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Research Article

Product Bessel Distributions of the First and Second Kinds

Saralees Nadarajah

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A new Bessel function distribution is introduced by taking the product of a Bessel function pdf of the first kind and a Bessel function pdf of the second kind. Various particular cases and expressions for moments are derived.

1. Introduction

Univariate Bessel function distributions have been used to model signal output processed by a radar receiver under various sets of conditions (see, e.g., McNolty [1]). There are two kinds of univariate Bessel function distributions. Bessel function distribution of the first kind has the pdf given by

$$f(x)=\frac{\left|1-c^2\right|^{m+1/2}x^m}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+1/2)}\exp\left(-\frac{cx}{b}\right)I_m\left(\frac{x}{b}\right) \tag{1.1}$$

for $x>0$, $b>0$, $c>1$ and $m>1$, where

$$I_m(x)=\frac{x^m}{\sqrt{\pi}2^m\Gamma(m+1/2)}\int_{-1}^{1}\left(1-t^2\right)^{m-1/2}\exp(\pm xt)dt \tag{1.2}$$

is the modified Bessel function of the first kind. Bessel function distribution of the second kind has the pdf given by

$$f(x)=\frac{\left|1-c^2\right|^{m+1/2}|x|^m}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+1/2)}\exp\left(-\frac{cx}{b}\right)K_m\left(\left|\frac{x}{b}\right|\right) \tag{1.3}$$

for $-\infty < x < \infty$, $b > 0$, $|c| < 1$, and $m > 1$, where

$$K_m(x) = \frac{\sqrt{\pi}x^m}{2^m\Gamma(m+1/2)}\int_1^\infty (t^2-1)^{m-1/2}\exp(-xt)dt \tag{1.4}$$

is the modified Bessel function of the second kind. In thispaper, we introduce a new Bessel function distribution with its pdf taken to be the product of two densities of the form (1.1) and (1.3), that is,

$$f(x) = Cx^{m+n}I_m\left(\frac{x}{b}\right)K_n\left(\frac{x}{\beta}\right) \tag{1.5}$$

for $x > 0$, $0 < \beta < b$, $m > 1$, and $n > 1$, where C denotes the normalizing constant. Application of [2, equation (2.16.28.1)] by Prudnikov et al. shows that one can determine C as

$$\frac{1}{C} = \frac{2^{m+n-1}\beta^{2m+n+1}}{b^m\Gamma(m+1)}\Gamma\left(m+n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right){}_2F_1\left(m+n+\frac{1}{2},m+\frac{1}{2};m+1;\frac{\beta^2}{b^2}\right), \tag{1.6}$$

where ${}_2F_1$ is the Gauss hypergeometric function defined by

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{x^k}{k!}, \tag{1.7}$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. Using special properties of the Gauss hypergeometric function, one can obtain simpler expressions for (1.6). For instance, if $m = n$, then (1.6) can be reduced to

$$\begin{aligned}\frac{1}{C} &= \pi^{-1/2}2^{2m-1}(b\beta)^{2m+1/2}\left(b^2-\beta^2\right)^{-m}\Gamma\left(\frac{2m+1}{2}\right)\\ &\quad\times\exp(mi\pi)Q_{m-1/2}^m\left(\frac{b^2+\beta^2}{2b\beta}\right),\end{aligned} \tag{1.8}$$

where $Q_\nu^\mu(\cdot)$ is the Legendre function defined by

$$Q_\nu^\mu(x) = \frac{\sqrt{\pi}\exp(i\mu\pi)\Gamma(\mu+\nu+1)}{2^{\nu+1}\Gamma(\nu+3/2)}x^{-\mu-\nu-1}\left(x^2-1\right)^{\mu/2}{}_2F_1\left(\frac{\mu+\nu+1}{2},\frac{\mu+\nu}{2};\nu+\frac{3}{2};\frac{1}{x^2}\right). \tag{1.9}$$

In the rest of thispaper, we derive various expressions for particular forms of (1.5) and its moments.

2. Particular cases

When m and n take half-integer values, one can reduce (1.5) to elementary forms. Note that

$$\begin{aligned}
I_{3/2}(x) &= \sqrt{\frac{2}{\pi}}\frac{x\cosh(x)-\sinh(x)}{x^{3/2}},\\
I_{5/2}(x) &= \sqrt{\frac{2}{\pi}}\frac{(x^2+3)\sinh(x)-3x\cosh(x)}{x^{5/2}},\\
I_{7/2}(x) &= \sqrt{\frac{2}{\pi}}\frac{x(x^2+15)\cosh(x)-3(2x^2+5)\sinh(x)}{x^{7/2}},\\
I_{9/2}(x) &= \sqrt{\frac{2}{\pi}}\frac{(x^4+45x^2+105)\sinh(x)-5x(2x^2+21)\cosh(x)}{x^{9/2}}
\end{aligned} \tag{2.1}$$

and, more generally, if $\nu - 1/2 \geq 1$ is an integer, then

$$\begin{aligned}
I_\nu(x) = {} & \sqrt{2}\sqrt{x\pi}\exp\left\{\frac{\pi i}{2}\left(\frac{1}{2}-\nu\right)\right\}\\
& \times\Bigg[\sinh\left(\frac{\pi x}{2}\left(\frac{1}{2}-\nu\right)-x\right)\times\sum_{k=0}^{[(2|\nu|-1)/4]}\frac{(|\nu|+2k-1/2)!}{(2k)!(|\nu|-2k-1/2)!(2x)^{2k}}\\
& \quad +\cosh\left(\frac{\pi x}{2}\left(\frac{1}{2}-\nu\right)-x\right)\sum_{k=0}^{[(2|\nu|-3)/4]}\frac{(|\nu|+2k+1/2)!(2x)^{-2k-1}}{(2k+1)!(|\nu|-2k-3/2)!}\Bigg].
\end{aligned} \tag{2.2}$$

Furthermore, note that

$$\begin{aligned}
K_{3/2}(x) &= \sqrt{\frac{\pi}{2}}\frac{\exp(-x)(x+1)}{x^{3/2}},\\
K_{5/2}(x) &= \sqrt{\frac{\pi}{2}}\frac{\exp(-x)(x^2+3x+3)}{x^{5/2}},\\
K_{7/2}(x) &= \sqrt{\frac{\pi}{2}}\frac{\exp(-x)(x^3+6x^2+15x+15)}{x^{7/2}},\\
K_{9/2}(x) &= \sqrt{\frac{\pi}{2}}\frac{\exp(-x)(x^4+10x^3+45x^2+105x+105)}{x^{9/2}}
\end{aligned} \tag{2.3}$$

and, more generally, if $\nu - 1/2 \geq 1$ is an integer, then

$$I_\nu(x) = \sqrt{\pi}\exp(-x)\sqrt{2x}\sum_{j=0}^{[|\nu|-1/2]}\frac{(j+|\nu|-1/2)!(2x)^{-j}}{j!(|\nu|-j-1/2)!}. \tag{2.4}$$

Thus, several particular forms of (1.5) can be obtained for half-integer values of m and n. For example, if $m = 3/2$ and $n = 3/2$, then (1.5) reduces to

$$f(x) = C(b\beta)^{3/2}\left\{\frac{x}{b}\cosh\left(\frac{x}{b}\right)-\sinh\left(\frac{x}{b}\right)\right\}\exp\left(-\frac{x}{\beta}\right)\left(\frac{x}{\beta}+1\right). \tag{2.5}$$

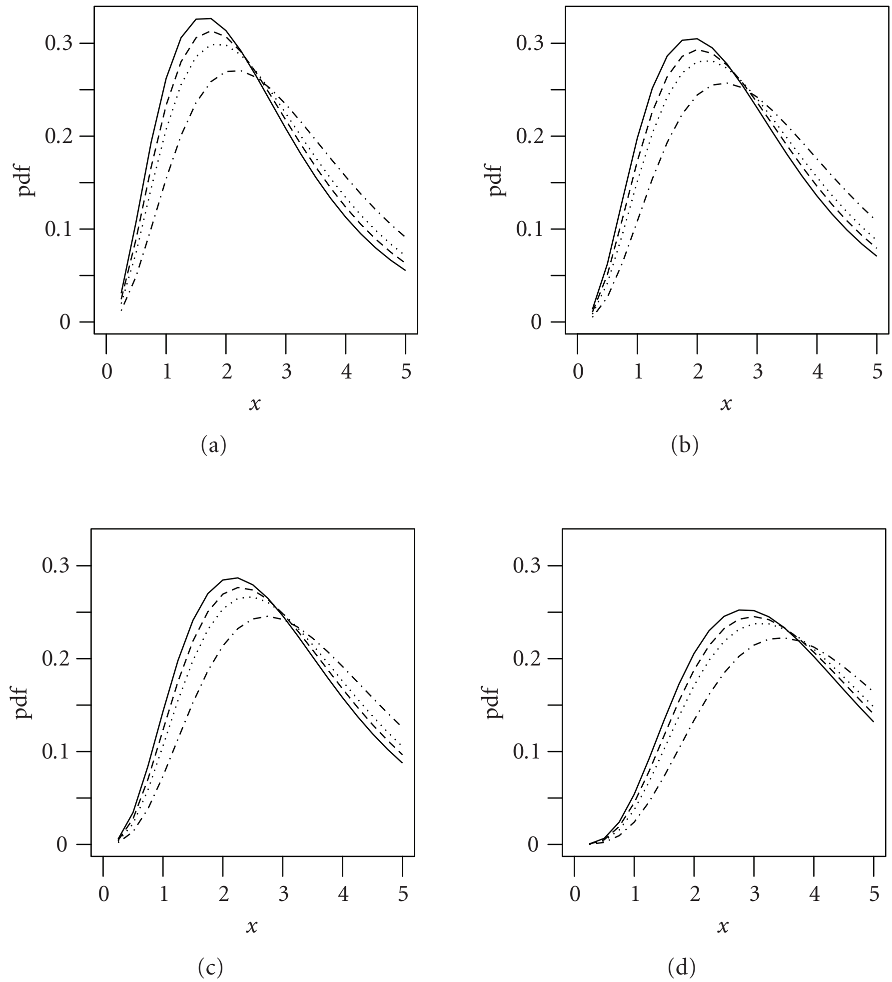

Figure 2.1. Plots of the pdf (1.5) for $b = 1$, $\beta = 1/2$, and (a) $m = 1.1$; (b) $m = 1.3$; (c) $m = 1.5$; and, (d) $m = 2$. The four curves in each plot from the left to the right correspond to $n = 1.1, 1.3, 1.5, 2$.

If $m = 3/2$ and $n = 5/2$, then (1.5) reduces to

$$f(x) = Cb^{3/2}\beta^{5/2}\left\{\frac{x}{b}\cosh\left(\frac{x}{b}\right) - \sinh\left(\frac{x}{b}\right)\right\}\exp\left(-\frac{x}{\beta}\right)\left(\frac{x^2}{\beta^2} + \frac{3x}{\beta} + 3\right). \tag{2.6}$$

Figure 2.1 illustrates possible shapes of the pdf (1.5) for selected values of m and n. The four curves in each plot correspond to selected values of n. Note that the shapes are unimodal and that the densities appear to shrink with increasing values of both m and n.

3. Moments

If X is a random variable with pdf (1.5), then its kth moment can be expressed as

$$E(X^k) = C\int_0^\infty x^{k+m+n} I_m\left(\frac{x}{b}\right) K_n\left(\frac{x}{\beta}\right) dx. \tag{3.1}$$

Application of [2, equation (2.16.28.1)] by Prudnikov et al. shows that (3.1) can be calculated as

$$\begin{aligned} E(X^k) = {} & \frac{C2^{k+m+n-1}\beta^{k+2m+n+1}}{b^m\Gamma(m+1)}\Gamma\left(m+n+\frac{k+1}{2}\right)\Gamma\left(m+\frac{k+1}{2}\right) \\ & \times {}_2F_1\left(m+n+\frac{k+1}{2}, m+\frac{k+1}{2}; m+1; \frac{\beta^2}{b^2}\right). \end{aligned} \tag{3.2}$$

Using special properties of the Gauss hypergeometric function, one can derive several simpler forms of (3.2) as discussed in the following. If $m = n$, then (3.2) reduces to

$$\begin{aligned} E(X^k) = {} & C\pi^{-1/2}2^{k+2m-1}(b\beta)^{k+2m+1/2}\left(b^2-\beta^2\right)^{-(k+2m)/2}\Gamma\left(\frac{k+2m+1}{2}\right) \\ & \times \exp\left(\frac{(k+2m)i\pi}{2}\right) Q_{m-1/2}^{(k+2m)/2}\left(\frac{b^2+\beta^2}{2b\beta}\right). \end{aligned} \tag{3.3}$$

If $k \geq 1$ is odd, then (3.2) can be reduced to the following elementary form:

$$\begin{aligned} E(X^k) = {} & \frac{C2^{k+m+n-1}b^{k+m+2n+1}\beta^{k+2m+n+1}}{\left(b^2-\beta^2\right)^{m+n+(k+1)/2}\Gamma(m+1)}\Gamma\left(m+n+\frac{k+1}{2}\right)\Gamma\left(m+\frac{k+1}{2}\right) \\ & \times {}_2F_1\left(m+n+\frac{k+1}{2}, \frac{1-k}{2}; m+1; \frac{\beta^2}{\beta^2-b^2}\right) \\ = {} & \frac{C2^{k+m+n-1}b^{k+m+2n+1}\beta^{k+2m+n+1}}{\left(b^2-\beta^2\right)^{m+n+(k+1)/2}\Gamma(m+1)}\Gamma\left(m+n+\frac{k+1}{2}\right)\Gamma\left(m+\frac{k+1}{2}\right) \\ & \times \sum_{j=0}^{(k-1)/2} \frac{(m+n+(k+1)/2)_j((1-k)/2)_j}{(m+1)_j}\left(\frac{\beta^2}{\beta^2-b^2}\right)^j. \end{aligned} \tag{3.4}$$

When k is even, one can reduce (3.2) to simpler forms when m and n take integer or half-integer values. If either both m and n are half-integers or m is an integer and n is a half-integer or m is a half-integer and n is an integer, then (3.2) can be reduced to an elementary form. On the other hand, if both m and n are integers, then one can express (3.2) in terms of the complete elliptical integral of the first kind and the complete elliptical integral of the second kind defined by

$$\begin{aligned} \text{EllipticK}(a) &= \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-a^2x^2}}dx, \\ \text{EllipticE}(a) &= \int_0^1 \frac{\sqrt{1-a^2x^2}}{\sqrt{1-x^2}}dx, \end{aligned} \tag{3.5}$$

respectively. For instance, if $m = 3/2$ and $n = 3/2$, then the first four even order moments are

$$
\begin{aligned}
E(X^2) &= 8C\beta^{15/2}\frac{-35-14x+x^2}{b^{3/2}(-1+x)^5},\\
E(X^4) &= 144C\beta^{19/2}\frac{-105-189x-27x^2+x^3}{b^{3/2}(-1+x)^7},\\
E(X^6) &= 5760C\beta^{23/2}\frac{-231-924x-594x^2-44x^3+x^4}{b^{3/2}(-1+x)^9},\\
E(X^8) &= 403200C\beta^{27/2}\frac{-429-3003x-4290x^2-1430x^3-65x^4+x^5}{b^{-3/2}(-1+x)^{-11}},
\end{aligned}
\tag{3.6}
$$

where $x = \beta^2/b^2$ and the normalizing constant $C = 2\beta^{11/2}(-5+x)/\{b^{3/2}(-1+x)^3\}$. If $m = 2$ and $n = 2$, then the first four even order moments are

$$
\begin{aligned}
E(X^2) = 15C\beta^9\{ &-23\,\text{EllipticK}(\sqrt{x})x - 87\,\text{EllipticK}(\sqrt{x})x^2 + 107\,\text{EllipticK}(\sqrt{x})x^3\\
&+ \text{EllipticK}(\sqrt{x})x^4 + 2\,\text{EllipticK}(\sqrt{x}) + 22\,\text{EllipticE}(\sqrt{x})x\\
&+ 216\,\text{EllipticE}(\sqrt{x})x^2 + 22\,\text{EllipticE}(\sqrt{x})x^3 - 2\,\text{EllipticE}(\sqrt{x})x^4\\
&- 2\,\text{EllipticE}(\sqrt{x})\}/\{x^2b^2(-1+x)^6\},\\
E(X^4) = 315C\beta^{11}\{ &-39\,\text{EllipticK}(\sqrt{x})x - 536\,\text{EllipticK}(\sqrt{x})x^2 + 158\,\text{EllipticK}(\sqrt{x})x^3\\
&+ 414\,\text{EllipticK}(\sqrt{x})x^4 + \text{EllipticK}(\sqrt{x})x^5 + 2\,\text{EllipticK}(\sqrt{x})\\
&+ 38\,\text{EllipticE}(\sqrt{x})x + 988\,\text{EllipticE}(\sqrt{x})x^2 + 988\,\text{EllipticE}(\sqrt{x})x^3\\
&+ 38\,\text{EllipticE}(\sqrt{x})x^4 - 2\,\text{EllipticE}(\sqrt{x})x^5\\
&- 2\,\text{EllipticE}(\sqrt{x})\}/\{x^2b^2(-1+x)^8\},\\
E(X^6) = 2835C\beta^{13}\{ &-295\,\text{EllipticK}(\sqrt{x})x - 8771\,\text{EllipticK}(\sqrt{x})x^2 - 8886\,\text{EllipticK}(\sqrt{x})x^3\\
&+ 12452\,\text{EllipticK}(\sqrt{x})x^4 + 5485\,\text{EllipticK}(\sqrt{x})x^5 + 5\,\text{EllipticK}(\sqrt{x})x^6\\
&+ 10\,\text{EllipticK}(\sqrt{x}) + 290\,\text{EllipticE}(\sqrt{x})x + 14546\,\text{EllipticE}(\sqrt{x})x^2\\
&+ 35884\,\text{EllipticE}(\sqrt{x})x^3 + 290\,\text{EllipticE}(\sqrt{x})x^5\\
&+ 14546\,\text{EllipticE}(\sqrt{x})x^4 - 10\,\text{EllipticE}(\sqrt{x})x^6\\
&- 10\,\text{EllipticE}(\sqrt{x})\}/\{(-1+x)^{10}x^2b^2\},\\
E(X^8) = 155925C\beta^{15}\{ &14\,\text{EllipticK}(\sqrt{x}) - 581\,\text{EllipticK}(\sqrt{x})x - 30336\,\text{EllipticK}(\sqrt{x})x^2\\
&- 86111\,\text{EllipticK}(\sqrt{x})x^3 + 19958\,\text{EllipticK}(\sqrt{x})x^4\\
&+ 80445\,\text{EllipticK}(\sqrt{x})x^5 + 16604\,\text{EllipticK}(\sqrt{x})x^6\\
&+ 7\,\text{EllipticK}(\sqrt{x})x^7 - 14\,\text{EllipticE}(\sqrt{x}) + 574\,\text{EllipticE}(\sqrt{x})x\\
&+ 47514\,\text{EllipticE}(\sqrt{x})x^2 + 214070\,\text{EllipticE}(\sqrt{x})x^3\\
&+ 47514\,\text{EllipticE}(\sqrt{x})x^5 + 214070\,\text{EllipticE}(\sqrt{x})x^4\\
&- 14\,\text{EllipticE}(\sqrt{x})x^7 + 574\,\text{EllipticE}(\sqrt{x})x^6\}/\{(-1+x)^{12}x^2b^2\},
\end{aligned}
\tag{3.7}
$$

where $x = \beta^2/b^2$ and the normalizing constant C satisfies

$$\begin{aligned}\frac{1}{C} = 3\beta^7 \{ &- 11\,\mathrm{EllipticK}(\sqrt{x})x + 8\,\mathrm{EllipticK}(\sqrt{x})x^2 + \mathrm{EllipticK}(\sqrt{x})x^3 + 2\,\mathrm{EllipticK}(\sqrt{x}) \\ &+ 10\,\mathrm{EllipticE}(\sqrt{x})x + 10\,\mathrm{EllipticE}(\sqrt{x})x^2 - 2\,\mathrm{EllipticE}(\sqrt{x})x^3 \\ &- 2\,\mathrm{EllipticE}(\sqrt{x})\}/\{x^2 b^2(-1+x)^4\}.\end{aligned} \tag{3.8}$$

References

[1] F. McNolty, "Applications of Bessel function distributions," *Sankhyā*, vol. 29, pp. 235–248, 1967.
[2] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series. Vol. 2*, Gordon & Breach Science Publishers, New York, NY, USA, 1986.

Saralees Nadarajah: School of Mathematics, University of Manchester, Manchester M13 9PL, UK
Email address: saralees.nadarajah@manchester.ac.uk