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Research Article

Best Simultaneous Approximation in Orlicz Spaces

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Let X be a Banach space and let $L^{\Phi}(I,X)$ denote the space of Orlicz X-valued integrable functions on the unit interval I equipped with the Luxemburg norm. In this paper, we present a distance formula $\text{dist}_{\Phi}(f_1,f_2,L^{\Phi}(I,G))$, where G is a closed subspace of X, and $f_1,f_2 \in L^{\Phi}(I,X)$. Moreover, some related results concerning best simultaneous approximation in $L^{\Phi}(I,X)$ are presented.

1. Introduction

A function $\Phi : (-\infty,\infty) \to [0,\infty)$ is called an Orlicz function if it satisfies the following conditions:

(1) Φ is even, continuous, convex, and $\Phi(0) = 0$;

(2) $\Phi(x) > 0$ for all $x \neq 0$;

(3) $\lim_{x\to 0}\Phi(x)/x = 0$ and $\lim_{x\to\infty}\Phi(x)/x = \infty$.

We say that a function Φ satisfies the Δ_2 condition if there are constants $k > 1$ and $x_0 > 0$ such that $\Phi(2x) \le k\Phi(x)$ for $x > x_0$. Examples of Orlicz functions that satisfy the Δ_2 conditions are widely available such as $\Phi(x) = |x|^p$, $1 \le p < \infty$, and $\Phi(x) = (1+|x|)\log(1+|x|) - |x|$. In fact, Orlicz functions are considered to be a subclass of Young functions defined in [1].

Let X be a Banach space and let (I,μ) be a measure space. For an Orlicz function Φ, let $L^{\Phi}(I,X)$ be the Orlicz-Bochner function space that consists of strongly measurable functions $f : I \to X$ with $\int_I \Phi(\alpha\|f\|)d\mu(t) < \infty$ for some $\alpha > 0$. It is known that $L^{\Phi}(I,X)$ is a Banach space under the Luxemburg norm

$$\|f\|_\Phi = \inf\left\{k > 0, \int_I \Phi\left(\frac{1}{k}\|f\|\right) d\mu(t) \le 1\right\}. \tag{1.1}$$

It should be remarked that if $\Phi(x) = |x|^p$, $1 \le p < \infty$, the space $L^\Phi(I,X)$ is simply the p-Lebesgue Bochner function space $L^p(I,X)$ with

$$\|f\|_\Phi = \Phi^{-1}\int_I \Phi(\|f\|) d\mu(t) = \left(\int_I \|f\|^p d\mu(t)\right)^{1/p} = \|f\|_p. \tag{1.2}$$

On the other hand, if $\Phi(x) = (1+|x|)\log(1+|x|) - |x|$, then the space $L^\Phi(I,X)$ is the well-known Zygmund space, $L\log L^+$. For excellent monographs on $L^\Phi(I,X)$, we refer the readers to [1–3].

For a function $F = (f_1, f_2) \in (L^\Phi(I,X))^2$, we define $\|F\|$ by

$$\|F\| = \big\| \|f_1(\cdot)\| + \|f_2(\cdot)\| \big\|_\Phi. \tag{1.3}$$

In this paper, for a given closed subspace G of X and $F = (f_1, f_2) \in (L^\Phi(I,X))^2$, we show the existence of a pair $G_0 = (g_0, g_0) \in (L^\Phi(I,G))^2$ such that

$$\|F - G_0\| = \inf_{g\in G}\|F - (g,g)\|. \tag{1.4}$$

If such a function g exists, it is called a best simultaneous approximation of $F = (f_1, f_2)$. The problem of best simultaneous approximation can be viewed as a special case of vector-valued approximation. Recent results in this area are due to Pinkus [4], where he considered the problem when a finite-dimensional subspace is a unicity space. Characterization results for linear problems were given in [5] based on the derivation of an expression for the directional derivative, and these results generalize the earlier results presented in [6]. Results on best simultaneous approximation in general Banach spaces may be found in [7, 8]. Related results on $L^p(I,X)$, $1 \le p < \infty$, are given in [9]. In [9], it is shown that if G is a reflexive subspace of a Banach space X, then $L^p(I,G)$ is simultaneously proximinal in $L^p(I,X)$. If $L^\Phi(I,X) = L^1(I,X)$, Abu-Sarhan and Khalil [10] proved that if G is a reflexive subspace of the Banach space X or G is a 1-summand subspace of X, then $L^1(I,G)$ is simultaneously proximinal in $L^1(I,X)$.

It is the aim of this work to prove a distance formula $\operatorname{dist}_\Phi(f_1, f_2, L^\Phi(I,G))$, where $f_1, f_2 \in L^\Phi(I,X)$, similar to that of best approximation. This will allow us to generalize some recent results on $L^1(I,X)$ to $L^\Phi(I,X)$.

Throughout this paper, X is a Banach space, Φ is an Orlicz function, and $L^\Phi(I,X)$ is the Orlicz-Bochner function space equipped with the Luxemburg norm.

2. Distance formula

Let G be a closed subspace of X. For $x, y \in X$, define

$$\operatorname{dist}(x, y, G) = \inf_{z\in G}\|x - z\| + \|y - z\|. \tag{2.1}$$

For $f_1, f_2 \in L^\Phi(I,X)$, we define $\operatorname{dist}_\Phi(f_1, f_2, L^\Phi(I,G))$ by

$$\begin{aligned}\operatorname{dist}_\Phi\left(f_1, f_2, L^\Phi(I,G)\right) &= \inf_{g\in L^\Phi(I,G)} \|(f_1, f_2) - (g,g)\| \\ &= \inf_{g\in L^\Phi(I,G)} \big\| \|f_1(\cdot) - g(\cdot)\| + \|f_2(\cdot) - g(\cdot)\| \big\|_\Phi.\end{aligned} \tag{2.2}$$

Our main result is the following.

Theorem 2.1. *Let G be a subspace of the Banach space X and let Φ be an Orlicz function that satisfies the Δ_2 condition. If $f_1, f_2 \in L^\Phi(I,X)$, then the function* $\operatorname{dist}(f_1(\cdot), f_2(\cdot), G)$ *belongs to $L^\Phi(I)$ and*

$$\|\operatorname{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi = \operatorname{dist}_\Phi\left(f_1, f_2, L^\Phi(I,G)\right). \tag{2.3}$$

Proof. Let $f_1, f_2 \in L^\Phi(I,X)$. Then there exist two sequences $(f_{n,1})$, $(f_{n,2})$ of simple functions in $L^\Phi(I,X)$ such that

$$\|f_{n,1}(t) - f_1(t)\| \longrightarrow 0, \quad \|f_{n,2}(t) - f_2(t)\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty \tag{2.4}$$

for almost all t in I. The continuity of $\operatorname{dist}(x, y, G)$ implies that

$$|\operatorname{dist}(f_{n,1}(t), f_{n,2}(t), G) - \operatorname{dist}(f_1(t), f_2(t), G)| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.5}$$

Set $H_n(t) = \operatorname{dist}(f_{n,1}(t), f_{n,2}(t), G)$. Then each H_n is a measurable function. Thus $\operatorname{dist}(f_1(\cdot), f_2(\cdot), G)$ is measurable and

$$\operatorname{dist}(f_1(t), f_2(t), G) \le \|f_1(t) - z\| + \|f_2(t) - z\| \tag{2.6}$$

for all z in G. Therefore,

$$\operatorname{dist}(f_1(t), f_2(t), G) \le \|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \tag{2.7}$$

for all $g \in L^\Phi(I,G)$. Thus

$$\|\operatorname{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi \le \big\| \|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \big\|_\Phi \tag{2.8}$$

for all $g \in L^\Phi(I,G)$. Hence $\operatorname{dist}(f_1(\cdot), f_2(\cdot), G) \in L^\Phi(I)$ and

$$\|\operatorname{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi \le \operatorname{dist}_\Phi\left(f_1, f_2, L^\Phi(I,G)\right). \tag{2.9}$$

Fix $\epsilon > 0$. Since the set of simple functions are dense in $L^\Phi(I,X)$, there exist simple functions f_i^* in $L^\Phi(I,X)$ such that $\|f_i - f_i^*\|_\Phi \le \epsilon/6$ for $i = 1,2$. Assume that $f_i^*(t) = \sum_{k=1}^n x_k^i \chi_{A_k}(t)$ with A_k's are measurable sets, $x_k^i \in X$, $k = 1,2,\ldots,n$, $i = 1,2$, $A_k \cap A_j = \phi$, $k \ne j$, and $\bigcup_{k=1}^n A_k = I$. We can assume that $\mu(A_k) > 0$ and $\Phi(1) \le 1$. For each $k = 1,2,\ldots,n$, let $y_k \in G$ be such that

$$\|x_k^1 - y_k\| + \|x_k^2 - y_k\| \le \operatorname{dist}(x_k^1, x_k^2, G) + \frac{\epsilon}{3}. \tag{2.10}$$

Set $g(t) = \sum_{k=1}^{n} y_k \chi_{A_k}(t)$ and

$$F(t) = \operatorname{dist}(f_1(t), f_2(t), G) + \|f_1(t) - f_1^*(t)\| + \|f_2(t) - f_2^*(t)\| + \frac{\epsilon}{3}. \tag{2.11}$$

Then

$$\begin{aligned}
&\int_I \Phi\left(\frac{\|f_1^*(t)-g(t)\| + \|f_2^*(t)-g(t)\|}{\|F\|_\Phi}\right) d\mu(t) \\
&\quad = \sum_{k=1}^{n} \int_{A_k} \Phi\left(\frac{\|f_1^*(t)-g(t)\| + \|f_2^*(t)-g(t)\|}{\|F\|_\Phi}\right) d\mu(t) \\
&\quad = \sum_{k=1}^{n} \int_{A_k} \Phi\left(\frac{\|x_k^1 - y_k\| + \|x_k^2 - y_k\|}{\|F\|_\Phi}\right) d\mu(t) \\
&\quad < \sum_{k=1}^{n} \int_{A_k} \Phi\left(\frac{\operatorname{dist}(x_k^1, x_k^2, G) + \epsilon/3}{\|F\|_\Phi}\right) d\mu(t) \\
&\quad = \int_I \Phi\left(\frac{\operatorname{dist}(f_1^*(t), f_2^*(t), G) + \epsilon/3}{\|F\|_\Phi}\right) d\mu(t) \\
&\quad \le \int_I \Phi\left(\frac{\|f_1(t) - f_1^*(t)\| + \|f_2(t) - f_2^*(t)\| + \operatorname{dist}(f_1(t), f_2(t), G) + \epsilon/3}{\|F\|_\Phi}\right) d\mu(t) \\
&\quad = \int_I \Phi\left(\frac{F(t)}{\|F\|_\Phi}\right) d\mu(t) \le 1.
\end{aligned} \tag{2.12}$$

Consequently,

$$\big\| \|f_1^*(\cdot) - g(\cdot)\| + \|f_2^*(\cdot) - g(\cdot)\| \big\|_\Phi \le \left\| \begin{matrix} \|f_1(\cdot) - f_1^*(\cdot)\| + \|f_2(\cdot) - f_2^*(\cdot)\| \\ + \operatorname{dist}(f_1(\cdot), f_2(\cdot), G) + \dfrac{\epsilon}{3} \end{matrix} \right\|_\Phi . \tag{2.13}$$

Notice that

$$\begin{aligned}
\operatorname{dist}_\Phi(f_1, f_2, L^\Phi(I,G)) &\le \operatorname{dist}_\Phi(f_1^*, f_2^*, L^\Phi(I,G)) + \|f_1 - f_1^*\|_\Phi + \|f_2 - f_2^*\|_\Phi \\
&< \frac{\epsilon}{3} + \big\| \|f_1^*(\cdot) - g(\cdot)\| + \|f_2^*(\cdot) - g(\cdot)\| \big\|_\Phi \\
&\le \frac{\epsilon}{3} + \left\| \begin{matrix} \operatorname{dist}(f_1(\cdot), f_2(\cdot), G) + \|f_1(\cdot) - f_1^*(\cdot)\| \\ + \|f_2(\cdot) - f_2^*(\cdot)\| + \dfrac{\epsilon}{3} \end{matrix} \right\|_\Phi \\
&\le \frac{2\epsilon}{3} + \|\operatorname{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi \\
&\quad + \|f_1(\cdot) - f_1^*(\cdot)\|_\Phi + \|f_2(\cdot) - f_2^*(\cdot)\|_\Phi \\
&\le \epsilon + \|\operatorname{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi,
\end{aligned} \tag{2.14}$$

which (since ϵ is arbitrary) implies that

$$\text{dist}_\Phi\left(f_1, f_2, L^\Phi(I,G)\right) \leq \|\text{dist}\left(f_1(\cdot), f_2(\cdot), G\right)\|_\Phi. \tag{2.15}$$

Hence by (2.9) and (2.15) the proof is complete. □

A direct consequence of Theorem 2.1 is the following result.

Theorem 2.2. *Let G be a closed subspace of the Banach space X and let Φ be an Orlicz function that satisfies the Δ_2 condition. For $g \in L^\Phi(I,G)$ to be a best simultaneous approximation of a pair of elements (f_1, f_2) in $L^\Phi(I,G)$, it is necessary and sufficient that $g(t)$ is a best simultaneous approximation of $(f_1(t), f_2(t))$ in G for almost all $t \in I$.*

3. Proximinality of $L^\Phi(I,G)$ in $L^\Phi(I,X)$

A closed subspace G of X is called 1-summand in X if there exists a closed subspace Y such that $X = G\bigoplus_1 Y$, that is, any element $x \in X$ can be written as $x = g + y, g \in G, y \in Y$, and $\|x\| = \|g\| + \|y\|$. It is known that a 1-summand subspace G of X is proximinal in X, and $L^1(I,G)$ is proximinal in $L^1(I,X)$, [11].

Our first result in this section is the following.

Theorem 3.1. *If G is simultaneously proximinal in X, then every pair of simple functions admits a best simultaneous approximation in $L^\Phi(I,G)$.*

Proof. Let f_1, f_2 be two simple functions in $L^\Phi(I,X)$. Then f_1, f_2 can be written as $f_1(s) = \sum_{k=1}^n u_k^1 \chi_{I_k}(s)$, $f_2(s) = \sum_{k=1}^n u_k^2 \chi_{I_k}(s)$, where I_k's are disjoint measurable subsets of I satisfying $\bigcup_{k=1}^n I_k = I$, and χ_{I_k} is the characteristic function of I_k. Since f_1 and f_2 represent classes of functions, we may assume that $\mu(I_k) > 0$ for each $1 \leq k \leq n$. By assumption, we know that for each $1 \leq k \leq n$ there exists a best simultaneous approximation w_k in G of the pair of elements $(u_k^1, u_k^2) \in X^2$ such that

$$\text{dist}\left(u_k^1, u_k^2, G\right) = \|u_k^1 - w_k\| + \|u_k^2 - w_k\|. \tag{3.1}$$

Set $g = \sum_{k=1}^n w_k \chi_{I_k}(s)$. Then, for any $\alpha > 0$ and $h \in L^\Phi(I,G)$, we obtain that

$$\begin{aligned}
\int_I \Phi\left(\frac{\|f_1(t) - h(t)\| + \|f_2(t) - h(t)\|}{\alpha}\right) d\mu(t) &= \sum_{k=1}^n \int_{I_k} \Phi\left(\frac{\|u_k^1 - h(t)\| + \|u_k^2 - h(t)\|}{\alpha}\right) d\mu(t) \\
&\geq \sum_{k=1}^n \int_{I_k} \Phi\left(\frac{\|u_k^1 - w_k\| + \|u_k^2 - w_k\|}{\alpha}\right) d\mu(t) \\
&= \int_I \Phi\left(\frac{\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\|}{\alpha}\right) d\mu(t).
\end{aligned} \tag{3.2}$$

Taking the infimum over all such α's, we have that

$$\big\| \|f_1(\cdot) - h(\cdot)\| + \|f_2(t) - h(\cdot)\| \big\|_\Phi \geq \big\| \|f_1(\cdot) - g(\cdot)\| + \|f_2(t) - g(\cdot)\| \big\|_\Phi \tag{3.3}$$

for all $h \in L^{\Phi}(I,G)$. Hence

$$\begin{aligned} \operatorname{dist}_{\Phi}(f_1, f_2, L^{\Phi}(I,G)) &= \| \| f_1(\cdot) - g(\cdot)\| + \|f_2(t) - g(\cdot)\| \|_{\Phi} \\ &\geq \| \| f_1(\cdot) - h(\cdot)\| + \|f_2(\cdot) - h(\cdot)\| \|_{\Phi}. \end{aligned} \tag{3.4}$$

□

Now we prove the following 2-dimensional analogous of [12, Theorem 4].

Theorem 3.2. *Let G be a closed subspace of the Banach space X and let Φ be an Orlicz function that satisfies the Δ_2 condition. If $L^1(I,G)$ is simultaneously proximinal in $L^1(I,X)$, then $L^{\Phi}(I,G)$ is simultaneously proximinal in $L^{\Phi}(I,X)$.*

Proof. Let $f_1, f_2 \in L^{\Phi}(I,X)$. Then $f_1, f_2 \in L^1(I,X)$; see [13]. By assumption, there exists $g \in L^1(I,G)$ such that

$$\| \| f_1(\cdot) - g(\cdot)\| + \|f_2(\cdot) - g(\cdot)\| \|_1 \leq \| \| f_1(\cdot) - h(\cdot)\| + \|f_2(\cdot) - h(\cdot)\| \|_1 \tag{3.5}$$

for every $h \in L^1(I,G)$. By Theorem 2.2 [10],

$$\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \leq \|f_1(t) - h(t)\| + \|f_2(t) - h(t)\| \tag{3.6}$$

for almost all t in I. But $0 \in G$. Thus

$$\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \leq \|f_1(t)\| + \|f_2(t)\| \tag{3.7}$$

or

$$\|g(t)\| \leq \|f_1(t)\| + \|f_2(t)\|. \tag{3.8}$$

Hence $g \in L^{\Phi}(I,G)$ and

$$\| \| f_1(\cdot) - g(\cdot)\| + \|f_2(\cdot) - g(\cdot)\| \|_{\Phi} \leq \| \| f_1(\cdot) - h(\cdot)\| + \|f_2(\cdot) - h(\cdot)\| \|_{\Phi} \tag{3.9}$$

for all $h \in L^1(I,G)$. □

Theorem 3.3. *Let G be a 1-summand subspace of the Banach space X. Then $L^{\Phi}(I,G)$ is simultaneously proximinal in $L^{\Phi}(I,X)$.*

The proof follows from Theorem 3.2 and [10, Theorem 2.4].

Theorem 3.4. *Let G be a reflexive subspace of the Banach space X. Then $L^{\Phi}(I,G)$ is simultaneously proximinal in $L^{\Phi}(I,X)$.*

The proof follows from Theorem 3.2 and [10, Theorem 3.2].

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