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Research Article

Necessary Conditions for the Solutions of Second Order Non-linear Neutral Delay Difference Equations to Be Oscillatory or Tend to Zero

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We find necessary conditions for every solution of the neutral delay difference equation $\Delta(r_n\Delta(y_n - p_n y_{n-m})) + q_n G(y_{n-k}) = f_n$ to oscillate or to tend to zero as $n \to \infty$, where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, and p_n, q_n, r_n are sequences of real numbers with $q_n \geq 0$, $r_n > 0$. Different ranges of $\{p_n\}$, including $p_n = \pm 1$, are considered in this paper. We do not assume that G is Lipschitzian nor nondecreasing with $xG(x) > 0$ for $x \neq 0$. In this way, the results of this paper improve, generalize, and extend recent results. Also, we provide illustrative examples for our results.

1. Introduction

In this paper, we present necessary conditions so that every solution of

$$\Delta(r_n\Delta(y_n - p_n y_{n-m})) + q_n G(y_{n-k}) = 0 \tag{1.1}$$

and of

$$\Delta(r_n\Delta(y_n - p_n y_{n-m})) + q_n G(y_{n-k}) = f_n \tag{1.2}$$

oscillates or tends to zero as $n \to \infty$, where Δ is the forward difference operator $\Delta y_n = y_{n+1} - y_n$, the sequences $\{p_n\}$, $\{f_n\}$, $\{q_n\}$, and $\{r_n\}$ are sequences of real numbers with $q_n \geq 0$ and $r_n > 0$. We assume that m, k are nonnegative constant integers, and $G \in C(\mathbb{R}, \mathbb{R})$. Various ranges of the sequence $\{p_n\}$ are considered.

Some of the following conditions will be assumed later this article.

(H0) G is Lipschitzian in every interval of form $[a, b]$, with $0 < a < b$.

(H1) $xG(x) > 0$ for $x \neq 0$ and G is nondecreasing.

(H2) $\sum_{n=0}^{\infty} q_n = \infty$.
(H3) $\sum_{n=0}^{\infty} 1/r_n = \infty$.
(H4) $\sum_{n=0}^{\infty} 1/r_n < \infty$.
(H5) $\sum_{n=1}^{\infty}(1/r_n)\sum_{i=0}^{n-1} q_i = \infty$.
(H6) There exists a bounded sequence $\{F_n\}$ such that $\Delta F_n = f_n$.

Difference equations occur as mathematical models of some real-world problems. To have a glimpse of the importance, utility, and development of the subject, one may refer [1–3]. In recent years, many authors have shown interest in the oscillation of neutral delay difference equations (NDDEs in short). For recent results and references, see the monograph by Agarwal [4], the papers [5–22], and the references cited there in. In this paper, neither (H0) nor (H1) is assumed for obtaining positive solution of (1.2). However, several authors use these conditions while they attempted the same problem for neutral equations of any order; see [6, 12–22]. To the best of our knowledge, no result regarding positive solutions of neutral equations (both differential and difference equations) of any order with $p_n \equiv -1$ is available in the literature. Even the papers written specially for $p_n = \pm 1$ do not have such a result [13, 14, 18, 20]. For difference equations, most ranges are covered in [12, 13, 15], but there is no result for $p_n = \pm 1$. In this paper, we have covered all ranges of p_n including those missing in [12, 13, 15]. Furthermore, the authors studying (1.1) assume either (H3) or (H4); see [6]. In this work, we are able to do away with these conditions. In particular, we show that either (H3) or (H5) is necessary for every solution of (1.1) or (1.2) to oscillate or to tend to zero as $n \to \infty$.

We remark that Thandapani et al. [23] have studied the m-order neutral delay difference equation

$$\Delta^m[y_n + p_n y_{n-s}] + \delta F(n, y_{n-k}) = 0. \tag{1.3}$$

They found conditions that are sufficient for every solution to oscillate or to tend to zero as $n \to \infty$, under various ranges for p_n. The results about our NDDE (even for $r_n = 1$) do not follow from the results presented in (1.3) with $m = 2$, because of the presence of the nonlinear term $F(n, y)$. Furthermore, our conditions are in certain sense opposite to those in [23].

We illustrate our results with suitable examples and show their significance over other results in the literature. Since $r_n \equiv 1$ is permissible, our results generalize and improve the results to second-order NDDEs in [12, 15].

Let $\tau = \max\{m, k\}$ and let N_0 be a fixed nonnegative integer. By a solution of (1.2), we mean a real sequence $\{y_n\}$ which is defined for all positive integer $n \geq N_0 - \tau$ and satisfies (1.2) for $n \geq N_0$. When an initial condition

$$y_n = a_n \quad \text{for } N_0 - \tau \leq n \leq N_0 \tag{1.4}$$

is given, (1.2) has a unique solution satisfying the given initial condition.

A solution $\{y_n\}$ of (1.2) is said to be oscillatory if for every positive integer $N_0 > 0$, there exists $n \geq N_0$ such that $y_n y_{n+1} < 0$; otherwise $\{y_n\}$ is said to be nonoscillatory.

We would like to present the following useful remarks.

Remark 1.1. (i) Since $r_n > 0$, only one of (H3) and (H4) holds but not both.

(ii) If (H3) holds, then (H2) implies (H5) but not conversely. This is justified from the example when $r_n = 3^{-n}$ and $q_n = 2^{-n}$.

(iii) If (H4) holds, then (H5) implies (H2) but not conversely. Indeed, this can be verified from the example when $r_n = n^3$ and $q_n \equiv 1$.

(iv) If (H2) and (H5) hold, then nothing can be said about (H3) and (H4). This can be seen from the example $r_n = n^2$ and $q_n \equiv 1$. In this case, (H2), (H5), (H4) hold but not (H3). Next, consider the example $r_n \equiv 1$ and $q_n \equiv 1$. Here (H2), (H3), (H5) hold but not (H4).

2. Positive solutions I

In this section, we assume that there exists a constant b, such that the sequence $\{p_n\}$ satisfies

(A1) $0 \leq p_n \leq b < 1$.

For our purpose, we need the following result.

Lemma 2.1 (Krasnoselskii's Fixed Point theorem [9]). *Let X be a Banach space and let S be a bounded closed convex subset of X. Let A, B be operators from S to X such that $Ax + By \in S$ for every pair of $x, y \in S$. If A is a contraction and B is completely continuous, then the equation*

$$Ax + Bx = x \tag{2.1}$$

has a solution in S.

Our first results read as follows.

Theorem 2.2. *Let (A1), (H4), and (H6) hold. If every solution of (1.2) oscillates or tends to zero as $n \to \infty$, then (H5) holds.*

Proof. We use the contraposition method. Assuming that (H5) does not hold, try to find a solution to (1.2) that does not oscillate and does not tend to zero. From the negation of (H5),

$$\sum_{i=1}^{\infty} \frac{1}{r_i} \sum_{j=0}^{i-1} q_j < \infty. \tag{2.2}$$

Using the continuity of G, we set

$$\mu = \max\left\{ |G(x)| : \frac{2(1-b)}{3} \leq x \leq \frac{4}{3} \right\}. \tag{2.3}$$

Then using (H4) and (H6), we obtain

$$\sum_{i=n}^{\infty} \frac{F_i}{r_i} < \infty. \tag{2.4}$$

From (2.2) and (2.4), we can find that $N_1 > 0$ such that for $n \geq N_1$,

$$\mu \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=0}^{i-1} q_j < \frac{1-b}{6}, \tag{2.5}$$

$$\sum_{i=n}^{\infty} \frac{|F_i|}{r_i} < \frac{1-b}{6}. \tag{2.6}$$

Let X be the Banach space consisting of bounded real sequences $x = \{x_n\}$, with the supremum norm

$$\|x\| = \sup\{|x_n| : n \geq N_1 - \tau\}. \tag{2.7}$$

In this space, we define the closed and convex set

$$S = \left\{ y \in X : \frac{2(1-b)}{3} \leq y_n \leq \frac{4}{3}, n \geq N_1 - \tau \right\}. \tag{2.8}$$

Now we define two operators A and B, from S to X, such that fixed points of $A + B$ are solutions of (1.2). For $y \in S$, define

$$(Ay)_n = \begin{cases} (Ay)_{N_1}, & N_1 - \tau \leq n \leq N_1, \\ p_n y_{n-m} + (1-b), & n \geq N_1, \end{cases} \tag{2.9}$$

$$(By)_n = \begin{cases} (By)_{N_1}, & N_1 - \tau \leq n \leq N_1, \\ \displaystyle\sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j G(y_{j-k}) - \sum_{i=n}^{\infty} \frac{F_i}{r_i}, & n \geq N_1. \end{cases} \tag{2.10}$$

Here we use the convention that $\sum_{j=n_1}^{n_2} \cdots = 0$ when $n_2 < n_1$.

First we show that if $x, y \in S$, then $Ax + By \in S$. With $x = \{x_n\}$ and $y = \{y_n\}$ in S, and $n \geq N_1$, we obtain

$$(Ax)_n + (By)_n = p_n x_{n-m} + (1-b) + \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j G(y_{j-k}) - \sum_{i=n}^{\infty} \frac{F_i}{r_i}. \tag{2.11}$$

Note that for $j \geq N_1$, the sequence $\{y_{j-k}\}$ is in S, so that $|G(y_{j-k})| \leq \mu$. Using (2.5), (2.6), and $0 \leq p_n \leq b < 1$, we have

$$\begin{aligned} (Ax)_n + (By)_n &< \frac{4b}{3} + (1-b) + \frac{1-b}{6} + \frac{1-b}{6} = \frac{4}{3}, \\ (Ax)_n + (By)_n &> 0 + (1-b) + 0 - \frac{1-b}{6} - \frac{1-b}{6} = \frac{2}{3}(1-b). \end{aligned} \tag{2.12}$$

Therefore, $2(1-b)/3 < (Ax)_n + (By)_n \leq 4/3$ so that $Ax + By$ belongs to S for all x, y in S.

Next we show that A is a contraction in S. In fact for x, y in S and $n \geq N_1$,

$$\|(Ax)_n - (Ay)_n\| \leq |p_n| \, |x_{n-m} - y_{n-m}| \leq b\|x - y\|. \tag{2.13}$$

This implies that A is a contraction, because $0 < b < 1$.

Next we show that B is completely continuous. As a first step, we show that B is continuous. Suppose that $x^l \equiv \{x_n^l\}$ is a sequence of points in S (with l taken from the index set) which converges to $x \equiv \{x_n\}$ in S as $l \to \infty$. Since S is closed, $x \in S$. For $n \geq N_1$, we have

$$|(Bx^l)_n - (Bx)_n| \leq \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j \, |G(x_{j-k}^l) - G(x_{j-k})|. \tag{2.14}$$

Since G is continuous, $|G(x_{j-k}^l) - G(x_{j-k})|$ approaches zero and is as $l \to \infty$. Hence, B is continuous. It remains to show that BS is relatively compact. Using [7, Theorem 3.3], we need only show that BS is uniformly cauchy. Let $x \equiv \{x_n\}$ be a sequence in S. Using (2.2) and (2.4), for $\epsilon > 0$, there exists $N^* \geq N_1$ such that, for $n \geq N^*$,

$$\sum_{i=n}^{\infty} \left|\frac{F_i}{r_i}\right| + \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j \mu < \frac{\epsilon}{2}. \tag{2.15}$$

Then for $n_2 > n_1 \geq N^*$,

$$|(Bx)_{n_2} - (Bx)_{n_1}| < \sum_{i=n_2}^{\infty} \left|\frac{F_i}{r_i}\right| + \sum_{i=n_1}^{\infty} \left|\frac{F_i}{r_i}\right| + \sum_{i=n_2}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j \mu + \sum_{i=n_1}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j \mu < 2\frac{\epsilon}{2} = \epsilon. \tag{2.16}$$

Thus, BS is uniformly cauchy. Hence, it is relatively compact. Then, by Lemma 2.1 there is an x^0 in S such that $Ax^0 + Bx^0 = x^0$; that is, for $y = x^0$ and $n \geq N_1$,

$$y_n = (A+B)y_n = p_n y_{n-m} + 1 - b + \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j G(y_{j-k}) - \sum_{i=n}^{\infty} \frac{F_i}{r_i}. \tag{2.17}$$

Applying the forward difference operator Δ, we obtain

$$\Delta(y_n - p_n y_{n-m}) + \frac{1}{r_n} \sum_{j=N_1}^{n-1} q_j G(y_{j-k}) = \frac{F_n}{r_n}. \tag{2.18}$$

Multiplying by r_n, and applying Δ again, with $\Delta F_n = f_n$, we obtain (1.2). Therefore, $(x^0)_n$ is a solution of (1.2) and is bounded below by $2(1-b)/3$; thus $(x^0)_n$ is nonoscillatory and does not approach zero as $n \to \infty$. This completes the proof. □

Corollary 2.3. *Let (A1), (H4), (H6) hold. If every solution of (1.2) oscillates or tends to zero as $n \to \infty$, then (H2) holds.*

The proof of this corollary follows from Remark 1.1(iii) and Theorem 2.2.

Theorem 2.4. *Let (A1) and (H3) hold. Assume that there exists $\alpha > 0$ such that for i large,*

$$r_i > \frac{1}{\alpha}, \tag{2.19}$$

$$\sum_{i=0}^{\infty} F_i < \infty \quad \text{with } \Delta F_n = f_n. \tag{2.20}$$

If every solution of (1.2) oscillates or tends to zero as $n \to \infty$, then (H5) holds.

Proof. Using (2.19) and (2.20), we obtain (2.4) and consequently get (2.6). The rest of the proof is similar to that of Theorem 2.2. □

Remark 2.5. Condition (2.20) implies (H6).

Corollary 2.6. *Let (A1), (2.19), (2.20) hold. If every solution of (1.2) oscillates or tends to zero as $n \to \infty$, then (H5) holds.*

Proof. By Remark 1.1(i), either (H3) or (H4) holds exclusively. If (H4) holds, using condition (2.20) and Theorem 2.2, we get the required solution to (1.2). If (H3) holds, using Theorem 2.4, we obtain the required solution. □

Remark 2.7. If in the proof of Theorem 2.2 we replace (2.2) by

$$\sum_{n=1}^{\infty} \frac{1}{r_n} \sum_{i=n}^{\infty} q_i < \infty, \tag{2.21}$$

then the theorem still holds. We just have to adjust the definition of the mapping B in (2.10). That is, for $n \geq N_1$,

$$(By)_n = -\sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} q_j G(y_{j-k}) - \sum_{i=n}^{\infty} \frac{F_i}{r_i}. \tag{2.22}$$

Then, if we take $r_n \equiv 1$, then condition (2.21) reduces to

$$\sum_{n=1}^{\infty} \sum_{i=n}^{\infty} q_i < \infty. \tag{2.23}$$

The above condition is required for the next result.

Corollary 2.8. *Inequality (2.23) is a sufficient condition for the second-order NDDE*

$$\Delta^2 (y_n - p_n y_{n-m}) + q_n G(y_{n-k}) = f_n \tag{2.24}$$

to have a solution bounded below by a positive constant, under assumptions (A1), (2.19), and (2.20).

The proof of the above corollary follows from Corollary 2.6.

Remark 2.9. We claim that the condition

$$\sum_{i=1}^{\infty} i q_i < \infty \tag{2.25}$$

implies (2.23). It is clear that (2.25) implies

$$M_n := \sum_{i=n}^{\infty}(i-n+1)q_i < \infty. \tag{2.26}$$

Note that $M_n \to 0$ as $n \to \infty$. Further

$$\begin{aligned}\Delta M_n = M_{n+1} - M_n &= \sum_{i=n+1}^{\infty}(i-n)q_i - \sum_{i=n}^{\infty}(i-n+1)q_i \\ &= \sum_{i=n+1}^{\infty}(i-n)q_i - \sum_{i=n}^{\infty}(i-n)q_i - \sum_{i=n}^{\infty}q_i = -\sum_{i=n}^{\infty}q_i.\end{aligned} \tag{2.27}$$

Then, summing from $n=1$ to $n=k-1$, we obtain

$$\sum_{n=1}^{k-1}\sum_{i=n}^{\infty}q_i = M_1 - M_k. \tag{2.28}$$

As $k \to \infty$, we obtain $\sum_{n=1}^{\infty}\sum_{i=n}^{\infty}q_i \to M_1 = \sum_{i=1}^{\infty} iq_i < \infty$. Hence, our claim holds.

Remark 2.10. Corollary 2.8 improves [16, Theorem 4.2] (for $m=2$ in their paper), because Parhi and Tripathy assumed G to be Lipschitzian and satisfy (H1). It may be noted in view of the above Remark 2.9, the condition we used, (2.23), is weaker than the condition (2.26) assumed in [16].

Remark 2.11. Corollary 2.6 of this paper improves and generalizes [6, Theorem 1] because we have removed the restrictions (H3) and (H1). Furthermore, in their theorem, $p_n \equiv p$, a constant, and m is an even positive integer.

Example 2.12. Consider the equation

$$\Delta^2\left(y_n - \frac{1}{n}y_{n-1}\right) + \frac{6(n-2)}{n(n-1)^2(n+1)(n+2)}y_{n-2} = 0, \quad n > 1 \tag{2.29}$$

which satisfies all the conditions of Theorem 2.4 and Corollaries 2.6 and 2.8. Hence, it has a solution, $y_n = 1 + 1/n$, which is nonoscillatory and does not tend to zero.

3. Positive solutions II

In the previous section, we obtained five results assuming condition (A1). In this section, obtain similar results for the following conditions:

(A2) $-1 < -b \le p_n \le 0$,
(A3) $-d \le p_n \le -c < -1$,
(A4) $1 < c \le p_n \le d$,

where b, c, and d are positive real numbers. Since the proofs are similar to the proofs in the previous section, we present only the sketch of the proofs, and leave some proofs as an exercise for the reader.

Theorem 3.1. *Let (A2), (H4), (H6) hold. If every solution of (1.2) oscillates or tends to zero as $n \to \infty$, then (H5) holds.*

Proof. We proceeding as in the proof of Theorem 2.2, with the following changes:

$$\mu = \max\{|G(x)| : 2(1-b) \le x \le 4\}. \tag{3.1}$$

Assuming (H4), (H6) and that (H5) does not hold, there exists N_1 such that for $n \ge N_1$,

$$\sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=0}^{i-1} q_j \mu < \frac{1-b}{2}, \qquad \sum_{i=n}^{\infty} \frac{|F_i|}{r_i} < \frac{1-b}{2}. \tag{3.2}$$

Let $S = \{y \in X : 2(1-b) \le y_n \le 4, n \ge N_1 - \tau\}$. Then, we define the operators A and B as follows:

$$(Ay)_n = \begin{cases} (Ay)_{N_1}, & N_1 - \tau \le n \le N_1, \\ p_n y_{n-m} + (3+b), & n \ge N_1; \end{cases}$$
$$(By)_n = \begin{cases} (By)_{N_1}, & N_1 - \tau \le n \le N_1, \\ \displaystyle\sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j G(y_{j-k}) - \sum_{i=n}^{\infty} \frac{F_i}{r_i}, & n \ge N_1. \end{cases} \tag{3.3}$$

Then as in Theorem 2.2, we prove the following: (i) $Ax + By \in S$, (ii) A is a contraction, and finally (iii) B is completely continuous. Then, by Lemma 2.1, there is a fixed point x_0 in S such that $Ax_0 + Bx_0 = x_0$ which is the required solution bounded below by $2(1-b) > 0$. □

Theorem 3.2. *Let (A2), (H3), (2.19), (2.20) hold. If every solution of (1.2) oscillates or tends to zero as $n \to \infty$, then (H5) holds.*

The proof of the above theorem is similar to that of Theorem 3.1.

Theorem 3.3. *Let (A3), (H4), (H6) hold. If every solution of (1.2) oscillates or tends to zero as $n \to \infty$, then (H5) holds.*

Proof. We proceed as in the proof of Theorem 2.2, with the following changes. If necessary, increment d so that $d \ge (c+2)/c$. Note that by (A3), we have $1 < c < d$ and $1/d < -1/p_{n+m} < 1/c$. Let ϵ be a positive constant with $\epsilon < (c-1)/2$. Let $h = (c-1) - \epsilon$, and $H = (d-1) + 2\epsilon/c$, so that $H > h > 0$. Let

$$\mu = \max\{|G(x)| : h \le x \le H\}. \tag{3.4}$$

Suppose that (H5) does not hold. Then from (H4) and (H6), it follows that (2.2) and (2.4) hold. Hence there exists $N_1 > 0$ such that for $n \ge N_1$,

$$\sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=0}^{i-1} q_j \mu < \frac{\epsilon}{2}, \qquad \sum_{i=n}^{\infty} \frac{|F_i|}{r_i} < \frac{\epsilon}{2}. \tag{3.5}$$

Let $S=\{y\in X: h\le y_n\le H \text{ for } n\ge N_1-\tau\}$. Then define the operators A and B as follows:

$$(Ay)_n=\begin{cases}(Ay)_{N_1}, & N_1-\tau\le n\le N_1\\ \dfrac{y_{n+m}}{p_{n+m}}-\dfrac{cd-1}{p_{n+m}}, & n\ge N_1;\end{cases}$$

$$(By)_n=\begin{cases}(By)_{N_1}, & N_1-\tau\le n\le N_1\\ -\dfrac{1}{p_{n+m}}\displaystyle\sum_{i=n+m}^{\infty}\frac{1}{r_i}\sum_{j=N_1}^{i-1}q_jG(y_{j-k}) & \\ +\dfrac{1}{p_{n+m}}\displaystyle\sum_{i=n+m}^{\infty}\frac{F_i}{r_i}, & n\ge N_1.\end{cases}\tag{3.6}$$

Then first we prove $Ax+By\in S$ when $x,y\in S$. With $x=\{x_n\}$ and $y=\{y_n\}$ in S, and $n\ge N_1$, we obtain

$$(Ax)_n+(By)_n=\frac{-1}{p_{m+n}}\left(-x_{n-m}+cd-1+\sum_{i=n+m}^{\infty}\frac{1}{r_i}\sum_{j=N_1}^{i-1}q_jG(y_{j-k})-\sum_{i=n+m}^{\infty}\frac{F_i}{r_i}\right).\tag{3.7}$$

Then

$$\begin{aligned}(Ax)_n+(By)_n&<\frac{1}{-p_{n+m}}(-x_{n+m}+cd-1+\epsilon)\\ &\le\frac{1}{c}(-h+cd-1+\epsilon)\\ &=\frac{1}{c}(-(c-1)+cd-1)+\frac{2\epsilon}{c}=(d-1)+\frac{2\epsilon}{c}=H.\end{aligned}\tag{3.8}$$

Also

$$\begin{aligned}(Ax)_n+(By)_n&>\frac{-1}{p_{n+m}}(-x_{n+m}+cd-1-\epsilon)\ge\frac{1}{d}(-H+cd-1-\epsilon)\\ &=\frac{1}{d}(-(d-1)+cd-1)-\frac{(c+2)\epsilon}{cd}\\ &=(c-1)-\frac{(c+2)\epsilon}{cd}\ge(c-1)-\epsilon=h,\quad\text{since }\frac{(c+2)}{c}\le d.\end{aligned}\tag{3.9}$$

Hence $Ax+By\in S$. Next, we show that A is a contraction in S. In fact for x,y in S and $n\ge N_1$,

$$||(Ax)_n-(Ay)_n||\le\left|\frac{1}{p_n}\right|\,|x_{n+m}-y_{n+m}|\le\frac{1}{c}\|x-y\|.\tag{3.10}$$

This implies that A is a contraction, because $0<1/c<1$. Finally, we show that B is completely continuous. For this as a first step, we observe B is obviously continuous. It is sufficient to show that BS is relatively compact. Using [7, Theorem 3.3], we need only show that BS is uniformly cauchy. Let $x \equiv \{x_n\}$ be a sequence in S. For $\eta > 0$, there exists $N^* \geq N_1$ such that, for $n \geq N^*$,

$$\sum_{i=n}^{\infty} \left| \frac{F_i}{r_i} \right| < \eta c/4, \qquad \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j \mu < \frac{\eta c}{4}. \tag{3.11}$$

Then for $n_2 > n_1 \geq N^*$, and using $(A3)$ and (3.11), we get

$$\begin{aligned} \left| (Bx)_{n_2} - (Bx)_{n_1} \right| &< \Bigg[\left| \frac{1}{p_{n_2+m}} \right| \sum_{i=n_2+m}^{\infty} \left| \frac{F_i}{r_i} \right| + \left| \frac{1}{p_{n_1+m}} \right| \sum_{i=n_1+m}^{\infty} \left| \frac{F_i}{r_i} \right| \\ &\quad + \left| \frac{1}{p_{n_2+m}} \right| \sum_{i=n_2+m}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j \mu + \left| \frac{1}{p_{n_1+m}} \right| \sum_{i=n_1+m}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j \mu \Bigg] \\ &< \frac{1}{c}(4)\left(\frac{\eta c}{4}\right) = \eta. \end{aligned} \tag{3.12}$$

Thus BS is uniformly cauchy. Hence, it is relatively compact. Then, by Lemma 2.1, there is a fixed point x_0 in S such that $Ax_0 + Bx_0 = x_0$ which is a solution of (1.2). This solution is bounded below by a positive constant; therefore it neither oscillates nor tends to zero. □

Theorem 3.4. *Let (A3), (H3), (2.19), (2.20) hold. If every solution of (1.2) oscillates or tends to zero as $n \to \infty$, then (H5) holds.*

The proof of the above theorem follows similar lines as in Theorem 3.3.

Corollary 3.5. *Let (A3), (2.19), (2.20) hold. If every solution of (1.1) oscillates or tends to zero as $n \to \infty$, then (H5) holds.*

Proof. In view of Remark 1.1(i), the proof follows lines similar to those in Theorems 3.3 and 3.4. □

The proofs under Condition (A4) are similar to those under Condition (A3). Hence we skip all the proofs, except the following one.

Theorem 3.6. *Let (A4), (H4), (H6) hold. If every bounded solution of (1.2) oscillates or tends to zero as $n \to \infty$, then (H5) holds.*

Proof. We proceed as in the proof of Theorem 3.3, with the following changes. If required decrement $c < 3$. Let $h = (d - 2c + 3)/d$ and $H = (d + c)/(c - 1)$. Then, $H > h > 0$. Let $\mu = \max\{|G(x)| : h \leq x \leq H\}$. Suppose that (H5) does not hold. Then from (H4) and (H6), one can find that $N_1 > 0$ such that for $n \geq N_1$,

$$\sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=0}^{i-1} q_j \mu < c - 1, \qquad \sum_{i=n}^{\infty} \frac{F_i}{r_i} < c - 1. \tag{3.13}$$

Then we define the operator A as

$$(Ay)_n = \begin{cases} (Ay)_{N_1}, & N_1 - \tau \leq n \leq N_1 \\ \dfrac{y_{n+m}}{p_{n+m}} + \dfrac{d+1}{p_{n+m}}, & n \geq N_1. \end{cases} \tag{3.14}$$

We define the operator B as in Theorem 3.3. We show that if $x, y \in S$, then $Ax + By \in S$. With $x = \{x_n\}$ and $y = \{y_n\}$ in S, and $n \geq N_1$, we obtain

$$(Ax)_n + (By)_n = \frac{1}{p_{m+n}}\left(x_{n+m} + d + 1 - \sum_{i=n+m}^{\infty} \frac{1}{r_i}\sum_{j=N_1}^{i-1} q_j G(y_{j-k}) + \sum_{i=n+m}^{\infty} \frac{F_i}{r_i}\right),$$

$$\begin{aligned} (Ax)_n + (By)_n &< \frac{1}{p_{n+m}}\left(x_{n+m} + d + 1 + \sum_{i=n+m}^{\infty} \frac{|F_i|}{r_i}\right) \\ &\leq \frac{1}{c}(H + (c-1) + (d+1)) = H \cdot \left(\text{since } H = \frac{d+c}{c-1}\right). \end{aligned} \tag{3.15}$$

Also

$$\begin{aligned} (Ax)_n + (By)_n &> \frac{1}{p_{n+m}}\left(d + 1 - \sum_{i=n+m}^{\infty} \frac{1}{r_i}\sum_{j=N_1}^{i-1} q_j G(y_{j-k}) + \sum_{i=n+m}^{\infty} \frac{|F_i|}{r_i}\right) \\ &\geq \frac{1}{d}(d + 1 - (c-1) - (c-1)) = \frac{d - 2c + 3}{d} = h. \end{aligned} \tag{3.16}$$

Hence $Ax + By \in S$. Then we prove A is a contraction and BS is relatively compact as in the proof of Theorem 3.3 and apply Lemma 2.1 to complete the proof. □

4. Positive solutions III

In this section, we find positive solutions for (1.2) when $p_n = \pm 1$. We consider the equations

$$\Delta(r_n \Delta(y_n + y_{n-m})) + q_n G(y_{n-k}) = f_n, \tag{4.1}$$

$$\Delta(r_n \Delta(y_n - y_{n-m})) + q_n G(y_{n-k}) = f_n. \tag{4.2}$$

For this purpose, we need the following result.

Lemma 4.1 (Schauder's Fixed Point Theorem [24]). *Let S be a closed, convex, and nonempty subset of a Banach space X. Let $B : S \to S$ be a continuous mapping such that $B(S)$ be a relatively compact subset of X. Then B has at least one fixed point in S. This means that there is an $x \in S$ such that $Bx = x$.*

For the proof of the next theorem, we would like to point out the following remark.

Remark 4.2. (i) Suppose that $a_i > 0$ for all i and $\sum_{i=1}^{\infty} a_i < \infty$. Then for any positive integer n and fixed positive integer m, it follows that $\sum_{i=n+m}^{\infty} a_i < \infty$. Then

$$\sum_{l=1}^{\infty} \sum_{i=n+(2l-1)m}^{n+2lm-1} a_i < \sum_{i=n+m}^{\infty} a_i < \infty. \tag{4.3}$$

(ii) Suppose that (H4) and (H6) hold but (H5) does not hold. Then (2.2) holds. Put $a_i = (1/r_i)\sum_{j=0}^{i-1} q_j$. Then by part (i) of this remark, we have

$$\sum_{l=1}^{\infty} \sum_{i=n+(2l-1)m}^{n+2lm-1} \frac{1}{r_i} \sum_{j=0}^{i-1} q_j < \infty. \tag{4.4}$$

Again using (H4) and (H6), we get (2.4), which implies (in view of the argument given above)

$$\sum_{l=1}^{\infty} \sum_{i=n+(2l-1)m}^{n+2lm-1} \frac{|F_i|}{r_i} < \infty. \tag{4.5}$$

Theorem 4.3. *Suppose (H4), (H6) hold. If every solution of (4.1) oscillates or tends to zero as $n \to \infty$, then (H5) holds.*

Proof. Suppose that (H5) does not hold. Then, (2.2) holds. From (H4) and (H6), we get (2.4). From Remark 4.2, it follows that (4.4) and (4.5) hold. Then we proceed as in the proof of Theorem 3.3 with the following changes. Let

$$\mu = \max\{|G(x)| : 2 \le x \le 4\}. \tag{4.6}$$

Then from (4.4) and (4.5), there exists $N_1 > 0$ such that for $n \ge N_1$,

$$\mu \sum_{l=1}^{\infty} \sum_{i=n+(2l-1)m}^{n+2lm-1} \frac{1}{r_i} \sum_{j=0}^{i-1} q_j < \frac{1}{2}, \qquad \sum_{l=1}^{\infty} \sum_{i=n+(2l-1)m}^{n+2lm-1} \frac{|F_i|}{r_i} < \frac{1}{2}. \tag{4.7}$$

Define $S = \{y \in X : 2 \le y_n \le 4, n \ge N_1 - \tau\}$, and a mapping B from S to X:

$$(By)_n = \begin{cases} (By)_{N_1}, & N_1 - \tau \le n \le N_1, \\ 3 + \sum_{l=1}^{\infty} \sum_{i=n+(2l-1)m}^{n+2lm-1} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j G(y_{j-k}) & \\ -\sum_{l=1}^{\infty} \sum_{i=n+(2l-1)m}^{n+2lm-1} \frac{F_i}{r_i}, & n \ge N_1. \end{cases} \tag{4.8}$$

Then for $y = y_n \in S$, we have $(By)_n \le 3 + (1/2) + (1/2) = 4$, and $(By)_n \ge 3 - (1/2) > 2$. Hence, $By \in S$. Then, we proceed as in the proof of Theorem 3.3, and prove that BS is relatively compact. Then, by Lemma 4.1, there is a fixed point y^0 in S such that $By_n^0 = y_n^0$. Hence,

$$y_n^0 = 3 + \sum_{l=1}^{\infty} \sum_{i=n+(2l-1)m}^{n+2lm-1} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j G(y_{j-k}^0) - \sum_{l=1}^{\infty} \sum_{i=n+(2l-1)m}^{n+2lm-1} \frac{F_i}{r_i}. \tag{4.9}$$

It follows, for $n \geq N_1$, that

$$y_n^0 + y_{n-m}^0 = 6 + \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=N_1}^{i-1} q_j G(y_{j-k}^0) - \sum_{i=n}^{\infty} \frac{F_i}{r_i}. \tag{4.10}$$

Applying Δ, multiplying by r_n, and applying Δ again, we arrive at (4.1), This solution is bounded below by a positive constant, so it does not oscillate and does not tend to zero as $n \to \infty$. □

Corollary 4.4. *Let (H4), (H6) hold. If every solution of (4.1) oscillates or tends to zero as $n \to \infty$, then (H2) holds.*

The proof of the above corollary follows from Remark 1.1(iii) and Theorem 4.3.

Theorem 4.5. *Let (H3), (2.19), (2.20) hold. If every solution of (4.1) oscillates or tends to zero as $n \to \infty$, then (H5) holds.*

The proof of the above theorem is similar to that of Theorem 4.3.

Example 4.6. Consider the equation

$$\Delta^2(y_n + y_{n-1}) + \frac{4(n-2)}{n(n+1)(n-1)(n+2)} G(y_{n-2}) = 0, \quad n > 1. \tag{4.11}$$

which satisfies all the conditions of Theorem 4.5. Hence, it admits a solution, $y_n \equiv 1$, which is not oscillatory and does not tend to zero. Here $G(u) = 1 - u$ is decreasing and does not satisfy (H1).

Theorem 4.7. *Suppose that (H6) holds. For each positive integer n, assume that*

$$\sum_{i=1}^{\infty} \sum_{l=n+mi}^{\infty} \frac{1}{r_l} \sum_{j=0}^{l-1} q_j < \infty, \tag{4.12}$$

$$\sum_{i=1}^{\infty} \sum_{l=n+mi}^{\infty} \frac{1}{r_l} < \infty. \tag{4.13}$$

Then (4.2) has a solution bounded below by a positive constant.

Proof. We proceeding as in the proof of Theorem 3.3 with the following changes. Let $\mu = \max\{|G(x)| : 2 \leq x \leq 4\}$. Then from (H6), (4.12), and (4.13), there exists $N_1 > 0$ such that for $n \geq N_1$,

$$\sum_{i=1}^{\infty} \sum_{l=n+mi}^{\infty} \frac{|F_l|}{r_l} < \frac{1}{2}, \qquad \sum_{i=1}^{\infty} \sum_{l=n+mi}^{\infty} \frac{1}{r_l} \sum_{j=0}^{l-1} q_j \mu < \frac{1}{2}. \tag{4.14}$$

Let $S = \{y \in X : 2 \le y_n \le 4, n \ge N_1 - \tau\}$. Then define the mapping

$$(By)_n = \begin{cases} (By)_{N_1}, & N_1 - \tau \le n \le N_1, \\ 3 - \sum_{i=1}^{\infty} \sum_{l=n+mi}^{\infty} \frac{1}{r_l} \sum_{j=N_1}^{l-1} q_j G(y_{j-k}) & \\ +\sum_{i=1}^{\infty} \sum_{l=n+mi}^{\infty} \frac{F_l}{r_l}, & n \ge N_1. \end{cases} \tag{4.15}$$

Then for $y = \{y_n\} \in S$, we have $(By)_n \le 3 + (1/2) < 4$ and $(By)_n \ge 3 - (1/2) - (1/2) = 2$. Hence, $By \in S$. Then using (4.12) and (4.13), we proceed as in the proof of Theorem 3.3 and prove that BS is relatively compact. Then, by Lemma 4.1, there is a fixed point y^0 in S such that $By_n^0 = y_n^0$. Hence,

$$y_n^0 = 3 - \sum_{i=1}^{\infty} \sum_{l=n+mi}^{\infty} \frac{1}{r_l} \sum_{j=N_1}^{l-1} q_j G(y_{j-k}^0) + \sum_{i=1}^{\infty} \sum_{l=n+mi}^{\infty} \frac{F_l}{r_l}. \tag{4.16}$$

For $n \ge N_1$, it follows that

$$y_n^0 - y_{n-m}^0 = \sum_{l=n}^{\infty} \frac{1}{r_l} \sum_{j=0}^{l-1} q_j G(y_{j-k}^0) - \sum_{l=n}^{\infty} \frac{F_l}{r_l}. \tag{4.17}$$

Applying Δ, multiplying by r_n, and applying Δ again, we arrive at (4.2). This solution is bounded below by 2 which is a positive constant. □

Remark 4.8. All the results in Sections 2 and 3 hold for NDDE (1.1) and the results of this section hold for the corresponding homogeneous NDDE associated with (4.1) and (4.2).

We close this article with an interesting example which illustrates our results, whereas most of the results available in the literature are not applicable to this example.

Example 4.9. Consider the equation

$$\Delta(r_n \Delta(y_n \pm p y_{n-1})) + \frac{1}{n^4} G(y_{n-2}) = \frac{G(1)}{n^4}, \quad n > 0, \tag{4.18}$$

where p is a constant in any range of $\{p_n\}$ considered in this paper. The sequence $\{r_n\}$ is positive and may satisfy (H3) or (4.13). If $\{r_n\}$ satisfies (4.13), then it satisfies (H4). The function G is continuous. The sequence $q_n = 1/n^4$ satisfies (2.2), (2.25), and (4.12) with a proper selection of r_n. To verify this, we refer to Remark 2.9. Here, $f_n = G(1)/n^4$. Hence, $F_n = -\sum_{i=n}^{\infty} G(1)/i^4$ which satisfies (2.20). Hence (H6) is satisfied. This NDDE satisfied the conditions of all the results of this paper. Hence, it admits a solution, $y_n \equiv 1$, which is bounded below by a positive constant. Since we have no restriction on G, most of the results available in the literature [6, 12, 15, 16] are not applicable to this NDDE; because G may not satisfy (H1).

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