Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 50591, 8 pages
doi:10.1155/2007/50591

Research Article

Generalizations of Morphic Group Rings

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Received 9 October 2006; Revised 20 December 2006; Accepted 5 February 2007

Recommended by Howard E. Bell

An element a in a ring R is called left morphic if there exists $b \in R$ such that $\mathbf{l}_R(a) = Rb$ and $\mathbf{l}_R(b) = Ra$. R is called left morphic if every element of R is left morphic. An element a in a ring R is called left π-morphic (resp., left G-morphic) if there exists a positive integer n such that a^n (resp., a^n with $a^n \neq 0$) is left morphic. R is called left π-morphic (resp., left G-morphic) if every element of R is left π-morphic (resp., left G-morphic). In this paper, the G-morphic problem and π-morphic problem of group rings are studied.

1. Introduction

An element a in a ring R is said to be left morphic if $R/Ra \cong \mathbf{l}_R(a)$, which is equivalent to that there exists $b \in R$ such that $\mathbf{l}_R(a) = Rb$ and $\mathbf{l}_R(b) = Ra$, where $\mathbf{l}_R(a)$ denotes the left annihilator of a in R. R is called left morphic if every element of R is left morphic. Right morphic elements and rings are defined analogously. Nicholson and Sánchez Campos introduced and investigated left morphic rings in [1] (see also [2–4] for more detailed discussion).

Left morphic rings are generalized to left π-morphic rings and left G-morphic rings by Huang and Chen [5]. An element $a \in R$ is called left π-morphic (resp., left G-morphic) if there exists a positive integer n such that a^n (resp., a^n with $a^n \neq 0$) is left morphic. R is called left π-morphic (resp., left G-morphic) if every element of R is left π-morphic (resp., left G-morphic). R is called π-morphic (resp., G-morphic) if it is left and right π-morphic (resp., left and right G-morphic). Moreover, they find examples which show that left π-morphic rings are proper generalizations of left morphic rings, and left G-morphic elements need not be left morphic.

Example 1.1 [5, Example 2.13]. Let $R = F[x,\sigma]/(x^2) = \{a + xb \mid a,b \in F\}$, where F is a field with an isomorphism σ from F to a subfield $\overline{F} \neq F$ and $cx = x\sigma(c)$ for all $c \in F$.

$S = R \oplus R$, then $\lambda = (1, xb) \in S$ (where $b \in F$, but $b \notin \overline{F}$) is left G-morphic, but not left morphic.

The question of when a group ring is morphic was studied by Chen et al. [6]. In this paper, we investigate when a group ring is π-morphic (resp., G-morphic). In Section 2, several general results about π-morphic and G-morphic group rings are obtained. In Section 3, necessary and sufficient conditions for RG to be left G-morphic are also given, where $R = \mathbb{Z}_n$, G is a finite Abelian group. In particular, we prove that if G is a finite Abelian group or a finite p-group, $r \geqslant 1$, then $\mathbb{Z}_{p^r}G$ is π-morphic.

All rings in this paper are associative rings with identity. Let R be a ring and let G be a group. We denote by RG the group ring of G over R. The following concepts in group rings play very important roles in our discussion and will be used frequently later. For any element $u = \Sigma a_i g_i \in RG$, where $a_i \in R$, $g_i \in G$, the augmentation of u, denoted by $\epsilon(u)$, is defined by $\epsilon(u) = \Sigma a_i$. The augmentation ideal of RG, denoted by $\Delta(G)$, is defined by $\Delta(G) = \{u \in RG \mid \epsilon(u) = 0\}$. If G is a cyclic group generated by g, then $\Delta(G) = RG(1-g)$. For any finite subgroup H of G, $\widehat{H}$ is defined to be $\widehat{H} = \sum_{\forall h \in H} h$. When H is a normal subgroup, $\widehat{H}$ is a central element in RG. For any group element $g \in G$ of finite order, define $\widehat{g}$ by $\widehat{g} = 1 + g + \cdots + g^{o(g)-1}$, where $o(g)$ is the order of g. It is not hard to verify that if $o(g) < \infty$, then $\mathbf{l}_{RG}(1-g) = RG\widehat{g}$, and if $|G| < \infty$, then $\mathbf{l}_{RG}(\widehat{G}) = \Delta(G)$. So if G is a finite cyclic group, then $\widehat{G}$ is always left morphic in RG. For more background knowledge about group rings, we refer readers to [7, 8].

2. General results

In this section, several general results about π-morphic and G-morphic group rings are given.

Theorem 2.1. *Let R be a ring and let G be a locally finite group. If RG is left π-morphic (resp., left G-morphic), then R is left π-morphic (resp., left G-morphic).*

Proof. For any $a \in R$, since a is left π-morphic (resp., left G-morphic) in RG, there exist a positive integer n (resp., $a^n \neq 0$) and $u \in RG$ such that $\mathbf{l}_{RG}(a^n) = RGu$ and $\mathbf{l}_{RG}(u) = RGa^n$. Let $u = \sum_{i=1}^{n} a_i g_i$ and $H = \langle g_1, \ldots, g_n \rangle$. Since G is a locally finite group, H is a finite group. Since $a^n u = u a^n = 0$, we have $a^n \epsilon(u) = \epsilon(a^n u) = 0$ and $\epsilon(u) a^n = \epsilon(u a^n) = 0$, where $\epsilon(u)$ is the augmentation of u. Thus $Rb \subseteq \mathbf{l}_R(a^n)$ and $Ra^n \subseteq \mathbf{l}_R(b)$, where $b = \epsilon(u)$. Next we show that in fact, $Rb = \mathbf{l}_R(a^n)$ and $Ra^n = \mathbf{l}_R(b)$. So a is left π-morphic (resp., left G-morphic) in R, and thus R is left π-morphic (resp., left G-morphic).

Let $x \in \mathbf{l}_R(a^n)$. Then $x \in \mathbf{l}_{RG}(a^n) = RGu$, so $x = vu$, $v \in RG$. Taking the augmentation on both sides, we obtain $x = \epsilon(x) = \epsilon(vu) = \epsilon(v)\epsilon(u) = \epsilon(v)b \in Rb$. Therefore, $\mathbf{l}_R(a^n) \subseteq Rb$, and thus $\mathbf{l}_R(a^n) = Rb$. Next, let $y \in \mathbf{l}_R(b)$. Then $yb = 0$. Let $\widehat{H} = \sum_{h \in H} h$. Since $u \in RH$, we have $\widehat{H}u = \epsilon(u)\widehat{H} = b\widehat{H}$. Thus $y\widehat{H}u = yb\widehat{H} = 0$, so $y\widehat{H} \in \mathbf{l}_{RG}(u) = RGa^n$. Hence $y\widehat{H} = \Sigma a_g g a^n$. Comparing the coefficients of the identity on both sides, we obtain that $y = a_e a^n \in Ra^n$, and so $\mathbf{l}_R(b) \subseteq Ra^n$. This implies that $\mathbf{l}_R(b) = Ra^n$. Therefore, a is left π-morphic (resp., left G-morphic) and so is R. □

Corollary 2.2. *If $G = H \times K$ is a locally finite group and RG is left π-morphic (resp., left G-morphic), then RH and RK are both left π-morphic (resp., left G-morphic).*

Proof. Note that $RG = R(H \times K) \cong (RH)K$. By Theorem 2.1, RH is left π-morphic (resp., left G-morphic). Similarly RK is left π-morphic (resp., left G-morphic). □

THEOREM 2.3. *Let G be a locally finite group. If RH is left π-morphic (resp., left G-morphic) for every finite subgroup H of G, then RG is left π-morphic (resp., left G-morphic).*

Proof. Let $u = \sum_{i=1}^{n} a_i g_i$. Now we show that u is left π-morphic (resp., left G-morphic) in RG. Denote $H = \langle g_1, \dots, g_n \rangle$. Since G is locally finite, H is a finite group. By the assumption, RH is left π-morphic (resp., left G-morphic). Since $u \in RH$, there exist a positive integer n (resp., $u^n \neq 0$) and $c \in RH$ such that $\mathbf{l}_{RH}(u^n) = RHc$ and $\mathbf{l}_{RH}(c) = RHu^n$. Since $u^n c = cu^n = 0$, we have $RGc \subseteq \mathbf{l}_{RG}(u^n)$ and $RGu^n \subseteq \mathbf{l}_{RG}(c)$. We next show that the other inclusions also hold.

Let $v \in \mathbf{l}_{RG}(u^n)$ and let $\{1, g_1', g_2', \dots\}$ be a left coset representative of H in G. That is, $G = H \cup g_1'H \cup g_2'H \cup \cdots$. Now v can be written as $v = \sum g_i' b_i$, where $b_i \in RH$. Since $0 = vu^n = \sum g_i'(b_i u^n)$ and $b_i u^n \in RH$, we obtain that $b_i u^n = 0$ for all i. So $b_i \in \mathbf{l}_{RH}(u^n) = RHc$, and thus $b_i = c_i c$ for some $c_i \in RH$. It follows that $v = \sum g_i' b_i = \sum (g_i' c_i) c \in RGc$, so $\mathbf{l}_{RG}(u^n) \subseteq RGc$, and thus $\mathbf{l}_{RG}(u^n) = RGc$. Similarly, we can prove that $\mathbf{l}_{RG}(c) = RGu^n$. This shows that u is left π-morphic (resp., left G-morphic) in RG, and therefore RG is left π-morphic (resp., left G-morphic). □

Recall that a group G is called a semidirect product of H by K, denoted by $G = H \rtimes K$, if H, K are subgroups of G such that (1) $H \trianglelefteq G$; (2) $HK = G$; (3) $H \cap K = 1$.

THEOREM 2.4. *Let $G = H \rtimes K$, $|H| < \infty$. If RG is left π-morphic (resp., left G-morphic), then RK is also left π-morphic (resp., left G-morphic).*

Proof. We show that for any $a \in RK$, a is left π-morphic (resp., left G-morphic) in RK. Since a is left π-morphic (resp., left G-morphic) in RG, there exist a positive integer n (resp., $a^n \neq 0$) and $u \in RG$ such that $\mathbf{l}_{RG}(a^n) = RGu$ and $\mathbf{l}_{RG}(u) = RGa^n$. Let $u = \sum u_i k_i$, where $u_i \in RH$, $k_i \in K$ (since $G = H \rtimes K$, the expression of u is unique) and $a^n = \sum a_j k_j$ where $a_j \in R$. Denote $b = \sum \epsilon(u_i) k_i$, so $b \in RK$. We will show that $\mathbf{l}_{RK}(a^n) = RKb$ and $\mathbf{l}_{RK}(b) = RKa^n$. So a is left π-morphic (resp., left G-morphic) in RK, and thus RK is left π-morphic (resp., left G-morphic).

Let $\omega : G \to G/H$ be the natural group homomorphism. We extend ω to a ring homomorphism (still denote it by ω). That is, $\omega : RG \to R(G/H)$ defined by $\omega(\sum a_i g_i) = \sum a_i \omega(g_i)$. Clearly, $\ker(\omega) \cap RK = \{0\}$ and $\omega(v) = \epsilon(v)$ for all $v \in RH$. Since $0 = a^n u$, we have $0 = \omega(a^n)\omega(u) = \omega(a^n)\omega(\sum u_i k_i) = \omega(a^n) \sum \epsilon(u_i)\omega(k_i) = \omega(a^n \sum \epsilon(u_i) k_i) = \omega(a^n b)$. Since $a^n b \in RK$, we conclude that $a^n b = 0$. Similarly, $ba^n = 0$. This shows that $RKb \subseteq \mathbf{l}_{RK}(a^n)$ and $RKa^n \subseteq \mathbf{l}_{RK}(b)$. We next show that the other inclusions also hold.

Let $x \in \mathbf{l}_{RK}(a^n)$. Then $x \in \mathbf{l}_{RG}(a^n) = RGu$. So $x = vu$. Let $v = \sum v_j k_j$ and $c = \sum \epsilon(v_j) k_j$, where $v_j \in RH$, $k_j \in K$. Then $\omega(x) = \omega(v)\omega(u) = \sum \epsilon(v_j)\omega(k_j) \sum \epsilon(u_i)\omega(k_i) = \omega(cb)$. Thus $x - cb \in \ker\omega \cap RK = \{0\}$. Therefore $x = cb \in RKb$. This shows that $\mathbf{l}_{RK}(a^n) \subseteq RKb$, and thus $\mathbf{l}_{RK}(a^n) = RKb$.

Let $y \in \mathbf{l}_{RK}(b)$. Then $yb = 0$. Since $H \trianglelefteq G$, $\widehat{H} = \sum_{h \in H} h$ is central in RG. Now we have $y\widehat{H}u = y\widehat{H} \sum u_i k_i = y \sum \widehat{H}\epsilon(u_i) k_i = y\widehat{H}b = yb\widehat{H} = 0$. So $y\widehat{H} \in \mathbf{l}_{RG}(u) = RGa^n$. Thus $\widehat{H}y = y\widehat{H} = wa^n$, where $w = \sum h_j u_j$, $h_j \in H$, $u_j \in RK$. Hence

$$\sum h_j y = \hat{H} y = wa^n = \sum h_j (u_j a^n). \tag{2.1}$$

Since $H \cap K = \{1\}$, the expression of wa^n is unique. Comparing the coefficients of the identity $h_0 = e$ in (2.1), we obtain $y = u_0 a^n \in RKa^n$. Thus $\mathbf{l}_{RK}(b) \subseteq RKa^n$, and therefore $\mathbf{l}_{RK}(b) = RKa^n$. □

From now on, we always assume that G is a finite group.

Proposition 2.5. *Assume that p is a prime number and $r > 1$. If $\mathbb{Z}_{p^r}G$ is left G-morphic, then p does not divide $|G|$.*

Proof. Assume that $p \mid |G|$. Then there exists $g \in G$ such that $o(g) = p$. Let $u = p^{r-1}\hat{G}$, where $\hat{G} = \sum_{g \in G} g$. Since u is left G-morphic in $\mathbb{Z}_{p^r}G$, there exists a positive integer n such that u^n is left morphic in $\mathbb{Z}_{p^r}G$. Since $u^2 = 0$, u is left morphic in $\mathbb{Z}_{p^r}G$. By Chen et al. [6, Theorem 2.7], this is impossible. So $p \nmid |G|$. □

Theorem 2.6. *Assume that p is a prime number and G is a finite p-group. $\mathbb{Z}_{p^r}G$ is left G-morphic if and only if G is a cyclic group and $r = 1$.*

Proof. "⇒" It follows from Proposition 2.5 that $r = 1$. Since $R = \mathbb{Z}_p$ is a field and G is a finite p-group, RG is a local ring by Nicholson theorem [9]. Because RG is left Artinian, the Jacobson radical $J(RG)$ is nilpotent. Since RG is left G-morphic, RG is left special by Huang and Chen [5, Theorem 2.8]. So it is left morphic. According to Chen et al. [6, Theorem 2.9], G is a cyclic group.

"⇐" If $G = \langle g \rangle$, clearly $\mathbb{Z}_p G$ is a special ring. Therefore it is left G-morphic. □

Theorem 2.7. *Assume that p is a prime number and G is a finite p-group, $r \geqslant 1$, then $\mathbb{Z}_{p^r}G$ is π-morphic.*

Proof. Since $R = \mathbb{Z}_{p^r}$ is local and G is a finite p-group, RG is a local ring by Nicholson'S theorem [9]. Because R is Artinian and G is a finite group, RG is Artinian by Connell [10, Theorem 1], and so the Jacobson radical $J(RG)$ is nilpotent. According to Huang and Chen [5, Lemma 2.10], every element of RG is either nilpotent or invertible. So RG is π-morphic. □

Remark 2.8. By Theorem 2.6, when $r > 1$ and G is a finite p-group, $\mathbb{Z}_{p^r}G$ is not left G-morphic, but by the above theorem, it is π-morphic.

3. Abelian group rings

In this section, we discuss when an Abelian group ring RG is left π-morphic (resp., left G-morphic).

Lemma 3.1 [6, Lemma 3.1]. $(R_1 \oplus R_2 \oplus \cdots \oplus R_s)G \cong \oplus_{i=1}^{s} R_i G$.

Lemma 3.2. *If $R = R_1 \oplus R_2 \oplus \cdots \oplus R_s$ is left π-morphic (resp., left G-morphic), then each R_i is left π-morphic (resp., left G-morphic).*

Proof. For any $r_i \in R_i$, $r = (0,\dots,0,r_i,0,\dots,0) \in R$. Since R is left π-morphic (resp., left G-morphic), there exist $u = (u_1,\dots,u_{i-1},u_i,\dots,u_s) \in R$, where $u_k \in R_k$, $k = 1,\dots,s$, and a positive integer n (resp., $r^n \neq 0$) such that $\mathbf{l}_R(u) = Rr^n$ and $\mathbf{l}_R(r^n) = Ru$, so we have $\mathbf{l}_{R_i}(u_i) = R_i r_i^n$ and $\mathbf{l}_{R_i}(r_i^n) = R_i u_i$. Then r_i is left π-morphic (resp., left G-morphic) in R_i, and thus R_i is left π-morphic (resp., left G-morphic). $\square$

Lemma 3.3. *Let D be a division ring and $s \geqslant 2$. The the following statements are equivalent:*

(1) *$D(C_{m_1} \times \cdots \times C_{m_s})$ is left G-morphic;*

(2) *$D(C_{m_i} \times C_{m_j})$ is left G-morphic for any $1 \leqslant i \neq j \leqslant s$;*

(3) *at most one of $m_1, m_2, \ldots, m_s$ is not invertible in D.*

Proof. We will prove (3)⇒(1)⇒(2)⇒(3).

"(3)⇒(1)" We may assume that $m_1, \ldots, m_{s-1}$ are invertible in D. So $|C_{m_1} \times \cdots \times C_{m_{s-1}}| = m_1 \times \cdots \times m_{s-1}$ is invertible in D. By Maschke's theorem, $D(C_{m_1} \times \cdots \times C_{m_{s-1}})$ is semisimple. It follows from [6, Lemma 3.5] that $D(C_{m_1} \times \cdots \times C_{m_{s-1}} \times C_{m_s})$ is strongly morphic, so it is G-morphic and (2.1) holds.

"(1)⇒(2)" Note that $D(C_{m_1} \times \cdots \times C_{m_s}) \cong D(C_{m_i} \times C_{m_j})(\prod_{k \neq i,j} C_{m_k})$ for any $1 \leqslant i \neq j \leqslant s$. It follows from Theorem 2.1 that $D(C_{m_i} \times C_{m_j})$ is left G-morphic.

"(2)⇒(3)" We prove it by contradiction. We may assume that m_1, m_2 are not invertible in D. Let $\mathrm{char}(D) = p > 0$. By assumption, p divides both m_1 and m_2. So we have $m_i = p^{r_i} t_i$, where $(t_i, p) = 1$, $r_i \geqslant 1$, $i = 1, 2$.

Note that $C_{m_1} \times C_{m_2} \cong (C_{p^{r_1}} \times C_{p^{r_2}}) \times (C_{t_1} \times C_{t_2})$, so $D(C_{m_1} \times C_{m_2}) \cong D(C_{p^{r_1}} \times C_{p^{r_2}}) \times (C_{t_1} \times C_{t_2})$. Since $D(C_{m_1} \times C_{m_2})$ is left G-morphic, $D(C_{p^{r_1}} \times C_{p^{r_2}})$ is left G-morphic by Theorem 2.1. Because $C_{p^{r_1}} \times C_{p^{r_2}}$ is a finite p-group, $D(C_{p^{r_1}} \times C_{p^{r_2}})$ is a local Artinian ring, so the Jacobson radical of this group ring is nilpotent. This ring is a left special ring, and then it is left morphic by Huang and chen [5, Theorem 2.8]. Thus $C_{p^{r_1}} \times C_{p^{r_2}}$ must be cyclic, a contradiction. $\square$

Proposition 3.4. *Let G be a finite Abelian group and $r > 1$. Then $\mathbb{Z}_{p^r} G$ is G-morphic if and only if $(p, |G|) = 1$.*

Proof. "⇐" By Chen et al. [6, Corollary 3.13], if $(p, |G|) = 1$, $\mathbb{Z}_{p^r} G$ is morphic, so it is G-morphic.

"⇒" By Proposition 2.5, if $r > 1$ and $\mathbb{Z}_{p^r} G$ is G-morphic, then $p \nmid |G|$, that is, $(p, |G|) = 1$. $\square$

Theorem 3.5. *Let G be a finite Abelian group. $\mathbb{Z}_n G$ is G-morphic if and only if for each prime number p if $p \mid (n, |G|)$, then $p^2 \nmid n$ and the Sylow p-subgroup G_p of G is cyclic.*

Proof. Let $G = C_{q_1^{t_1}} \times \cdots \times C_{q_m^{t_m}}$, $t_i \geqslant 1$ be a finite Abelian group and let $\alpha = q_1 \cdots q_m$. Suppose that $\mathbb{Z}_n G$ is G-morphic. Let $(n, |G|) = p_1^{r_1} \cdots p_s^{r_s}$. If $r_i > 1$ for some i (i.e., $p_i^2 \mid n$), then $n = p_i^{s_i} n_1$, where $s_i \geqslant r_i > 1$ and $(n_1, p_i) = 1$. Thus $\mathbb{Z}_n G \cong \mathbb{Z}_{p_i^{s_i}} G \oplus \mathbb{Z}_{n_1} G$. Since $\mathbb{Z}_n G$ is G-morphic, $\mathbb{Z}_{p_i^{s_i}} G$ is also G-morphic by Lemma 3.2. By Proposition 3.4, $(p_i, |G|) = 1$. However, $p_i \mid (n, |G|)$. This leads to a contradiction. Thus $r_i \leqslant 1$ for all i. Next we show that $p_i^2 \nmid \alpha$. Otherwise, assume that $p_i^2 \mid \alpha$. There exists $k \neq l$ such that $q_k = q_l = p_i$. Hence $G \cong C_{q_k^{t_k}} \times C_{q_l^{t_l}} \times H$. Since $p_i \mid n$ and $p_i^2 \nmid n$, we have $n = p_i n_1$ with $(p_i, n_1) = 1$. So $\mathbb{Z}_n G \cong$

$\mathbb{Z}_{p_i}G \oplus \mathbb{Z}_{n_1}G$. By Lemma 3.2, $\mathbb{Z}_{p_i}G$ is G-morphic. Since $\mathbb{Z}_{p_i}G \cong \mathbb{Z}_{p_i}(C_{q_k^{t_k}} \times C_{q_l^{t_l}})H$, we conclude that $\mathbb{Z}_{p_i}(C_{q_k^{t_k}} \times C_{q_l^{t_l}}) = \mathbb{Z}_{p_i}(C_{p_i^{t_k}} \times C_{p_i^{t_l}})$ is G-morphic. This contradicts the result of Theorem 2.6. Therefore, $p_i^2 \nmid \alpha$, and thus G_{p_i} is cyclic. □

Remark 3.6. According to Proposition 3.4 and Theorem 3.5, the following group rings are not G-morphic:

$$\mathbb{Z}_4C_2, \qquad \mathbb{Z}_4C_4, \qquad \mathbb{Z}_4(C_2 \times C_2), \qquad \mathbb{Z}_2(C_2 \times C_2), \qquad \mathbb{Z}_2(C_2 \times C_4). \tag{3.1}$$

But by Theorem 2.7, the above group rings are all π-morphic.

Lemma 3.7. *Let R be a ring and let G be a group. If $a \in R$ is left morphic in R, then a is left morphic in RG.*

Proof. If $a \in R$ is left morphic, there exists $b \in R$ such that $\mathbf{l}_R(a) = Rb$ and $\mathbf{l}_R(b) = Ra$. Since $ba = ab = 0$, we have $RGb \subseteq \mathbf{l}_{RG}(a)$ and $RGa \subseteq \mathbf{l}_{RG}(b)$. We next show that the other inclusions also hold.

Let $x \in \mathbf{l}_{RG}(a)$, $x = \sum r_j g_j$, where $r_j \in R$, $g_j \in G$. Then $\sum r_j g_j a = 0$ or $\sum (r_j a) g_j = 0$, so all $r_j a = 0$. Thus $r_j \in Rb$ and $r_j = r_j' b$, $r_j' \in R$. Therefore, $x = \sum (r_j' b) g_j = \sum r_j' g_j b \in RGb$. This shows that $\mathbf{l}_{RG}(a) \subseteq RGb$, and thus $\mathbf{l}_{RG}(a) = RGb$.

Using a similar proof, we can show that $\mathbf{l}_{RG}(b) \subseteq RGa$, and thus $\mathbf{l}_{RG}(b) = RGa$. So a is left morphic in RG. □

Recall that if $n = p^u n_1$, $(n_1, p) = 1$, we denote that $p^u \| n$.

Lemma 3.8. *Let p be a prime number, $r \geqslant 1$, $p^r \| m$, and $1 \leqslant n \leqslant m$.*
(1) *If $(p, n) = 1$, then $p^r \mid C_m^n$.*
(2) *If $p^t \| n$, $r \geqslant t$, then $p^{r-t} \mid C_m^n$.*

Proof. Let $m = m_1 p^r$, $(m_1, p) = 1$. Then

$$C_m^n = \frac{m(m-1)\cdots(m-(n-1))}{1\cdots(n-1)n} = \frac{m}{n} C_{m-1}^{n-1} = \frac{m_1 p^r}{n} C_{m-1}^{n-1}. \tag{3.2}$$

(1) If $(p, n) = 1$, then $(p^r, n) = 1$, so $p^r \mid C_m^n$.
(2) If $p^t \| n$, $t \leqslant r$, then $n = n_1 p^t$, where $(p, n_1) = 1$, so

$$C_m^n = \frac{m_1 p^r}{n} C_{m-1}^{n-1} = \frac{m_1 p^r}{n_1 p^t} C_{m-1}^{n-1} = \frac{m_1 p^{r-t}}{n_1} C_{m-1}^{n-1}. \tag{3.3}$$

We have $p^{r-t} \mid C_m^n n_1$. Since $(p, n_1) = 1$, $(p^{r-t}, n_1) = 1$, so $p^{r-t} \mid C_m^n$. □

Proposition 3.9. *Let p be a prime number and let G be a finite Abelian group. If for some $r, t \geqslant 1$, $x \in \mathbb{Z}_{p^r}(C_{p^t} \times G) = \mathbb{Z}_{p^r}(\langle g \rangle \times G)$, then $x^{p^r} \in \mathbb{Z}_{p^r}(C_{p^{t-1}} \times G) = \mathbb{Z}_{p^r}(\langle g^p \rangle \times G)$.*

Proof. For $x \in \mathbb{Z}_{p^r}(C_{p^t} \times G) = (\mathbb{Z}_{p^r}G)C_{p^t} = (\mathbb{Z}_{p^r}G)\langle g \rangle$, $x = r_0 + r_1 g + \cdots + r_{p^t-1}g^{p^t-1}$, where $r_i \in \mathbb{Z}_{p^r}G$. Since

$$\begin{aligned}
&(x_1 + x_2 + \cdots + x_s)^k \\
&\quad = \sum_{k_1=0}^{k} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{s-1}=0}^{k_{s-2}} C_k^{k_1} C_{k_1}^{k_2} \cdots C_{k_{s-2}}^{k_{s-1}} x_1^{k-k_1} x_2^{k_1-k_2} \cdots x_{s-1}^{k_{s-2}-k_{s-1}} x_s^{k_{s-1}}, \\
&x^{p^r} = (r_0 + r_1 g + \cdots + r_{p^t-1} g^{p^t-1})^{p^r} \\
&\quad = \sum_{n_1=0}^{p^r} \sum_{n_2=0}^{n_1} \cdots \sum_{n_{p^t-1}=0}^{n_{p^t-2}} C_{p^r}^{n_1} C_{n_1}^{n_2} \cdots C_{n_{p^t-2}}^{n_{p^t-1}} r_0^{p^r-n_1} (r_1 g)^{n_1-n_2} \cdots (r_{p^t-1} g^{p^t-1})^{n_{p^t-1}}.
\end{aligned} \tag{3.4}$$

□

Claim 3.10. Let n_i be the first number in $n_1, \ldots, n_{p^t-1}$ such that n_i is not divisible by p. Then $p^r \mid C_{p^r}^{n_1} C_{n_1}^{n_2} \cdots C_{n_{i-1}}^{n_i}$.

Proof. If $i = 1$, then $(n_1, p) = 1$, and by Lemma 3.8, $p^r \mid C_{p^r}^{n_1}$.

Now we set $i > 1$. Let $n_k = n_k' p^{u_k}$, $1 \leqslant k \leqslant i-1$, where $(n_k', p) = 1$. Since $C_{n_{k-1}}^{n_k} = C_{n_{k-1}}^{n_{k-1}-n_k}$, we can assume that $u_k \leqslant u_{k-1}$. By Lemma 3.8, we have $p^{u_{k-1}-u_k} \mid C_{n_{k-1}}^{n_k}$, $1 \leqslant k \leqslant i-1$, and $p^{u_{i-1}} \mid C_{n_{i-1}}^{n_i}$ because $(p, n_i) = 1$. So

$$p^{(r-u_1)+(u_1-u_2)+\cdots+(u_{i-2}-u_{i-1})+u_{i-1}} \mid C_{p^r}^{n_1} C_{n_1}^{n_2} \cdots C_{n_{i-1}}^{n_i}. \tag{3.5}$$

Hence, $p^r \mid C_{p^r}^{n_1} C_{n_1}^{n_2} \cdots C_{n_{i-1}}^{n_i}$.

By the above claim, if there exists n_i such that $p \nmid n_i$, then $C_{p^r}^{n_1} C_{n_1}^{n_2} \cdots C_{n_{i-1}}^{n_i} = 0$ in $\mathbb{Z}_{p^r}$. So assume that $p \mid n_j$, $j = 1, \ldots, p^t - 1$, and then we have

$$\begin{aligned}
x^{p^r} &= \sum_{p \mid n_1, 0 \leqslant n_1 \leqslant p^r} \sum_{p \mid n_2, 0 \leqslant n_2 \leqslant n_1} \cdots \sum_{p \mid n_{p^t-1}, 0 \leqslant n_{p^t-1} \leqslant n_{p^t-2}} \\
&\qquad \times C_{p^r}^{n_1} C_{n_1}^{n_2} \cdots C_{n_{p^t-2}}^{n_{p^t-1}} r_0^{p^r-n_1} (r_1 g)^{n_1-n_2} \cdots (r_{p^t-1} g^{p^t-1})^{n_{p^t-1}} \\
&= \sum c_i (g^p)^i \in (\mathbb{Z}_{p^r}G)\langle g^p \rangle = (\mathbb{Z}_{p^r G}) C_{p^{t-1}}.
\end{aligned} \tag{3.6}$$

□

THEOREM 3.11. *If p is a prime number, $r \geqslant 1$, and G is a finite Abelian group, then $\mathbb{Z}_{p^r}G$ is π-morphic.*

Proof

Case 1. If $(p, |G|) = 1$, then $(p^r, |G|) = 1$. By Chen et al. [6, Corollary 3.13], $\mathbb{Z}_{p^r}G$ is morphic, so $\mathbb{Z}_{p^r}G$ is π-morphic.

Case 2. If $p \mid |G|$, then $G = C_{p^{t_1}} \times \cdots \times C_{p^{t_s}} \times H$, where $(p, |H|) = 1$. Now if $x \in \mathbb{Z}_{p^r}G = \mathbb{Z}_{p^r}(C_{p^{t_2}} \times \cdots \times C_{p^{t_s}} \times H)C_{p^{t_1}}$, then $x^{p^r} \in \mathbb{Z}_{p^r}(C_{p^{t_2}} \times \cdots \times C_{p^{t_s}} \times H)C_{p^{t_1-1}}$ by Proposition 3.9. So we have $x^{k_1} \in \mathbb{Z}_{p^r}(C_{p^{t_2}} \times \cdots \times C_{p^{t_s}} \times H)$ for some k_1. Continuing the process, we get $x^n \in \mathbb{Z}_{p^r}H$ for some n. By Chen et al. [6, Corollary 3.13], $\mathbb{Z}_{p^r}H$ is morphic. So x^n is morphic in $\mathbb{Z}_{p^r}H$. Thus x^n is morphic in $\mathbb{Z}_{p^r}G$ by Lemma 3.7. Hence x is π-morphic in $\mathbb{Z}_{p^r}G$. □

Acknowledgments

The authors would like to thank the referee for careful reading of the paper and for pointing out some typos and minor errors. The research was supported by the National Natural Science Foundation of China (no. 10571026), the Natural Science Foundation of Jiangsu Province (no. BK2005207), and the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutes of MOE, China. The authors would like to thank Professor Yuanlin Li for reading of the paper and making some valuable comments and suggestions.

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