphi(x) = y^{-1}$, $\varphi(y) = x\rho_1^{r_2+r_3}\rho_2^{s_2+s_3}$, $\varphi(c) = y^{-1}c\rho_1^{u_1}\rho_2^{u_2}$.

Case 1.2. Here $\theta(x_3)^{p_3}$ is the only t-involution among the elements $\theta(x_i)^{p_i}$. The group G is generated by the elements $x = \theta(x_1x_2)^3\theta(x_1)$, $y = \theta(x_1x_2)^2\theta(x_2)$, $c = \theta(x_1x_2)^{-1}$ and two central involutions ρ_1, ρ_2 which satisfy the relations $c^6 = \rho_1^{r_3}\rho_2^{s_3}$, $cyc^{-1} = x^{-1}y\rho_1^{r_2}\rho_2^{s_2}$, $cxc^{-1} = y\rho_1^{r_1}\rho_2^{s_1}$ and $[x,y] = \rho_1^{r_1+r_3}\rho_2^{s_1+s_3}$. Since $y^2 = cy^2c^{-1} = (x^{-1}y\rho_1^{r_2}\rho_2^{s_2})^2 = x^{-2}y^2\rho_1^{r_1+r_3}\rho_2^{s_1+s_3}$, it follows that $x^2 = y^2 = \rho_1^{r_1+r_3}\rho_2^{s_1+s_3}$. Thus x, y, ρ_1, and ρ_2 generate a subgroup of G isomorphic to H_7 and $G \simeq (H_7 \ltimes \langle c\rangle)/\langle c^6\rho_1^{r_3}\rho_2^{s_3}\rangle$. Any generating vector of G is equivalent to $(c^3x, c^2y, c\rho_1^{u_1+u_3}\rho_2^{u_2+u_3} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3})$, where u_i are defined by (3.2) (see Table 3.9).

Case 1.3. Now G is generated by elements $c=\theta(x_1)$, $x=\theta(x_1)^2\theta(x_2)$, $y=\theta(x_1)^2\theta(x_2)^2\theta(x_1)^2$ and two central involutions ρ_1, ρ_2 satisfying the relations $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_1^{r_2}\rho_2^{s_2}$, $[x,y] = \rho_1^{r_1+r_2+r_3+u_1+u_3}\rho_2^{s_1+s_2+s_3+u_2+u_3}$, $c^3 = \rho_1^{r_1}\rho_2^{s_1}$. Thus $y^2 = x^2 = [x,y]$ and so the elements x, y, ρ_1, and ρ_2 generate a subgroup of G isomorphic to H_1 if $r_1 + r_2 + r_3 + u_1 + u_3 \equiv 0(2)$ and $s_1 + s_2 + s_3 + u_2 + u_3 \equiv 0(2)$ or to H_7 otherwise. Furthermore, $G \cong H_i \ltimes \langle c\rangle$ if

Table 3.8

Case 1.1	$\tau = \sigma(\Lambda)$	Presentation of G	u_1	u_2	u_3	Conditions
a.i, $7 \le i \le 9$	$[4, 8^2, 2^{(g-17)/16}]$	$(H_i \ltimes \langle c : c^8 \rangle)/\langle c^4\rho_2 \rangle$; $cxc^{-1} = y^{-1}\rho_3$, $cyc^{-1} = x$	$\frac{\nu(p)-7}{8}$	$\frac{\nu(q)-5}{8}$	4	$\nu(p) \equiv 7(8)$, $\nu(q) \equiv 5(8)$
b.i, $7 \le i \le 9$	$[4, 8^2, 2^{(g-17)/16}]$	$(H_i \ltimes \langle c : c^8 \rangle)/\langle c^4\rho_2 \rangle$; $cxc^{-1} = y^{-1}$, $cyc^{-1} = x$	$\frac{\nu(p)-3}{8}$	$\frac{\nu(q)-9}{8}$	4	$\nu(p) \equiv 3(8)$, $\nu(q) \equiv 1(8)$
c.i, $7 \le i \le 9$	$[2, 4, 8, 2^{(g-5)/16}]$	$H_i \ltimes \langle c : c^4 \rangle$; $cxc^{-1} = y^{-1}$, $cyc^{-1} = x$	4	$\frac{\nu(p)-3}{8}$	$\frac{\nu(q)-3}{8}$	$\nu(p) \equiv 3(8)$, $\nu(q) \equiv 3(8)$
d.i, $7 \le i \le 9$	$[4^2, 8, 2^{(g-13)/16}]$	$H_i \ltimes \langle c : c^4 \rangle$; $cxc^{-1} = y^{-1}\rho_2$, $cyc^{-1} = x$	4	$\frac{\nu(p)-7}{8}$	$\frac{\nu(q)-3}{8}$	$\nu(p) \equiv 7(8)$, $\nu(q) \equiv 3(8)$
e.i, $7 \le i \le 9$	$[2, 8^2, 2^{(g-9)/16}]$	$(H_i \ltimes \langle c : c^8 \rangle)/\langle c^4\rho_2 \rangle$; $cxc^{-1} = y^{-1}\rho_2$, $cyc^{-1} = x$	$\frac{\nu(p)-3}{8}$	$\frac{\nu(q)-5}{8}$	4	$\nu(p) \equiv 3(8)$, $\nu(q) \equiv 5(8)$

Table 3.9

1.2	$\tau = \sigma(\Lambda)$	Presentation of G	u_1	u_2	u_3	Conditions
a	$[2, 3, 12, 2^{(g-5)/24}]$	$(H_7 \ltimes \langle c : c^{12} \rangle)/\langle c^6\rho_1 \rangle$; $cxc^{-1} = y$, $cyc^{-1} = x^{-1}y$	0	$\frac{\nu(p)-3}{12}$	$\frac{\nu(q)-3}{12}$	$p \equiv 3(12)$, $q \equiv 3(12)$
b	$[2, 6, 12, 2^{(g-13)/24}]$	$(H_7 \ltimes \langle c : c^{12} \rangle)/\langle c^6\rho_1 \rangle$; $cxc^{-1} = y$, $cyc^{-1} = x^{-1}y\rho_2$	0	$\frac{\nu(p)-7}{12}$	$\frac{\nu(q)-3}{12}$	$\nu(p) \equiv 7(12)$, $\nu(q) \equiv 3(12)$
c	$[4, 3, 12, 2^{(g-17)/24}]$	$(H_7 \ltimes \langle c : c^{12} \rangle)/\langle c^6\rho_3 \rangle$; $cxc^{-1} = y\rho_2$, $cyc^{-1} = x^{-1}y$	$\frac{\nu(q)-3}{12}$	$\frac{\nu(p)-9}{12}$	0	$\nu(p) \equiv 9(12)$, $\nu(q) \equiv 3(12)$
d	$[4, 6, 12, 2^{(g-25)/24}]$	$(H_7 \ltimes \langle c : c^{12} \rangle)/\langle c^6\rho_3 \rangle$; $cxc^{-1} = y\rho_2$, $cyc^{-1} = x^{-1}y\rho_1$	$\frac{\nu(q)-7}{12}$	$\frac{\nu(p)-9}{12}$	0	$\nu(p) \equiv 9(12)$, $\nu(q) \equiv 7(12)$
e	$[4, 6, 12, 2^{(g-25)/24}]$	$(H_7 \ltimes \langle c : c^{12} \rangle)/\langle c^6\rho_3 \rangle$; $cxc^{-1} = y\rho_2$, $cyc^{-1} = x^{-1}y\rho_2$	$\frac{\nu(q)-3}{12}$	$\frac{\nu(p)-13}{12}$	0	$\nu(p) \equiv 1(12)$, $\nu(q) \equiv 3(12)$

Table 3.10

1.3	$\tau = \sigma(\Lambda)$	Presentation of G	u_1	u_2	u_3	Conditions
a.1	$[3^2, 6, 2^{(g-5)/12}]$	$H_7 \ltimes \langle c : c^3 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}$	0	$\frac{\nu(p)-3}{6}$	$\frac{\nu(q)-3}{6}$	$p \equiv 3(12)$, $q \equiv 3(12)$
a.2	$[3^2, 6, 2^{(g-5)/12}]$	$H_1 \ltimes \langle c : c^3 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}$	$\frac{\nu(p)-3}{6}$	$\frac{\nu(q)-3}{6}$	0	$p \equiv 9(12)$, $q \equiv 9(12)$
a.3	$[3^2, 6, 2^{(g-5)/12}]$	$H_7 \ltimes \langle c : c^3 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}$	$\frac{\nu(p)-3}{6}$	$\frac{\nu(q)-3}{6}$	0	$\nu(p) \equiv 3(12)$, $\nu(q) \equiv 9(12)$
c.1	$[3, 6^2, 2^{(g-9)/12}]$	$H_7 \ltimes \langle c : c^3 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_2$	$\frac{\nu(q)-3}{6}$	$\frac{\nu(p)-5}{6}$	0	$\nu(p) \equiv 5(12)$, $\nu(q) \equiv 3(12)$
c.2	$[3, 6^2, 2^{(g-9)/12}]$	$H_7 \ltimes \langle c : c^3 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_1$	$\frac{\nu(p)-5}{6}$	$\frac{\nu(q)-3}{6}$	0	$\nu(p) \equiv 11(12)$, $\nu(q) \equiv 9(12)$
c.3	$[3, 6^2, 2^{(g-9)/12}]$	$H_1 \ltimes \langle c : c^3 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_1$	$\frac{\nu(p)-5}{6}$	$\frac{\nu(q)-3}{6}$	0	$\nu(p) \equiv 5(12)$, $\nu(q) \equiv 9(12)$
c.4	$[3, 6^2, 2^{(g-9)/12}]$	$H_7 \ltimes \langle c : c^3 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_2$	0	$\frac{\nu(p)-5}{6}$	$\frac{\nu(q)-3}{6}$	$\nu(p) \equiv 11(12)$, $\nu(q) \equiv 3(12)$
d.1	$[6^3, 2^{(g-13)/12}]$	$(H_1 \ltimes \langle c : c^6 \rangle)/\langle c^3\rho_1 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_2$	$\frac{\nu(p)-5}{6}$	$\frac{\nu(q)-5}{6}$	0	$p \equiv 5(12)$, $q \equiv 5(12)$
d.2	$[6^3, 2^{(g-13)/12}]$	$(H_7 \ltimes \langle c : c^6 \rangle)/\langle c^3\rho_1 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_3$	0	$\frac{\nu(p)-5}{6}$	$\frac{\nu(q)-5}{6}$	$p \equiv 11(12)$, $q \equiv 11(12)$
d.3	$[6^3, 2^{(g-13)/12}]$	$(H_7 \ltimes \langle c : c^6 \rangle)/\langle c^3\rho_2 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_1$	$\frac{\nu(q)-5}{6}$	$\frac{\nu(p)-5}{6}$	0	$\nu(p) \equiv 5(12)$, $\nu(q) \equiv 11(12)$

Table 3.10. Continued.

1.3	$\tau = \sigma(\Lambda)$	Presentation of G	u_1	u_2	u_3	Conditions
d.4	$[6^3, 2^{(g-13)/12}]$	$(H_7 \ltimes \langle c : c^6 \rangle)/\langle c^3\rho_2 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_2$	0	$\frac{\nu(p)-7}{6}$	$\frac{\nu(q)-3}{6}$	$\nu(p) \equiv 7(12)$, $\nu(q) \equiv 3(12)$
d.5	$[6^3, 2^{(g-13)/12}]$	$(H_1 \ltimes \langle c : c^6 \rangle)/\langle c^3\rho_1 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_1$	$\frac{\nu(p)-7}{6}$	$\frac{\nu(q)-3}{6}$	0	$\nu(p) \equiv 1(12)$, $\nu(q) \equiv 9(12)$
d.6	$[6^3, 2^{(g-13)/12}]$	$(H_7 \ltimes \langle c : c^6 \rangle)/\langle c^3\rho_1 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_1$	$\frac{\nu(p)-7}{6}$	$\frac{\nu(q)-3}{6}$	0	$\nu(p) \equiv 7(12)$, $\nu(q) \equiv 9(12)$
d.7	$[6^3, 2^{(g-13)/12}]$	$(H_7 \ltimes \langle c : c^6 \rangle)/\langle c^3\rho_2 \rangle$; $cxc^{-1} = x^{-1}y$, $cyc^{-1} = x^{-1}\rho_2$	$\frac{\nu(q)-3}{6}$	$\frac{\nu(p)-7}{6}$	0	$\nu(p) \equiv 1(12)$, $\nu(q) \equiv 3(12)$

Table 3.11

Case 1.4	$\tau = \sigma(\Lambda)$	Presentation of G	u_1	u_2	u_3	Conditions
a.i, $4 \le i \le 7$	$[2^3, 4, 2^{(g-5)/8}]$	$H_i \ltimes \langle c : c^2 \rangle$; $cxc = x^{-1}$, $cyc = y^{-1}$	0	$\frac{\nu(p)-3}{4}$	$\frac{\nu(q)-3}{4}$	$p \equiv 3(4)$, $q \equiv 3(4)$
b.i, $4 \le i \le 7$	$[2^2, 4^2, 2^{(g-9)/8}]$	$H_i \ltimes \langle c : c^2 \rangle$; $cxc = x^{-1}$, $cyc = y^{-1}\rho_2$	$\frac{\nu(p)-3}{4}$	$\frac{\nu(q)-5}{4}$	4	$\nu(p) \equiv 3(4)$, $\nu(q) \equiv 1(4)$
c.i, $4 \le i \le 7$	$[2, 4^3, 2^{(g-13)/8}]$	$H_i \ltimes \langle c : c^2 \rangle$; $cxc = x^{-1}\rho_2$, $cyc = y^{-1}\rho_2$	4	$\frac{\nu(p)-7}{4}$	$\frac{\nu(q)-3}{4}$	$p \equiv 3(4)$, $q \equiv 3(4)$
d.i, $1 \le i \le 3$	$[2, 4^3, 2^{(g-13)/8}]$	$H_i \ltimes \langle c : c^2 \rangle$; $cxc = x^{-1}\rho_1$, $cyc = y^{-1}\rho_2$	$\frac{\nu(p)-5}{4}$	$\frac{\nu(p)-5}{4}$	4	$\nu(p) \equiv 1(4)$, $\nu(q) \equiv 1(4)$
e.i, $4 \le i \le 7$	$[4^4, 2^{(g-17)/8}]$	$(H_i \ltimes \langle c : c^4 \rangle)/\langle c^2\rho_2 \rangle$; $cyc = y^{-1}$	$\frac{\nu(q)-3}{4}$	$\frac{\nu(p)-9}{4}$	4	$\nu(p) \equiv 3(4)$, $\nu(q) \equiv 1(4)$
f.i, $4 \le i \le 7$	$[4^4, 2^{(g-17)/8}]$	$(H_i \ltimes \langle c : c^4 \rangle)/\langle c^2\rho_1 \rangle$; $cyc = y^{-1}\rho_1\rho_2$, $cxc = x^{-1}$	$\frac{\nu(p)-7}{4}$	$\frac{\nu(p)-5}{4}$	4	$p \equiv 3(4)$, $q \equiv 3(4)$

Table 3.12

Case	Presentation of $\widetilde{G}$	Bran. data	N	Generating vector
0.1	G is trivial	$(-)$	1	—
0.2	$Z_N = \langle \widetilde{x} : \widetilde{x}^N \rangle$	(N^2)	any	$(\widetilde{x}, \widetilde{x}^{-1})$
0.3	$D_{N/2} = \langle \widetilde{x}, \widetilde{y} : \widetilde{x}^2, \widetilde{y}^2, (\widetilde{x}\widetilde{y})^{N/2} \rangle$	$\left(2^2, \dfrac{N}{2}\right)$	even	$(\widetilde{x}, \widetilde{y}, (\widetilde{x}\widetilde{y})^{-1})$
0.4	$A_4 = \langle \widetilde{x}, \widetilde{y} : \widetilde{x}^2, \widetilde{y}^3, (\widetilde{x}\widetilde{y})^3 \rangle$	$(2, 3^2)$	12	$(\widetilde{x}, \widetilde{y}, (\widetilde{x}\widetilde{y})^{-1})$
0.5	$S_4 = \langle \widetilde{x}, \widetilde{y} : \widetilde{x}^2, \widetilde{y}^3, (\widetilde{x}\widetilde{y})^4 \rangle$	$(2, 3, 4)$	24	$(\widetilde{x}, \widetilde{y}, (\widetilde{x}\widetilde{y})^{-1})$
0.6	$A_5 = \langle \widetilde{x}, \widetilde{y} : \widetilde{x}^2, \widetilde{y}^3, (\widetilde{x}\widetilde{y})^5 \rangle$	$(2, 3, 5)$	60	$(\widetilde{x}, \widetilde{y}, (\widetilde{x}\widetilde{y})^{-1})$

$\rho_1^{r_1}\rho_2^{s_1} = 1$ or $G \cong (H_i \ltimes \langle c \rangle)/c^3\rho_1^{r_1}\rho_2^{s_1}$ if $\rho_1^{r_1}\rho_2^{s_1} \neq 1$. The generating vector of G has a form $(c, c^{-2}x, x^{-1}c\rho_1^{u_1+u_3}\rho_2^{u_2+u_3}, \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3})$. The numbers u_i are integers if and only if exactly one of elements $\theta(x_1)^3$, $\theta(x_2)^3$, or $\theta(x_3)^3$ is the only t-involution however the all possibilities provides the equivalent actions (see Table 3.10).

Case 1.4. Here $\theta(x_i)^2$ is the t-involution for exactly one i in range $1 \le i \le 4$ and it is no importance for which of them. Here G is generated by elements $x = \theta(x_1)\theta(x_2)$, $y = \theta(x_3)\theta(x_1)$, $c = \theta(x_1)$ and two central involutions ρ_1, ρ_2 which satisfy the relations $cxc^{-1} = x^{-1}\rho_1^{r_1+r_2}\rho_2^{s_1+s_2}$, $cyc^{-1} = y^{-1}\rho_1^{r_1+r_3}\rho_2^{s_1+s_3}$, $[x, y] = \rho_1^{r_1+\cdots+r_4}\rho_2^{s_1+\cdots+s_4}$. The generating vector of G has the form $(c, c^{-1}x, yc^{-1}, cy^{-1}x^{-1}\rho_1^{u_1+u_3}\rho_2^{u_2+u_3} \mid \rho_1^{u_1}, \rho_2^{u_2}, \rho_3^{u_3})$. Using pairs $(\psi_{1,2}, \varphi)$ and (ϕ, μ), where φ, ϕ, and μ are defined by $\varphi(x) = x$, $\varphi(y) = yx\rho_1^{r_1}\rho_2^{s_1}$, $\varphi(c) = c^{-1}x$, $\varphi(\rho_i) = \rho_i$, $\phi(x_1) = x_1^{-1}$, $\phi(x_2) = x_1x_3^{-1}x_1^{-1}$, $\phi(x_3) = x_1x_2^{-1}x_1^{-1}$, $\phi(x_4) = x_1x_2x_3x_4^{-1}x_3^{-1}x_2^{-1}x_1^{-1}$, $\phi(x_5) = x_r^{-1}, \dots, \phi(x_r) = x_5^{-1}$ and by $\mu(c) = c^{-1}$, $\mu(x) = y\rho_1^{r_1+r_3}\rho_2^{s_1+s_3}$, $\mu(y) = x\rho_1^{r_1+r_2}\rho_2^{s_1+s_2}$, $\mu(\rho_j) = \rho_j$, we can prove that the action of G is equivalent to one of those listed in Table 3.11. □

3.3. Classification of conformal actions on a surface $X_0^{p,q}$. By Lemma 3.1, for any integers p, q, g in range $q \ge p \ge 0$ and $2q - 1 < g \le 2p + 2q + 1$, there exists a Riemann surface admitting p- and q-involutions whose product is a $(g - p - q)$-hyperelliptic involution. According to [8, Theorem 3.7], these p- and q-involutions are central and unique in the full automorphism group G if $g > 3q + 1$. By analyzing the all possible solutions of (3.1) for $k = 0$ and the classification of spherical groups given in [10], we conclude that the corresponding group $\widetilde{G}$ is isomorphic to one of the groups listed in Table 3.9. By arguments similar to those in the proofs of the previous theorems, we obtain the topological classification of actions induced by $\widetilde{G}$ on a surface $X_0^{p,q}$ with central p- and q-involutions.

THEOREM 3.7. *The topological type of the action on a surface $X_0^{p,q}$ of genus $g > 2q - 1$ is determined by the finite group of automorphisms G, the signature of Λ given in Table 3.13 and the generating vector listed in Table 3.14, where ν denotes a permutation of the set $\{p, q, t\}$. In particular, these tables determine the actions on any pq-hyperelliptic Riemann surface of genus g in range $2p + 2q - 2 \le g \le 2p + 2q + 1$ for $5 \le p < q < 2p - 3$.*

Table 3.13

Case	$\tau = \sigma(\Lambda)$	Presentation of G	Conditions
0.1	$[2^{g+3}]$	$\langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.2.a	$[N,N,2^{(g+3)/N}]$	$\langle x : x^N\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.2.b	$[N,2N,2^{(g+2)/N}]$	$\langle x : x^{2N}\rangle \oplus \langle z : z^2\rangle$	$N \equiv 1(2)$
0.2.c	$[2N,2N,2^{(g+1)/N}]$	$\langle x : x^{2N}\rangle \oplus \langle z : z^2\rangle$	None
0.2.c'	$[2N,2N,2^{(g+1)/N}]$	$\langle x : x^{2N}\rangle \oplus \langle z : z^2\rangle$	$N \equiv 1(2)$
0.3.a	$[2,2,N/2,2^{(g+3)/N}]$	$\langle x,y : x^2,y^2,(xy)^{N/2}\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	$N \equiv 0(2), N \geq 4$
0.3.b	$[2,4,N/2,2^{(g+3)/N-1/2}]$	$\langle x,y : x^4,y^{N/2},(xy)^2,(x^{-1}y)^2\rangle \oplus \langle z : z^2\rangle$	$N \equiv 0(2), N \geq 4$
0.3.c	$[4,4,N/2,2^{(g+3)/N-1}]$	$\langle x,y : x^4,x^2y^2,(xy)^{N/2}\rangle \oplus \langle z : z^2\rangle$	$N \equiv 0(2), N \geq 4$
0.3.c'	$[4,4,N/2,2^{(g+3)/N-1}]$	$\langle x,y : x^4,y^4,(xy)^{N/2},(x^2y)^2y^2,(y^2x)^2x^2\rangle$	$N \equiv 0(4)$
0.3.d	$[2,2,N,2^{(g+1)/N}]$	$\langle x,y : x^2,y^2,(xy)^N\rangle \oplus \langle z : z^2\rangle$	$N \equiv 0(2), N \geq 4$
0.3.e	$[2,4,N,2^{(g+1)/N-1/2}]$	$\langle x,y : x^2,y^N,xyxy^{N/2+1}\rangle \oplus \langle z : z^2\rangle$	$N \equiv 0(2), N \geq 4$
0.3.e'	$[2,4,N,2^{(g+1)/N-1/2}]$	$\langle x,y : x^2,y^4,(xy)^N,(y^2x)^2,((xy)^{N/2}x)^2\rangle$	$N \equiv 0(2), N \geq 4$
0.3.f	$[4,4,N,2^{(g+1)/N-1}]$	$\langle x,y : x^2y^{N/2},y^N,x^{-1}yxy\rangle \oplus \langle z : z^2\rangle$	$N \equiv 0(2), N \geq 4$
0.3.f'	$[4,4,N,2^{(g+1)/N-1}]$	$\langle x,y : x^4,x^2y^2,(xy)^N\rangle$	$N \equiv 0(2), N \geq 4$
0.3.f"	$[4,4,N,2^{(g+1)/N-1}]$	$\langle x,y : x^4,x^2(xy)^{N/2},y^4,y^2xy^2x^{-1}\rangle$	$N \equiv 0(4)$
0.4.a	$[2,3,3,2^{(g+3)/12}]$	$\langle x,y : x^2,y^3,(xy)^3\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.4.b	$[2,3,6,2^{(g-1)/12}]$	$\langle x,y : x^2,y^3,(xy)^3\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.4.c	$[2,6,6,2^{(g-5)/12}]$	$\langle x,y : x^2,y^3,(xy)^3\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.4.d	$[4,3,3,2^{(g-3)/12}]$	$\langle x,y : x^4,y^3,(xy)^3,yx^2y^{-1}x^2\rangle \oplus \langle z\rangle$	None
0.4.e	$[4,3,6,2^{(g-7)/12}]$	$\langle x,y : x^4,y^3,(xy)^3,yx^2y^{-1}x^2\rangle \oplus \langle z\rangle$	None
0.4.f	$[4,6,6,2^{(g-11)/12}]$	$\langle x,y : x^4,y^3,(xy)^3,yx^2y^{-1}x^2\rangle \oplus \langle z : z^2\rangle$	None
0.5.a	$[2,3,4,2^{(g+3)/24}]$	$\langle x,y : x^2,y^3,(xy)^4\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.5.b	$[2,6,4,2^{(g-5)/24}]$	$\langle x,y : x^2,y^3,(xy)^4\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.5.c	$[2,3,8,2^{(g-3)/24}]$	$\langle x,y : x^2,y^3,(xy)^8,(xy)^4(yx)^4\rangle \oplus \langle z : z^2\rangle$	None
0.5.d	$[2,6,8,2^{(g-11)/24}]$	$\langle x,y : x^2,y^3,(xy)^8,(xy)^4(yx)^4\rangle \oplus \langle z : z^2\rangle$	None
0.5.e	$[4,3,4,2^{(g-9)/24}]$	$\langle x,y : x^4,y^3,(xy)^4,yx^2y^{-1}x^2\rangle \oplus \langle z : z^2\rangle$	None
0.5.f	$[4,6,4,2^{(g-17)/24}]$	$\langle x,y : x^4,y^3,(xy)^4,yx^2y^{-1}x^2\rangle \oplus \langle z : z^2\rangle$	None
0.5.g	$[4,3,8,2^{(g-15)/24}]$	$\langle x,y : x^4,y^3,(xy)^8,x^2(xy)^4\rangle \oplus \langle z : z^2\rangle$	None
0.5.g'	$[4,3,8,2^{(g-15)/24}]$	$\langle x,y : x^4,y^3,(xy)^8,x^2yx^2y^{-1},(xy)^4x(xy)^4x^{-1}\rangle$	None
0.5.h	$[4,6,8,2^{(g-23)/24}]$	$\langle x,y : x^4,y^3,(xy)^8,x^2(xy)^4\rangle \oplus \langle z : z^2\rangle$	None
0.5.h'	$[4,6,8,2^{(g-23)/24}]$	$\langle x,y : x^4,y^3,(xy)^8,x^2yx^2y^{-1},(xy)^4x(xy)^4x^{-1}\rangle$	None
0.6.a	$(2,3,5,2^{(g+3)/60})$	$\langle x,y : x^2,y^3,(xy)^5\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.6.b	$[2,3,10,2^{(g-9)/60}]$	$\langle x,y : x^2,y^3,(xy)^5\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.6.c	$[2,6,10,2^{(g-29)/60}]$	$\langle x,y : x^2,y^3,(xy)^5\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.6.d	$[2,6,5,2^{(g-17)/60}]$	$\langle x,y : x^2,y^3,(xy)^5\rangle \oplus \langle z : z^2\rangle \oplus \langle w : w^2\rangle$	None
0.6.e	$[4,3,5,2^{(g-27)/60}]$	$\langle x,y : x^4,y^3,(xy)^5,yx^2y^{-1}x^2\rangle \oplus \langle z : z^2\rangle$	None
0.6.f	$[4,3,10,2^{(g-39)/60}]$	$\langle x,y : x^4,y^3,(xy)^5,yx^2y^{-1}x^2\rangle \oplus \langle z : z^2\rangle$	None
0.6.g	$[4,6,5,2^{(g-47)/60}]$	$\langle x,y : x^4,y^3,(xy)^5,yx^2y^{-1}x^2\rangle \oplus \langle z : z^2\rangle$	None
0.6.h	$[4,6,10,2^{(g-59)/60}]$	$\langle x,y : x^4,y^3,(xy)^5,yx^2y^{-1}x^2\rangle \oplus \langle z : z^2\rangle$	None

Table 3.14

Case	Generating vector
0.1	$(\mid z^{F_{\nu(p)}/2}, w^{F_{\nu(q)}/2}, (zw)^{F_{\nu(t)}/2})$
0.2.a	$(xz^{u_1+u_3}w^{u_2+u_3}, x^{-1} \mid z^{F_{\nu(p)}/2N}, w^{F_{\nu(q)}/2N}, (zw)^{F_{\nu(t)}/2N})$
0.2.b	$(x^{N+1}, x^{-1} \mid (x^N)^{(F_{\nu(p)}-2)/2N}, z^{F_{\nu(q)}/2N}, (x^N z)^{F_{\nu(t)}/2N})$
0.2.c	$(x^{1+N(u_1+u_3)}z^{u_2+u_3}, x^{-1} \mid (x^N)^{(F_{\nu(p)}-4)/2N}, z^{F_{\nu(q)}/2N}, (x^N z)^{F_{\nu(t)}/2N})$
0.2.c’	$(x, x^{N-1}z \mid (x^N)^{(F_{\nu(p)}-2)/2N}, z^{(F_{\nu(q)}-2)/2N}, (x^N z)^{F_{\nu(t)}/2N})$
0.3.a	$(xz^{u_1+u_3}, yw^{u_2+u_3}, (xy)^{-1} \mid z^{F_{\nu(p)}/2N}, w^{F_{\nu(q)}/2N}(zw)^{F_{\nu(t)}/2N})$
0.3.b	$((xy)^{-1}z^{u_2+u_3}, x^{1+2(u_1+u_3)}, y \mid (x^2)^{F_{\nu(p)}/2N-1/2}z^{F_{\nu(q)}/2N}(x^2z)^{F_{\nu(t)}/2N})$
0.3.c	$(x^{1+2(u_1+u_3)}, yz^{u_2+u_3}, (xy)^{-1} \mid (x^2)^{F_{\nu(p)}/2N-1}, z^{F_{\nu(q)}/2N}, (x^2z)^{F_{\nu(t)}/2N})$
0.3.c’	$(x^{1+2(u_1+u_3)}, y^{1+2(u_2+u_3)}, (xy)^{-1} \mid (x^2)^{F_{\nu(p)}/2N-1/2}, (y^2)^{F_{\nu(q)}/2N-1/2}, (x^2y^2)^{F_{\nu(t)}/2N})$
0.3.d	$(xz^{u_2+u_3}, y(xy)^{N/2(u_1+u_3)}, (xy)^{-1} \mid ((xy)^{N/2})^{(F_{\nu(p)}-4)/2N}, z^{F_{\nu(q)}/2N}, ((xy)^{N/2}z)^{F_{\nu(t)}/2N})$
0.3.e	$(xz^{u_2+u_3}, (yx)^{-1+2(u_1+u_3)}, y \mid (y^{N/2})^{(F_{\nu(p)}-4)/2N-1/2}, z^{F_{\nu(q)}/2N}, (y^{N/2}z)^{F_{\nu(t)}/2N})$
0.3.e’	$(x(xy)^{N/2(u_2+u_3)}, y^{1+2(u_1+u_3)}, (xy)^{-1} \mid (y^2)^{(F_{\nu(p)})/2N-1/2}, ((xy)^{N/2})^{(F_{\nu(q)}-4)/2N}, (y^2(xy)^{N/2})^{F_{\nu(t)}/2N})$
0.3.f	$(x^{1+2(u_1+u_3)}, (yx)^{-1}z^{u_2+u_3}, y \mid (x^2)^{(F_{\nu(p)}-4)/2N-1}, z^{F_{\nu(q)}/2N}, (x^2z)^{F_{\nu(t)}/2N})$
0.3.f’	$(x^{1+2(u_1+u_3)}, y(xy)^{N/2(u_2+u_3)}, (xy)^{-1} \mid (x^2)^{F_{\nu(p)}/2N-1}, ((xy)^{N/2})^{(F_{\nu(q)}-4)/2N}, (x^2(xy)^{N/2})^{F_{\nu(t)}/2N})$
0.3.f”	$(x^{1+2(u_1+u_3)}, y^{1+2(u_2+u_3)}, (xy)^{-1} \mid (x^2)^{(F_{\nu(p)}-4)/2N-1/2}, (y^2)^{F_{\nu(q)}/2N-1/2}, (x^2y^2)^{F_{\nu(t)}/2N})$
0.4.a	$(xz^{u_1+u_3}w^{u_2+u_3}, y, (xy)^{-1} \mid z^{F_{\nu(p)}/24}, w^{F_{\nu(q)}/24}, (zw)^{F_{\nu(t)}/24})$
0.4.b	$(xz^{u_1+u_3+1}w^{u_2+u_3}, y, (xy)^{-1}z, z^{(F_{\nu(p)}-8)/24}, w^{F_{\nu(q)}/24}(zw)^{F_{\nu(t)}/24})$
0.4.c	$(xz^{u_1+u_3+1}w^{u_2+u_3+1}, yz, (xy)^{-1}w, z^{(F_{\nu(p)}-8)/24}, w^{(F_{\nu(q)}-8)/24}, (zw)^{F_{\nu(t)}/24})$
—	$(xz^{u_1+u_3}w^{u_2+u_3}, yz, (xy)^{-1}z \mid z^{(F_{\nu(p)}-16)/24}, w^{F_{\nu(q)}/24}, (zw)^{F_{\nu(t)}/24})$
4.d	$(x^{1+2(u_1+u_3)}z^{u_2+u_3}, y, (xy)^{-1} \mid (x^2)^{(F_{\nu(p)}-12)/24}, z^{F_{\nu(q)}/24}, (x^2z)^{F_{\nu(t)}/24})$
0.4.e	$(x^{3+2(u_1+u_3)}z^{u_2+u_3}, y, y^{-1}x \mid (x^2)^{(F_{\nu(p)}-20)/24}, z^{F_{\nu(q)}/24}, (x^2z)^{F_{\nu(3)}/24})$
—	$(x^{1+2(u_1+u_3)}z^{u_2+u_3+1}, y, (xy)^{-1}z \mid (x^2)^{(F_{\nu(p)}-12)/24}, z^{(F_{\nu(q)}-8)/24}, (x^2z)^{F_{\nu(t)}/24})$
0.4.f	$(x^{1+2(u_1+u_3)}z^{u_2+u_3}, x^2y, y^{-1}x \mid (x^2)^{(F_{\nu(p)}-28)/24}, z^{F_{\nu(q)}/24}, (x^2z)^{F_{\nu(t)}/24})$
—	$(x^{1+2(u_1+u_3)}z^{u_2+u_3}, yz, (xy)^{-1}z \mid (x^2)^{(F_{\nu(p)}-12)/24}, z^{(F_{\nu(q)}-16)/24}, (x^2z)^{F_{\nu(t)}/24})$
—	$(x^{3+2(u_1+u_3)}z^{u_2+u_3+1}, x^2y, (xy)^{-1}z \mid (x^2)^{(F_{\nu(p)}-20)/24}, z^{(F_{\nu(q)}-8)/24}, (x^2z)^{F_{\nu(t)}/24})$
—	$(x^{3+2(u_1+u_3)}z^{u_2+u_3}, zy, y^{-1}xz \mid (x^2)^{(F_{\nu(p)}-12)/24}, z^{(F_{\nu(q)}-8)/24}, (x^2z)^{(F_{\nu(t)}-8)/24})$
0.5.a	$(xz^{u_1+u_3}w^{u_2+u_3}, y, (xy)^{-1} \mid z^{F_{\nu(p)}/48}, w^{F_{\nu(q)}/48}, (zw)^{F_{\nu(t)}/48})$
0.5.b	$(xz^{u_1+u_3+1}, yz, (xy)^{-1}w^{u_2+u_3} \mid z^{(F_{\nu(p)}-16)/48}w^{F_{\nu(q)}/48}, (zw)^{F_{\nu(t)}/48})$
0.5.c	$(xz^{u_2+u_3}, y, (xy)^{-1+4(u_1+u_3)} \mid ((xy)^4)^{(F_{\nu(p)}-12)/48}, z^{F_{\nu(q)}/48}, ((xy)^4z)^{F_{\nu(t)}/48})$
0.5.d	$(xz^{u_2+u_3}, y(xy)^4, (xy)^{3+4(u_1+u_3)} \mid ((xy)^4)^{(F_{\nu(p)}-28)/48}, z^{F_{\nu(q)}/48}((xy)^4z)^{F_{\nu(t)}/48})$
—	$(xz^{u_2+u_3+1}, yz, (xy)^{-1+4(u_1+u_3)} \mid ((xy)^4)^{(F_{\nu(1)}-12)/48}, z^{(F_{\nu(2)}-16)/48}, ((xy)^4z)^{F_{\nu(3)}/48})$
0.5.e	$(x^{1+2(u_1+u_3)}, y, (xy)^{-1}z^{u_2+u_3} \mid (x^2)^{(F_{\nu(p)}-24)/48}, z^{F_{\nu(q)}/48}, (x^2z)^{F_{\nu(3)}/48})$

Table 3.14. Continued.

Case	Generating vector
0.5.f	$(x^{3+2(u_1+u_3)}, x^2y, (xy)^{-1}z^{u_2+u_3} \mid (x^2)^{(F_{\nu(p)}-40)/48}, z^{F_{\nu(q)}/48}, (x^2z)^{F_{\nu(t)}/48})$
[2pt]—	$(x^{1+2(u_1+u_3)}, yz, (xy)^{-1}z^{u_2+u_3+1} \mid (x^2)^{(F_{\nu(p)}-24)/48}, z^{(F_{\nu(q)}-16)/48}, (x^2z)^{F_{\nu(t)}/48})$
[2pt]5.g	$(x^{1+2(u_1+u_3)}, y, (xy)^{-1}z^{u_2+u_3} \mid (x^2)^{(F_{\nu(p)}-36)/48}, z^{F_{\nu(q)}/4}, (x^2z)^{F_{\nu(t)}/4})$
[2pt]0.5.g’	$(x^{1+2(u_1+u_3)}, y, (xy)^{-1+4(u_2+u_3)} \mid (x^2)^{(F_{\nu(p)}-24)/48}, ((xy)^4)^{(F_{\nu(q)}-12)/48}, (x^2(xy)^4)^{F_{\nu(t)}/48})$
[2pt]0.5.h	$(x^{3+2(u_1+u_3)}, x^2y, (xy)^{-1}z^{u_2+u_3} \mid (x^2)^{(F_{\nu(p)}-52)/48}, z^{F_{\nu(q)}/48}, (x^2z)^{F_{\nu(t)}/48})$
[2pt]—	$(x^{1+2(u_1+u_3)}, yz, (xy)^{-1}z^{u_2+u_3+1} \mid (x^2)^{(F_{\nu(p)}-36)/48}, z^{(F_{\nu(q)}-16)/48}, (x^2z)^{F_{\nu(t)}/48})$
[2pt]0.5.h’	$(x^{3+2(u_1+u_3)}, x^2y, (xy)^{-1+4(u_2+u_3)} \mid (x^2)^{(F_{\nu(p)}-40)/48}, ((xy)^4)^{(F_{\nu(q)}-12)/48}, (x^2(xy)^4)^{F_{\nu(t)}/48})$
—	$(x^{1+2(u_1+u_3)}, y(xy)^4, (xy)^{3+4(u_2+u_3)} \mid (x^2)^{(F_{\nu(p)}-24)/48}, ((xy)^4)^{(F_{\nu(q)}-28)/48}, (x^2(xy)^4)^{F_{\nu(3)}/48})$
0.6.a	$(xz^{u_1+u_3}w^{u_2+u_3}, y, (xy)^{-1} \mid z^{F_{\nu(p)}/120}, w^{F_{\nu(q)}/120}, (zw)^{F_{\nu(t)}/120})$
0.6.b	$(xz^{u_1+u_3+1}w^{u_2+u_3}, y, (xy)^{-1}z \mid z^{(F_{\nu(p)}-24)/120}, w^{F_{\nu(q)}/120}, (zw)^{F_{\nu(t)}/120})$
0.6.c	$(xz^{u_1+u_3}w^{u_2+u_3}, yz, (xy)^{-1}z \mid z^{(F_{\nu(1)}-64)/120}, w^{F_{\nu(2)}/120}, (zw)^{F_{\nu(3)}/120})$
—	$(xz^{u_1+u_3+1}w^{u_2+u_3+1}, yw, (xy)^{-1}z \mid z^{(F_{\nu(1)}-24)/120}, w^{(F_{\nu(2)}-40)/120}, (zw)^{F_{\nu(3)}/120})$
0.6.d	$(xz^{u_1+u_3+1}w^{u_2+u_3}, yz, (xy)^{-1} \mid z^{(F_{\nu(p)}-40)/120}, w^{F_{\nu(q)}/120}, (zw)^{F_{\nu(t)}/120})$
0.6.e	$(x^{1+2(u_1+u_3)}z^{u_2+u_3}, y, (xy)^{-1} \mid (x^2)^{(F_{\nu(p)}-60)/120}, z^{F_{\nu(q)}/120}, (x^z)^{F_{\nu(t)}/120})$
0.6.f	$(x^{3+2(u_1+u_3)}z^{u_2+u_3}, y, y^{-1}x \mid (x^2)^{(F_{\nu(p)}-84)/120}, z^{F_{\nu(q)}/120}, (x^2z)^{F_{\nu(t)}/120})$
—	$(x^{1+2(u_1+u_3)}z^{u_2+u_3+1}, y, (xy)^{-1}z \mid (x^2)^{(F_{\nu(p)}-60)/120}, z^{(F_{\nu(q)}-24)/120}, (x^2z)^{F_{\nu(t)}/120})$
0.6.g	$(x^{3+2(u_1+u_3)}z^{u_2+u_3+1}, yx^2, (xy)^{-1} \mid (x^2)^{(F_{\nu(p)}-100)/120}, z^{F_{\nu(q)}/120}, (x^2z)^{F_{\nu(t)}/120})$
—	$(x^{1+2(u_1+u_3)}z^{u_2+u_3+1}, yz, (xy)^{-1} \mid (x^2)^{(F_{\nu(p)}-60)/120}, z^{(F_{\nu(q)}-40)/120}, (x^2z)^{F_{\nu(t)}/120})$
0.6.h	$(x^{1+2(u_1+u_3)}z^{u_2+u_3}, x^2y, y^{-1}x \mid (x^2)^{(F_{\nu(p)}-124)/120}, z^{F_{\nu(q)}/120}, (x^2z)^{F_{\nu(t)}/120})$
—	$(x^{3+2(u_1+u_3)}z^{u_2+u_3+1}, x^2y, (xy)^{-1}z \mid (x^2)^{(F_{\nu(p)}-100)/120}, z^{(F_{\nu(q)}-24)/120}, (x^2z)^{F_{\nu(3)}/120})$
—	$(x^{3+2(u_1+u_3)}z^{u_2+u_3+1}, zy, y^{-1}x \mid (x^2)^{(F_{\nu(p)}-84)/120}, z^{(F_{\nu(q)}-40)/120}, (x^2z)^{F_{\nu(t)}/120})$

4. Full actions on pq-hyperelliptic Riemann surfaces

Theorem 4.1. *For any integers $2 \le p < q < 2p$, let X be a pq-hyperelliptic Riemann surface of genus $g > 3q+1$ in range $2p+2q-10 \le g \le 2p+2q+1$ except $X_1^{p,q}$ of genus $2p+2q-3$ and $X_2^{p,q}$ of genus $2p+2q-7$. A group G of order $4N$ is the full automorphism group of X if and only if G appears in one of Tables 3.3, 3.7, 3.8–3.11, 3.13 in such a way that the corresponding triple* (case;u;g) *is different from* (0.2.c;1;$N-1$), (0.2.c;2;$2N-1$), (0.2.c’;1;$N-1$), (0.3.f’;0;$N-1$), (1.5.d.4;1;5), (1.5.d.5;1;5).

Proof. Let $G = \Lambda/\Gamma$ be a group of automorphisms of a surface $X = \mathscr{H}/\Gamma$ determined in theorem. If the signature τ of Λ does not appear in the first column of [11, Tables 1.5.1 or 1.5.2], then Λ can be chosen to be a maximal [11] and so G can be assumed to be the full group of automorphisms of X. In the other case, Λ is always contained in an NEC group Λ' and signatures τ' of such groups are given in the second column of the corresponding row. However, most τ' gives rise to the action on a surface which does not satisfy the conditions of theorem or its genus g' is distinct to g. Consequently, we need only to examine the cases when Λ has one of the signatures: $[2,2N,2N]$, $[2,2,2N,2N]$, $[4,4,N]$, or $(1;2)$. If for every epimorphism $\theta : \Lambda \to G$ whose kernel Γ has signature $(g;-)$ and $\mathcal{H}/\Gamma$ is pq-hyperelliptic, there exists a Fuchsian group Λ', a group G', group embeddings $i : \Lambda \hookrightarrow \Lambda'$, $j : G \hookrightarrow G'$, and an epimorphism $\theta' : \Lambda' \to G'$ such that $1 \neq [\Lambda' : \Lambda] = [G : G']$ and $\theta' \cdot i = j \cdot \theta$, then $G = \Lambda/\Gamma \subsetneq \Lambda'/\Gamma = G' \subseteq \text{Aut}(X)$ for all pq-hyperelliptic surfaces X of genus g on which G acts as a group of automorphisms.

First assume that $\tau = [2,2N,2N]$ and $\tau' = [2,4,2N]$. Let Λ' be a Fuchsian group with the signature τ' containing Λ and let y_1, y_2, y_3 be its canonical generators. Clearly $x_1 = y_2^2$, $x_2 = y_3$, and $x_3 = y_1y_3y_1$ belong to Λ and have orders 2, $2N$, and $2N$, respectively. Moreover, it is easy to see that $x_1x_2x_3 = 1$ and x_1, x_2, x_3 generate a normal subgroup of Λ' of index 2. So x_1, x_2, x_3 form a system of canonical generators of Λ and consequently, the embedding $i : \Lambda \hookrightarrow \Lambda'$ is induced by the assignment: $i(x_1) = y_2^2$, $i(x_2) = y_3$, $i(x_3) = y_1y_3y_1$. By theorems of the previous section, G has the presentation 0.2.c or 0.2.c' with $u = 1$. Thus $G \cong Z_{2N} \oplus Z_2 = \langle x\rangle \oplus \langle z\rangle$ and the generating vector of G has the form $v = (z, zx, x^{-1})$ or $v = (x^N z, x, x^{N-1}z)$, respectively. The case 0.3.e' with $u = 0$ determines the presentation of G'. So $G' \cong \langle a,b : a^2, b^4, (ab)^{2N}, (b^2a)^2, ((ab)^N a)^2\rangle$ and the generating vector of G' has the form $v' = (a, b, (ab)^{-1})$. It is easy to check that the assignment $j(x) = ba$, $j(z) = b^2$ in the case 0.2.c or $j(x) = (ab)^{-1}$, $j(z) = (ab)^N b^2$ in the case 0.2.c' gives a group monomorphisms $j : G \to G'$ consistent with an epimorphisms $\theta : \Lambda \to G$ and $\theta' : \Lambda' \to G'$ representing by the generating vectors v and v'.

Next assume that $\tau = [2,2,2N,2N]$. Then G has the presentation 0.2.c or 0.2.c' with $u = 2$. So $G \cong Z_{2N} \oplus Z_2 = \langle x\rangle \oplus \langle z\rangle$ and $v = (z,z,x,x^{-1})$ or $v = (z, x^N, x, x^{N-1}z)$, respectively. The signature $\tau' = [2,2,2,2N]$ appears in case 0.3.d with $u = 1$. Thus $G' \cong \langle a,b : a^2, b^2, (ab)^{2N}\rangle \oplus \langle c : c^2\rangle$ and $v' = (c, ca, b, (ab)^{-1})$. In the case 0.2.c, we can define i and j consistent with θ and θ' by the assignments $i(x_1) = y_1$, $i(x_2) = y_2y_1y_2$, $i(x_3) = y_3y_4y_3$, $i(x_4) = y_4$, and $j(x) = ab$, $j(z) = c$, respectively. In the case 0.2.c', the required i, j do not exist since otherwise $i(x_3)$ and $i(x_4)$ would be conjugated with y_4 and by the equations $j\theta(x_i) = \theta' i(x_i)$, both images $j(x)$ and $j(z)$ would belong to the subgroup of G' generated by ab and consequently, $j(G)$ would have the order not exceeding $2N$, a contradiction.

If $\tau = [4,4,N]$ and $\tau' = [2,4,2N]$, then G and G' have the presentations 0.3.f' and 0.3.e', respectively, with $u = 0$. So $G \cong \langle x,y : x^4, x^2y^2, (xy)^N\rangle$, $v = (x, y, (xy)^{-1})$ while $G' \cong \langle a,b : a^2, b^4, (ab)^{2N}, (b^2a)^2, ((ab)^N a)^2\rangle$ and $v' = (a, b, (ab)^{-1})$. We can define the embeddings i and j by the assignments $i(x_1) = y_2$, $i(x_2) = y_1y_2y_1$, $i(x_3) = (y_2y_1)^{-2}$ and $j(x) = b$, $j(y) = aba$, respectively.

Finally, if $\tau = (1;2)$ and $\tau' = [2,2,2,4]$, then G has the presentation 1.5.d.4 or 1.5.d.5 with $u = 1$ and it is isomorphic to H_4 or H_5 while G' has the presentation 1.4.a.4 or 1.4.a.5 with $u = 0$ and so it is isomorphic to $H_4 \ltimes Z_2$ or to $H_5 \ltimes Z_2$. Now i and j are defined by $i(x_1) = y_4^2$, $i(a) = y_1y_2$, $i(b) = y_3y_1$ and by $j(x) = x$, $j(y) = y$, respectively. □

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