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Research Article

Lebesgue Measurability of Separately Continuous Functions and Separability

V. V. Mykhaylyuk

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A connection between the separability and the countable chain condition of spaces with L-property (a topological space X has L-property if for every topological space Y, separately continuous function $f : X \times Y \to \mathbb{R}$ and open set $I \subseteq \mathbb{R}$, the set $f^{-1}(I)$ is an F_σ-set) is studied. We show that every completely regular Baire space with the L-property and the countable chain condition is separable and constructs a nonseparable completely regular space with the L-property and the countable chain condition. This gives a negative answer to a question of M. Burke.

1. Introduction

A function $f : X \to \mathbb{R}$ defined on a topological space X is called *a first Baire class function* if there exists a sequence $(f_n)_{n=1}^{\infty}$ of continuous functions $f_n : X \to \mathbb{R}$ which converges pointwise to f on X; and *a first Lebesgue class function* if $f^{-1}(G)$ is an F_σ-set for every open set $G \subseteq \mathbb{R}$. Standard reasons (see [1, page 394]) show that every first Baire class function is a first Lebesgue class function.

Investigations of Baire and Lebesgue classifications of separately continuous functions were started by Lebesgue in [2] and were continued in papers of many mathematicians (see [3]).

We say that a topological space X has *the B-property* (*the L-property*) if for every topological space Y each separately continuous function $f : X \times Y \to \mathbb{R}$ is a first Baire class function (a first Lebesgue class function).

It is known [4, 5] that any topological space X has the B-property (the L-property) if and only if the evaluation function $c_X : X \times C_p(X) \to \mathbb{R}$, $c_X(x, y) = y(x)$ is a first Baire class function (a first Lebesgue class function), where $C_p(X)$ means the space of continuous on X functions with the pointwise convergence topology.

Baire and Lebesgue classifications of separately continuous function were investigated in [6]. In particular, it was shown in [6] that any completely regular space X with the B-property and the countable chain condition is separable (topological space X has a countable chain condition (CCC) if every system of disjoint open-in-X sets is at most countable). In this connection the following question arose in [6, Problem 4.6].

Question 1. Is every completely regular space X with the L-property and the countable chain condition a separable space?

In this paper, we show that if a space X is a Baire space, then Question 1 has a positive answer and construct an example which gives a negative answer to the question in general case.

2. Density of Baire spaces with the L-property

The minimal cardinal $\aleph \geq \aleph_0$ for which any system of disjoint open in a topological space X sets has the cardinality at most $\aleph$ is called *a Souslin number of* X and is denoted by $c(X)$. Note that the countable chain condition of X means that $c(X) = \aleph_0$. It is easy to see that $c(X) \leq d(X)$, where $d(X)$ is the density of X.

The following result implies that for a Baire space X Question 1 has a positive answer.

Theorem 2.1. *Let X be a completely regular Baire space with the L-property. Then $c(X) = d(X)$.*

Proof. Since the evaluation function c_X is a first Lebesgue class function, the set $E = \{(x,y) : y(x) = 0\}$ is a G_δ-set in $X \times Y$, where $Y = C_p(X)$. Choose a sequence $(W_n)_{n=1}^\infty$ of open-in-$X \times Y$ sets W_n such that $E = \bigcap_{n=1}^\infty W_n$. Denote by y_0 the null-function on Y. For every $n \in \mathbb{N}$ and an $x \in X$ find open neighborhoods $U(x,n)$ and $V(x,n)$ of x and y_0 in X and Y, respectively, such that $U(x,n) \times V(x,n) \subseteq W_n$.

Fix an $n \in \mathbb{N}$ and show that there exists a set $A_n \subseteq X$ with $|A_n| \leq c(X) = \aleph$ such that the open set $G_n = \bigcup_{x \in A_n} U(x,n)$ is dense in X. Consider a system $\mathcal{U}$ of all open-in-X nonempty sets U such that $U \subseteq U(x,n)$ for some $x \in X$ and choose a maximal system $\mathcal{U}' \subseteq \mathcal{U}$ which consists of disjoint sets. It is clear that $|\mathcal{U}'| \leq \aleph$. For every $U \in \mathcal{U}'$ find an $x = x(U) \in X$ such that $U \subseteq U(x,n)$ and put $A_n = \{x(U) : U \in \mathcal{U}'\}$. Then $|A_n| \leq |\mathcal{U}'| \leq \aleph$. Besides, it follows from the maximality of $\mathcal{U}'$ that G_n is dense in X.

Since X is a Baire space, the set $X_0 = \bigcap_{n=1}^\infty G_n$ is dense in X. For every $n \in \mathbb{N}$ and $x \in X$ choose a finite set $B(x,n) \subseteq X$ such that $y \in V(x,n)$ for each $y \in Y$ with $y|_{B(x,n)} = y_0|_{B(x,n)}$. Put $B = \bigcup_{n \in \mathbb{N}} \bigcup_{x \in A_n} B(x,n)$. Note that $|B| \leq \aleph_0 \cdot \aleph = \aleph$.

Show that B is dense in X. Since X is a completely regular space, it is enough to prove that y_0 is a unique continuous on X function which equals to 0 at every point from B. Let $y \in Y$ be a function such that $y(b) = 0$ for every $b \in B$. Fix a point $x \in X_0$ and an integer $n \in \mathbb{N}$. Find $a \in A_n$ such that $x \in U(a,n)$. Then $B(a,n) \subseteq B$ implies $y \in V(a,n)$. Therefore, $(x,y) \in W_n$. Thus $X_0 \times \{y\} \subseteq \bigcap_{n=1}^\infty W_n = E$, that is, $y(x) = 0$ for every $x \in X_0$. Hence $y = y_0$ because X_0 is dense in X.

Thus $d(X) \leq |B| \leq c(X)$. Therefore, $c(X) = d(X)$. □

Corollary 2.2. *Every completely regular Baire space with the L-property and the countable chain condition is a separable space.*

3. Nonseparable spaces with the L-property and CCC

The following notion was introduced in [4], where some properties of spaces with the B-property were studied.

A topological space X with a topology τ is called *quarter-stratifiable* if there exists a function $g : \mathbb{N} \times X \to \tau$ such that

(i) $X = \bigcup_{x \in X} g(n,x)$ for every $n \in \mathbb{N}$;

(ii) if $x \in g(n,x_n)$ for each $n \in \mathbb{N}$, then $x_n \to x$.

The following result follows from [7, Proposition 2.1].

Proposition 3.1. *Every quarter-stratifiable space X has the L-property.*

A topological space X is called *σ-discrete* if there exists an increasing sequence $(X_n)_{n=1}^{\infty}$ of closed discrete subspaces X_n of X such that $X = \bigcup_{n=1}^{\infty} X_n$.

Proposition 3.2. *Every σ-discrete space is a quarter-stratifiable space.*

Proof. Let $(X_n)_{n=1}^{\infty}$ be an increasing sequence of closed discrete subspaces X_n of X such that $X = \bigcup_{n=1}^{\infty} X_n$. For every $n \in \mathbb{N}$ and $x \in X_n$ denote by $U(x,n)$ an open-in-X neighborhood of x such that $U(x,n) \cap X_n = \{x\}$. We define a function $g : \mathbb{N} \times X \to \tau$, where τ is the topology of X, by $g(x,n) = U(x,n)$ if $x \in X_n$ and $g(x,n) = X \setminus X_n$ if $x \notin X_n$. It is easy to see that g satisfies (i) and (ii). □

Show now that Question 1 has a negative answer.

Theorem 3.3. *There exists a completely regular nonseparable space with the L-property and with the countable chain condition.*

Proof. Let Γ_0 be a set with $|\Gamma_0| \geq \aleph_1$, let $(a_n)_{n=1}^{\infty}$ be a sequence of distinct points $a_n \notin \Gamma_0$, $\Gamma_n = \Gamma_0 \cup \{a_k : 1 \leq k \leq n\}$, and let $\mathcal{A}_n$ be a system of all subsets $A \subseteq \Gamma_{n-1}$ such that $|A| = n$. Denote by X_n a set of all function $x \in \{0,1\}^{\Gamma}$ such that $x = \chi_{A \cup \{a_n\}}$ for some $A \in \mathcal{A}_n$, where χ_B means the characteristic function of B, and put $X = \bigcup_{n=1}^{\infty} X_n$.

Show that X is a σ-discrete space. For every $n \in \mathbb{N}$ put $Y_n = \bigcup_{k=1}^{n} X_k$. Fix an integer $n \in \mathbb{N}$ and for each $1 \leq k \leq n$ put $G_k = \{x \in X : x(a_k) = 1,\ x(a_i) = 0,\ k < i \leq n\}$. It is easy to see that $G_k \cap Y_n = X_k$. Since all spaces X_k are discrete, Y_n is discrete in X too. Besides, Y_n is closed in X. Thus, X has the L-property by Propositions 3.1 and 3.2.

Note that X is dense in $Y = \{0,1\}^{\Gamma}$. Indeed, let $A \subseteq \Gamma$ be a finite set and $y : A \to \{0,1\}$. Choosing $n \geq |A|$ with $A \subseteq \Gamma_n$ find $x \in X_{n+1}$ such that $x|_A = y$. Then $c(X) = \aleph_0$ since $c(Y) = \aleph_0$ and X is dense in Y.

It remains to note that X is nonseparable because for every separable subspace Z of X there exists a countable set $B \subseteq \Gamma$ such that $z(\gamma) = 0$ for every $\gamma \in \Gamma \setminus B$. □

This example shows that there exists a quarter-stratifiable space which has not the B-property. Thus, Proposition 3.1 cannot be generalized for spaces with the B-property.

A family $(A_i : i \in I)$ of sets A_i is called *pointwise finite* if $\bigcap_{i\in J} A_i = \emptyset$ for each infinite set $J \subseteq I$. A cardinal

$$p(X) = \sup\{|\mathscr{A}| : \mathscr{A} \text{ is a pointwise finite family of nonempty open-in-}X\text{ sets}\} \tag{3.1}$$

is called *a point-finite cellularity of a topological space* X. Clearly $c(X) \le p(X)$. Besides, it is known that $p(X) = c(X)$ for each Baire space X. Therefore, the following question arises naturally from Theorem 2.1 and the fact that $p(X) = |\Gamma| > \aleph_0$ for the space X from Theorem 3.3.

Question 2. Is every completely regular space X with the L-property and $p(X) = \aleph_0$ a separable space?

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V. V. Mykhaylyuk: Department of Mathematical Analysis, Chernivtsi National University, Kotsjubyns'koho 2, 58012 Chernivtsi, Ukraine
Email address: mathan@chnu.cv.ua