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Review Article

Extending Hall's Theorem into List Colorings: A Partial History

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Recommended by Eugene H. Dshalalow

In 1988, A. J. W. Hilton and P. D. Johnson Jr. found a natural generalization of the condition in Philip Hall's celebrated theorem on systems of distinct representatives. This generalization was formed in the relatively new theory of list colorings of graphs. Here we give an account of a strand of development arising from this generalization, concentrating on extensions of Hall's theorem. New results are presented concerning list colorings of independence systems and colorings of graphs with nonnegative measurable functions on positive measure spaces.

1. List colorings of graphs

Throughout, G will be a finite simple graph, with vertex set $V = V(G)$ and edge set $E = E(G)$, C will be an infinite set (of "colors" or symbols), and $\mathcal{F}$ will be the collection of finite subsets of C. A *list assignment* to G is a function $L : V \to \mathcal{F}$. If L is a list assignment to G, a *proper L-coloring* of G is a function $\varphi : V \to C$ satisfying, for all $u, v \in V$,

(i) $\varphi(u) \in L(u)$;

(ii) if $uv \in E$ then $\varphi(u) \neq \varphi(v)$.

Notice that (ii) is equivalent to

(ii)' for each $\sigma \in C$, $\varphi^{-1}(\sigma) = \{v \in V \mid \varphi(v) = \sigma\}$ is independent

(this means that no two vertices of $\varphi^{-1}(\sigma)$ are adjacent in G).

For example, in Figure 1.1 there are two graphs with list assignments indicated (set brackets are omitted). In one case there are two proper colorings from the lists assigned, while in the other there is no proper coloring.

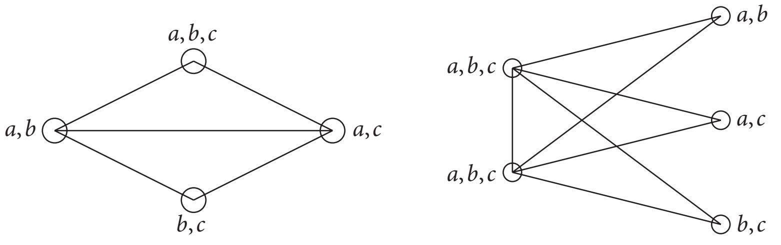
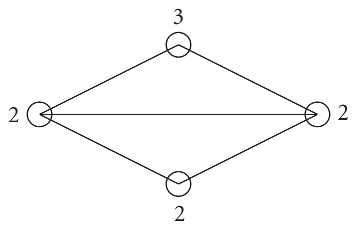


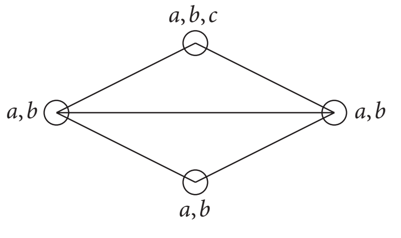
Figure 1.1

Graph list colorings were first studied by Vizing [1] and then, independently, by Erdös et al. [2]. Vizing's paper was in Russian, and was unnoticed in the West until the publication of [2]. Erdös et al., were inspired by Dinitz's problem, posed by Jeff Dinitz at one of the Southeastern Combinatorics Conferences: if the cells of an $n \times n$ array are assigned sets of size n, can representatives of these sets necessarily be found for the cells so that no representative occurs more than once in any row or column of the array? Clearly this question can be posed as a question about graph list coloring.

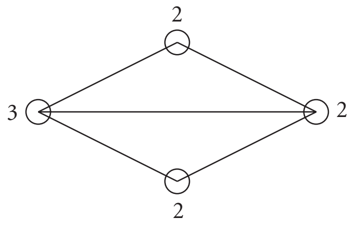
The early focus of attention was on how big (or "long") the lists would have to be to ensure the existence of a proper coloring. The central definition in [2] is this: supposing f to be a function from V into the positive integers, G is said to be f-*choosable* if there is a proper L-coloring of G whenever L satisfies $|L(v)| \geq f(v)$ for all $v \in V$. For example, the first graph in Figure 1.1 (K_4-minus-on-edge, or $\theta(1,2,2)$), in spite of its proper colorability from the list assignment there, is not f-choosable, if f is defined by



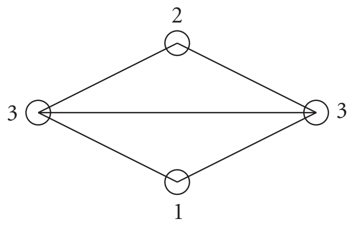
Here is a list assignment showing that $\theta(1,2,2)$ is not f-choosable:



(we leave it as a recreation to show that $\theta(1,2,2)$ is



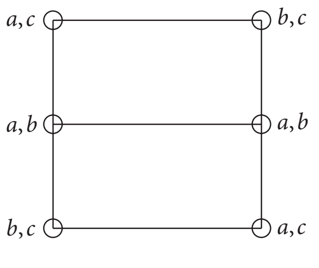
choosable, but not



choosable).

The graph parameter of greatest importance in list-coloring theory is the *choice number* or *list chromatic number*, denoted here by ch; $\mathrm{ch}(G)$ is the smallest constant function f such that G is f-choosable. It is easy to see that the choice number of each graph in Figure 1.1 is 3, which also happens to be the chromatic number of these graphs. Since the chromatic number χ is definable as ch is defined, with the restriction that the list assignment is to be constant, it is clear that $\chi(G) \leq \mathrm{ch}(G)$ for all G. The extremal question of when $\chi(G) = \mathrm{ch}(G)$ is and most likely will continue to be an inspirational challenge. Dinitz's problem, restated, is: does equality hold when G is the line graph of $K_{n,n}$? Galvin's theorem [3] solves Dinitz's problem and more: $\chi(G) = \mathrm{ch}(G)$ when G is the line graph of a bipartite multigraph. One fundamental unanswered question about list coloring was first posed by Vizing [1]: does $\chi(G) = \mathrm{ch}(G)$ hold whenever G is a line graph? (At least one of us believes that the answer to this question is probably no, and that the question has survived largely because of the difficulty of finding choice numbers.)

The smallest graph whose choice number exceeds its chromatic number is $K_{3,3}$-minus-two-independent-edges, depicted below with a list assignment that shows that its choice number is > 2. (It follows from Brooks-theorem-for-the-choice-number, proven in [2] and [1], that the choice number of this graph is 3.)



2. List size is not everything: Hall's Condition

In 1988, A. J. W. Hilton noticed something about list-coloring problems that had probably been noticed numerous times before, but evidently never before with a substantive outcome: given G and L, the problem of finding a proper L-coloring of G can be thought of as the problem of finding a "system of G-distinct representatives of the sets $L(v)$, $v \in V(G)$," that is, an indexed collection $[\varphi_v;\ v \in V(G)]$ such that for all $u,v \in V(G)$, $\varphi_v \in L(v)$ and if $uv \in E(G)$ then $\varphi_v \neq \varphi_u$.

The venerable grandmother of all systems-of-distinct-representatives results is Hall's theorem [4], which asserts that, given a finite indexed collection of sets, a certain obviously necessary condition for the existence of a system of distinct representatives of the sets in the collection is sufficient for the existence. Since a proper L-coloring of the complete graph K_n is, in fact, a system of distinct representatives of the sets $L(v)$, $v \in V(K_n)$, and since the vertex set of K_n can serve as an index set as well as any other finite set, it turns out that Hall's theorem can be viewed as a theorem about list colorings of graphs, over 40 years before these were defined.

Theorem 2.1 (Hall's theorem). *Suppose that L is a list assignment to K_n. There is a proper L-coloring of K_n if and only if, for all $U \subseteq V(K_n)$,*

$$\left| \bigcup_{u\in U} L(u) \right| \geq |U|. \tag{2.1}$$

There may be a myriad of ways to generalize the condition in Hall's theorem to a statement applicable to list colorings of graphs, but what follows seemed in 1988, and seems now, to suit the purpose in a natural way. Suppose that L is a list assignment to G and H is a subgraph of G. For each $\sigma \in C$, let $\alpha(\sigma,L,H)$ denote the vertex independence number of the subgraph of H induced by $\{v \in V(H) \mid \sigma \in L(v)\}$; that is, $\alpha(\sigma,L,H)$ is the largest size of a set of mutually nonadjacent vertices of H, among those bearing σ on their lists. G and L *satisfy Hall's condition* (HC) if and only if, for each subgraph H of G,

$$\sum_{\sigma\in C} \alpha(\sigma,L,H) \geq |V(H)|. \tag{$*$}$$

Since removing edges does not decrease vertex independence numbers, for G and L to satisfy HC it suffices that $(*)$ holds for all induced subgraphs H of G. If G is a complete graph on vertex set V, and $U \subseteq V$, then the subgraph H of G induced by U is complete, and so, for each $\sigma \in C$,

$$\alpha(\sigma,L,H) = \begin{cases} 1 & \text{if } \sigma \in \bigcup_{u\in U} L(u), \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

That is, $\alpha(\cdot,L,H)$ is the characteristic function of $\bigcup_{u\in U} L(u)$, and so $\sum_{\sigma\in C}\alpha(\sigma,L,H) = |\bigcup_{u\in U} L(u)|$. Thus Hall's condition boils down to the condition in Hall's theorem when G is complete.

If there is a proper L-coloring of G, then for each subgraph H of G there is a proper L-coloring of H; since, for each $\sigma \in C$, $\alpha(\sigma,L,H)$ is an upper bound on the number of vertices of H on which σ could appear in a proper L-coloring of H, it is clear that $(*)$ must hold. That is, HC is necessary for the existence of a proper coloring.

We will now give an apparently much more complicated proof of this elementary fact; the investment of pain will be repaid later. For $S \subseteq C$ let $\mathrm{char}_s : C \to \{0,1\}$ denote the characteristic function of S. (χ_S is more conventional notation, but, in graph theory, χ is reserved for the chromatic number.)

Lemma 2.2. *With G, L, and H as above, for each $\sigma \in C$,*

$$\alpha(\sigma,L,H) = \max\left[\sum_{v\in S}\mathrm{char}_{L(v)}(\sigma) : S \text{ is an independent set of vertices of } H\right]. \tag{2.3}$$

The proof is straightforward.

Proposition 2.3. *If there is a proper L-coloring of G then G and L satisfy HC.*

Proof. Suppose that φ is a proper L-coloring of G and H is a subgraph of G. Then

$$\begin{aligned}
|V(H)| &= \sum_{v\in V(H)}\sum_{\sigma\in C}\mathrm{char}_{\{\varphi(v)\}}(\sigma) = \sum_{\sigma\in C}\sum_{v\in\varphi^{-1}(\sigma)\cap V(H)}\mathrm{char}_{\{\varphi(v)\}}(\sigma)\\
&\le \sum_{\sigma\in C}\sum_{v\in\varphi^{-1}(\sigma)\cap V(H)}\mathrm{char}_{L(v)}(\sigma) \quad [\text{because } \{\varphi(v)\}\subseteq L(v)\ \forall v]\\
&\le \sum_{\sigma\in C}\max_{\substack{S\subseteq V(H)\\ S \text{ independent}}}\sum_{v\in S}\mathrm{char}_{L(v)}(\sigma) \qquad (2.4)\\
&\qquad [\text{because } \varphi^{-1}(\sigma)\cap V(H) \text{ is independent, for each } \sigma\in C]\\
&= \sum_{\sigma\in C}\alpha(\sigma,L,H) \quad (\text{Lemma 2.2}).
\end{aligned}$$

□

3. Hall's condition and graph colorings, 1989–2003

The content of Hall's theorem is that when G is complete, Hall's condition is sufficient for the existence of a proper L-coloring of G. The first question that arose after the formulation of Hall's condition (for line graphs in [5] and for all graphs in [6]) was: which graphs share this property of the complete graphs? The early hope was that the class of such graphs would be quite large.

This hope was smothered at birth by the easy discovery of small graphs without the property. The smallest such graph is C_4, depicted in Figure 3.1 with a list assignment satisfying Hall's condition from which no proper coloring is possible.

(A note on verifying HC: in principle, one has to verify the inequality $(*)$ for all induced subgraphs H of G, but the necessity of HC for a proper coloring very often diminishes this burden to tolerable proportions. For instance, in Figure 3.1, it is very easy to see that $G - v$ is properly L-colorable for each $v \in V(G)$, which takes care of $(*)$ for all induced subgraphs H of G except G itself. Then one calculates $\alpha(a,L,G) = \alpha(b,L,G) = 1$, $\alpha(c,L,G) = 2$, so $\sum_{\sigma\in C}\alpha(\sigma,L,G) = 4 \ge 4 = |V(G)|$.)

Observe that one can draw the edge in Figure 3.1 from the vertex with list a, b to the vertex with list c to obtain another small graph for which HC is not sufficient for a proper

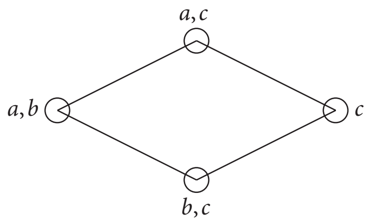


Figure 3.1

coloring, our old friend K_4-minus-an-edge. But it turns out that what is going on with these two graphs, and the list assignment in Figure 3.1, is essentially the only thing about a graph that makes it possible to have a list assignment satisfying HC from which no proper coloring of the graph is possible. The main assertion of the following theorem was proven in [6], with the forbidden-induced subgraph characterization added in [7]. A *block* of a graph is a subgraph maximal with respect to the properties of being connected and containing no cut vertex (of itself).

Theorem 3.1 (Theorem HJW [6, 7]). *The following are equivalent:*
(a) *for every list assignment L to G such that G and L satisfy HC, there is a proper L-coloring of G;*
(b) *every block of G is a clique;*
(c) *G has no induced subgraph isomorphic to C_n, for any $n \geq 4$, nor to K_4-minus-an-edge.*

So it turned out that Hall's theorem did not extend very far in the world of graph list-colorings, just the short stroll from complete graphs to the graphs whose components are made from complete graphs by sticking them together at cut vertices. There ensued efforts—still going on, by the way—to exploit Hall's condition as a tool in list-coloring theory by combining it with other conditions. For instance, the restatement of Ryser's theorem on completing Latin squares given in [5] can be reconstrued to say that for a special kind of graph with a special kind of list assignment, Hall's condition is sufficient for the existence of a proper coloring. For another instance, much work has been done on the *Hall number*: the Hall number $h(G)$ of a graph G is the smallest positive integer m such that there is a proper L-coloring of G whenever $|L(v)| \geq m$ for all $v \in V(G)$ and G and L satisfy Hall's condition. (See [7–9].)

However, we will leave the account of these hybrid ventures for another time, for it turns out that there is much more to tell about "direct" extensions of Hall's theorem. Most of these are founded on an early improvement of Hall's theorem probably noticed independently by numbers of mathematicians, but certainly by Rado [10] and then Halmos and Vaughan [11]. We state it here in a form fairly close to the original.

Theorem 3.2 (Theorem RHV). *Suppose that $A_1,\dots,A_n$ are sets, $k_1,\dots,k_n$ are positive integers, and for all $J \subseteq \{1,\dots,n\}$,*

$$\left|\bigcup_{j\in J} A_j\right| \geq \sum_{j\in J} k_j. \tag{3.1}$$

Then there exist pairwise disjoint $B_1,\ldots,B_n$ *such that* $B_i \subseteq A_i$, *and* $|B_i| = k_i$, $i = 1, \ldots, n$.

Hall's theorem is the special case $k_i = 1$, $i = 1,\ldots,n$; Theorem RHV is easily derived from this special case. Suppose that Hall's theorem is true (and it is), and $[A_i,\ k_i;\ i = 1,\ldots,n]$ satisfy the hypothesis of Theorem RHV. Make a new "list" or indexed family of sets by repeating A_i k_i times for each i. The new indexed family satisfies the hypothesis of Hall's theorem and so there is a "system" of distinct representatives of the new indexed family of sets. Collect the k_i distinct representatives of the k_i copies of A_i into a set B_i, for each i, and Theorem RHV is proved.

Both Hall's theorem and Theorem RHV can be given as biconditional statements; in Theorem RHV the condition claimed to be sufficient for the existence of the B_i is obviously necessary.

In Theorem RHV the index set is finite, although the sets are allowed to be infinite. There is a corollary for infinite indexed families.

Theorem 3.3 (Theorem RHV′). *Suppose that* $[A_i;\ i \in I]$ *is an indexed family of finite sets and* $[k_i;\ i \in I]$ *is a corresponding family of positive integers. There is a family* $[B_i;\ i \in I]$ *of pairwise disjoint sets such that* $B_i \subseteq A_i$ *and* $|B_i| = k_i$ *for each* $i \in I$ *if and only if, for each finite* $J \subseteq I$, $|\bigcup_{j\in J} A_j| \geq \sum_{j\in J} k_j$.

Theorem RHV′ may be proved by proving the sufficiency in the special case $k_i = 1$ for all $i \in I$ from Hall's theorem by a "compactness" argument, and then deriving RHV′ from that as RHV was derived from Hall's theorem.

Theorems RHV and RHV′ have engendered at least two offspring. The first is due to Bollobás and Varopoulos [12].

Theorem 3.4 (Theorem BV). *Suppose* $(X, \mathcal{M}, \mu)$ *is an atomless positive measure space,* $[A_i;\ i \in I]$ *is an indexed family of subsets of* X *of finite measure, and* $[\lambda_i;\ i \in I]$ *is a corresponding family of positive real numbers. There is a family* $[B_i;\ i \in I]$ *of pairwise disjoint measurable subsets of* X *such that* $B_i \subseteq A_i$ *and* $\mu(B_i) = \lambda_i$ *for each* $i \in I$ *if and only if, for each finite* $J \subseteq I$, $\mu(\bigcup_{j\in J} A_j) \geq \sum_{j\in J} \lambda_j$.

Clearly the statement of Theorem BV owes a lot to Theorem RHV′, and, in fact, the proof owes a lot to Theorem RHV; the proof uses a discretization argument plus RHV to get "close" to the desired family $[B_i;\ i \in I]$ and then "takes the limit" to get to $[B_i;\ i \in I]$ using some razzle-dazzle from functional analysis.

But also, Theorem BV throws light back on Theorem RHV′. On closer inspection, Theorem RHV′ turns out to involve a measure space, namely, any set X containing $\bigcup_{i\in I} A_i$ equipped with the *counting measure*; the counting measure of any $A \subseteq X$ is $|A|$, the number of elements in A (with all infinite cardinals registered as simply ∞). Thus viewed, the statements of Theorems BV and RHV′ are almost perfectly analogous, not merely closely related. The perfection of the analogy is marred only by the requirement that the k_i in Theorem RHV′ are whole numbers. But this slight difference turns out to be quite significant; we can find no theorem of which these two are special cases, no theorem that encompasses weighted counting measures, for instance, or measure spaces with atoms together with atomless oceans. The situation is curious: two almost perfectly analogous theorems involving measure spaces at the opposite extremes of the spectrum of measure spaces, with nothing for the measure spaces in between. Or so it seems.

The other development partially inspired by Theorem RHV—with another component coming from the idea of "n-tuple colorings" pioneered by Stahl [13]—is the natural generalization of list colorings to list "multicolorings." The constituents of the situation are G, L, C and $\mathcal{F}$ as before, and a function $\kappa : V(G) \to \mathbb{P} = \{1,2,\dots\}$. Given these, a *proper* (L,κ)*-coloring* of G is a function $\varphi : V(G) \to \mathcal{F}$ such that for all $u, v \in V(G)$:

(i) $\varphi(v) \subseteq L(v)$;

(ii) $uv \in E(G)$ implies $\varphi(u) \cap \varphi(v) = \emptyset$;

(iii) $|\varphi(v)| = \kappa(v)$.

As before, observe that (ii) is equivalent to

(ii)′ for every $\sigma \in C$, $\{w \in V(G) \mid \sigma \in \varphi(w)\}$ is independent.

We will say that G, L, and κ satisfy Hall's condition (HC) if and only if, for each induced subgraph H of G,

$$\sum_{\sigma \in C} \alpha(\sigma, L, H) \geq \sum_{v \in V(H)} \kappa(v). \tag{**}$$

Clearly this is a generalization of the earlier version of Hall's condition, in which $\kappa \equiv 1$. Theorem RHV can be seen to say that when G is a clique, HC is sufficient for the existence of a proper (L,κ)-coloring of G, and it is straightforward to see that HC is always a necessary condition for the existence of a proper (L,κ)-coloring of G. (A formal proof of this can be obtained from the proof of Proposition 2.3 by replacing L by (L,κ), $|V(H)|$ by $\sum_{v\in V(H)} \kappa(v)$, $\varphi^{-1}(\sigma)$ by $\{v \in V(G) \mid \sigma \in \varphi(v)\}$, and $\{\varphi(v)\}$ by $\varphi(v)$. Also, see Proposition 4.2, later in this paper.)

The quest for the analogue of Theorem HJW for multicolorings of graphs was greatly impeded by a fixation on Theorem HJW; that is, most interested parties thought that the graphs for which HC would be sufficient for a proper (L,κ)-coloring would be those of which every block is a clique.

Then Cropper found a counter example [14], which he subsequently reduced to the incredibly simple example depicted in Figure 3.2.

This breakthrough led to the discovery of the correct analogue of Theorem HJW by Cropper et al. [15], given below in Theorem CGL. It is notable that the Cropper Claw example turns out to be, in a sense to be left unexplained, the only impediment to proper coloring in the presence of Hall's condition added by allowing κ to be other than the constant function 1.

Theorem 3.5 (Theorem CGL [15]). *The following are equivalent:*

(a) *for every list assignment L to G and $\kappa : V(G) \to \mathbb{P}$ such that G, L, and κ satisfy HC, there is a proper (L,κ)-coloring of G;*

(b) *every block of G is a clique and every cut vertex of G lies in exactly two blocks;*

(c) *G has no induced subgraph isomorphic to C_n for any $n \geq 4$, nor to K_4-minus-an-edge, nor to $K_{1,3}$;*

(d) *G is the line graph of a forest.*

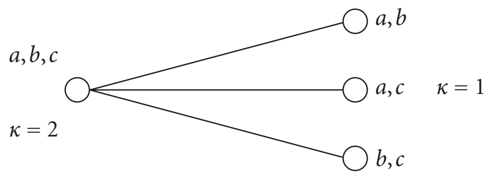


Figure 3.2. The Cropper Claw; every block a clique, but HC is not sufficient for a proper (L,κ)-coloring.

Actually [15] contains a stronger theorem which answers a question posed in [16], and which contains Theorem HJW.

Theorem 3.6 (Theorem CGL′). *A pair (G,κ), where $\kappa: V(G) \to \mathbb{P}$, has the property that there is a proper (L,κ)-coloring of G for every L such that G, L, and κ satisfy HC if and only if every block of G is a clique and for every cut-vertex v of G which is in 3 or more blocks of G, $\kappa(v) = 1$.*

In what follows, graphs satisfying (a)–(d) of Theorem CGL will be called CGL graphs. Also, we will sometimes use the charming terminology that may have originated with Cropper and Lehel, whereby a list assignment is called a *color supply function* and any $\kappa: V(G) \to \mathbb{P}$ is a *color demand function*.

4. Color supply from measure spaces

The next development after Theorems CGL and CGL′ was the attempt to prove a theorem bearing the same relation to Theorem CGL as Theorem BV (for finite index sets) bears to Theorem RHV. To formulate such a theorem, we will have to enlarge the scope of previous definitions. Suppose that $(X,\mathcal{M},\mu)$ is a positive measure space (here $\mathcal{M}$ is a σ-algebra of measurable sets and μ is the measure, assigning elements of $[0,\infty]$ to elements of $\mathcal{M}$), G is a simple graph (as always), and $\lambda: V(G) \to [0,\infty)$ is a function, the "color demand" function in this context. A list assignment, or color supply function, will be a function $L: V(G) \to \mathcal{M}$. A *proper* (L,λ)*-coloring* of G is a function $\varphi: V(G) \to \mathcal{M}$ satisfying, for all $u,v \in V(G)$,

(i) $\varphi(u) \subseteq L(v)$;
(ii) $uv \in E(G)$ implies $\varphi(u) \cap \varphi(v) = \emptyset$;
(iii) $\mu(\varphi(v)) \geq \lambda(v)$.

Again, (ii) is equivalent to

(ii)′ for each $x \in X$, $\{v \in V(G) \mid x \in \varphi(v)\}$ is an independent set of vertices.

For a subgraph H of G, a color supply function $L: V(G) \to \mathcal{M}$, and $x \in X$, $\alpha(x,L,H)$ is defined just as it was when $X = C$ and μ was the counting measure: $\alpha(x,L,H)$ is the largest size of an independent set of vertices contained in $\{v \in V(H) \mid x \in L(v)\}$. The following is a restatement of Lemma 2.2 in this wider context; its proof is omitted.

Lemma 4.1. *With G, L, $(X,\mathcal{M},\mu)$, and H as above, for each $x \in X$,*

$$\alpha(x,L,H) = \max\left[\sum_{v\in S} \operatorname{char}_{L(v)}(x) : S \text{ is an independent set of vertices of } H\right]. \tag{4.1}$$

Since each set $L(v)$, $v \in V(G)$, is measurable, it follows that for each H, the (integer-valued, nonnegative) function $x \mapsto \alpha(x,L,H)$ is a measurable function on X. Thus the integral in (***), below, is well defined.

Given G, $(X,\mathcal{M},\mu)$, $L : V(G) \to \mathcal{M}$, and $\lambda : V(G) \to [0,\infty)$, we will say that G, L, and λ satisfy Hall's condition if and only if, for each (induced) subgraph H of G,

$$\int_X \alpha(x,L,H)d\mu \geq \sum_{v\in V(H)} \lambda(v). \qquad (***)$$

The following is a generalization of Proposition 2.3; the proof is essentially the same, but we give it anyway. It is essential to realize that for any $S \in \mathcal{M}$, $\int_X \operatorname{char}_S(x)d\mu = \mu(S)$.

Proposition 4.2. *Hall's condition on G, L, and λ is necessary for the existence of a proper (L,λ)-coloring of G.*

Proof. Suppose that φ is a proper (L,λ)-coloring of G, and that H is an induced subgraph of G. Then φ restricted to $V(H)$ is a proper (L,λ)-coloring of H (really a proper (L',λ')-coloring of H, with L', λ' denoting the restrictions of L, λ, respectivly, to $V(H)$); we verify (***). Since $\mu(\varphi(v)) \geq \lambda(v)$ for all $v \in V(G)$, we have

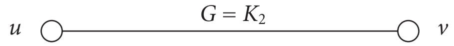
$$\begin{aligned}
\sum_{v\in V(H)} \lambda(v) \leq \sum_{v\in V(H)} \mu(\varphi(v)) &= \sum_{v\in V(H)} \int_X \operatorname{char}_{\varphi(v)}(x)d\mu \\
&= \int_X \left(\sum_{v\in V(H)} \operatorname{char}_{\varphi(v)}(x)\right)d\mu = \int_X \max_{\substack{S\subseteq V(H)\\ S \text{ independent}}} \left(\sum_{v\in S} \operatorname{char}_{\varphi(v)}(x)\right)d\mu \\
&\qquad [\text{because for each } x \in X, \{v \in V(H) \mid \operatorname{char}_{\varphi(v)}(x) = 1\} \text{is independent}] \\
&\leq \int_X \max_{\substack{S\subseteq V(H)\\ S \text{ independent}}} \left(\sum_{v\in S} \operatorname{char}_{L(v)}(x)\right)d\mu = \int_X \alpha(x,L,H)d\mu.
\end{aligned} \qquad (4.2)$$

□

Again the great question is: when is the necessary condition for proper coloring sufficient? But there does not seem to be a neat answer covering all choices of $(X,\mathcal{M},\mu)$, in the same way that Theorem CGL takes care of the cases $X = C$, μ = counting measure, when λ is confined to be integer valued. Consider the situation in Figure 4.1, in which the graph is K_2, the measure space is any infinite set with the counting measure, the color demand at each vertex is 1/2, and the color supply at each vertex is the singleton $\{a\}$.

If H is any proper induced subgraph of G, that is, H is either the single vertex u or the single vertex v, then there is a proper (L,λ)-coloring of H, so (***) holds. It is verified in Figure 4.1 that (***) holds when $H = G$. But the only measurable subset of $\{a\}$ with measure $\geq 1/2$ is $\{a\}$ itself, so no proper coloring is possible.

Clearly the failure of Hall's condition to suffice for a proper coloring in this example is for silly, fixable reasons. When $X = C$ and μ is the counting measure, there is a proper (L,λ)-coloring if and only if there is a proper $(L,\lceil\lambda\rceil)$-coloring; in the example $\lceil\lambda\rceil = \lceil 1/2\rceil = 1$, and K_2, L, and 1 do not satisfy HC. In the case of a counting measure all



$$L(u) = L(v) = \{a\}$$
$$\lambda(u) = \lambda(v) = 1/2$$
$$\int_C \alpha(\sigma, L, G)\,d\mu = \sum_{\sigma\in C} \mathrm{char}_{\{a\}}(\sigma) = 1 = \sum_{v\in V(G)} \lambda(v)$$

Figure 4.1. Hall's condition satisfied by K_2, L, λ, but no proper coloring is possible.

earlier results, like Theorems CGL and CGL$'$, can be restated in this expanded context by replacing κ by $\lceil \lambda \rceil$.

But what about other measure spaces? Fixing the "Figure 4.1 impediment" to obtain a version of Theorem CGL for a given weighted counting measure, for instance, is possible, although the statement will be lengthy and awkward. For the whole class of weighted counting measures a theorem can be formulated, but we are better off not actually formulating it. But for the typical atomless measure space, all is well. To avoid certain trivialities we will require our atomless space to be *semifinite*, which means that every set of infinite measure has a measurable subset of finite positive measure.

Theorem 4.3 (Theorem HJ [17]). *Suppose that $(X, \mathcal{M}, \mu)$ is a positive atomless semifinite measure space, with $\mu(X) > 0$. A graph G has the property that there is a proper (L,λ)-coloring of G whenever $G, L : V(G) \to \mathcal{M}$ and $\lambda : V(G) \to [0,\infty)$ satisfy Hall's condition, if and only if G is a CGL graph.*

The "only if" statement is quite straightforward to prove, but we will not do so here. As for the "if" statement, a discretization argument together with Theorem CGL gets us close to a proper coloring when HC is satisfied (in fact gets us a proper coloring when each inequality (***) is satisfied strictly), and then the argument is finished in [17] by adapting the functional analytic arguments of Bollobás and Varopoulos [12]. Undoubtedly the proof can be executed more constructively, avoiding the functional analysis, using more direct measure theoretic techniques, and we propose it as a worthy problem to provide such a proof.

It is also shown in [17] that no obvious analogue of Theorem CGL$'$ holds when the measure space involved is atomless and semifinite. With $(X, \mathcal{M}, \mu)$ as in Theorem HJ, the question is: for which pairs (G,λ), $\lambda : V(G) \to [0,\infty)$ is there a proper (L,λ)-coloring of G whenever G, L, and λ satisfy HC? If G is a CGL graph, then the pair (G,λ) has the stated property regardless of λ, by Theorem HJ. To restate [17, Proposition 3], if G is *not* a CGL graph, the only ways that (G,λ) could have the stated property are

(i) the subgraph of G induced by $\{v \in V(G) \mid \lambda(v) > 0\}$ is a CGL graph and/or

(ii) $\mu(X) < \sum_{v\in V(G)} \lambda(v)$, which opens the possibility that there are no list assignments $L : V(G) \to \mathcal{M}$ such that G, L, and λ satisfy HC, whereby (G,λ) would have the stated property vacuously.

Thus there is still a patch of unexplored territory here: what if G is not a CGL graph, $\lambda(v) > 0$ for all $v \in V(G)$, $\mu(X) < \sum_{v\in V(G)} \lambda(v)$, and there exist L such that G, L, λ satisfy HC? Can it be that there is a proper (L,λ)-coloring for every such L?

5. Coloring graphs with functions

Every set is identifiable with a function, its characteristic function on the überset of which it is thought to be a subset. There is a natural way to analogize the original list-coloring setting so that both the lists and the colorings are functions. Let $(X,\mathcal{M},\mu)$ be, as before, a positive measure space, and let $\mathcal{F}(X)$ denote the space of nonnegative measurable functions on X. A functional list assignment to a simple graph G will be a function $L : V(G) \to \mathcal{F}(X)$. Suppose that L is such a list assignment and $\lambda : V(G) \to [0,\infty)$ is a (color demand) function. A *proper functional* (L,λ)*-coloring* of G is a function $\varphi : V(G) \to \mathcal{F}(X)$ satisfying, for all $u,v \in V(G)$:

(i) $\varphi(v)(x) \leq L(v)(x)$ for all $x \in X$;

(ii) if $uv \in E(G)$ then $\varphi(v)\varphi(u) \equiv 0$ on X;

(iii) $\int_X \varphi(v)(x)d\mu \geq \lambda(v)$.

For $f \in \mathcal{F}(X)$, let $\operatorname{supp}(f) = \{x \in X \mid f(x) > 0\}$, the *support* of f. Clearly (ii) is equivalent to: if $uv \in E(G)$ then $\operatorname{supp}(\varphi(v)) \cap \operatorname{supp}(\varphi(u)) = \varnothing$, and thus (ii) is equivalent to:

(ii)′ for each $x \in X$, $\{v \in V(G) \mid \varphi(v)(x) > 0\}$ is independent.

If $Z,Y \subseteq X$, then $Z \subseteq Y$ if and only if $\operatorname{char}_Z \leq \operatorname{char}_Y$, pointwise on X. Thus condition (i) is a natural replacement of the corresponding condition $(\varphi(v) \subseteq L(V))$ in earlier definitions of a proper list coloring, in which both L and φ are assignments of sets.

In this setting there is greater freedom in assigning "lists" than before, and also greater freedom in extracting "colorings" from those lists, so it is not instantly apparent whether it is easier or harder, or sometimes one, sometimes the other, to color graphs in accordance with the requirements of this new setting. However, one relation between the new and the old is clear.

Lemma 5.1. *Suppose that $(X,\mathcal{M},\mu)$ is a positive measure space and $L : V(G) \to \mathcal{F}(X)$ and $\lambda : V(G) \to [0,\infty)$ are color supply and demand functions, respectively, on G. If, for each $v \in V(G)$, $L(v)$ is the characteristic function of some measurable set $Y(v) \subseteq X$, then there is a proper functional (L,λ)-coloring of G if and only if there is a proper (Y,λ)-coloring of G, as defined at the beginning of Section 4.*

Proof. The "if" claim is clear. Suppose that φ is a proper functional (L,λ)-coloring of G. Let $\psi : V(G) \to \mathcal{M}$ be defined by $\psi(v) = \operatorname{supp}(\varphi(v))$. Since, for each $v \in V(G)$, $\varphi(v) \leq L(v) = \operatorname{char}_{Y(v)}$, pointwise on X, it follows that $\psi(v) \subseteq Y(v)$ and $\varphi(v) \leq \operatorname{char}_{\psi(v)} \leq L(v)$; from the last we see that

$$\lambda(v) \leq \int_X \varphi(v)(x)d\mu \leq \int_X \operatorname{char}_{\psi(v)}(x)d\mu = \mu(\psi(v)). \tag{5.1}$$

Finally, by previous remarks, if $uv \in E(G)$ then $\operatorname{supp}(\varphi(u)) \cap \operatorname{supp}(\varphi(v)) = \varnothing$, that is,

$$\psi(u) \cap \psi(v) = \varnothing. \tag{5.2}$$

Thus ψ is a proper (Y,λ)-coloring of G. □

There is a Hall's condition for $L : V(G) \to \mathcal{F}(x)$, $\lambda : V(G) \to [0,\infty)$, but it requires a definition that would be a bolt from the blue were it not reasonably and naturally descended from its ancestors—see Lemmas 2.2 and 4.1. Given $L : V(G) \to \mathcal{F}(X)$ and a subgraph H of G, we define, for each $x \in X$,

$$\alpha(x,L,H) = \max_{\substack{S\subseteq V(H)\\ S \text{ independent}}} \sum_{v\in S} L(v)(x). \tag{5.3}$$

We will say that G, L, and λ *satisfy Hall's Condition* if and only if, for each induced subgraph H of G,

$$\int_X \alpha(x,L,H)\,d\mu \geq \sum_{v\in V(H)} \lambda(v). \tag{****}$$

Proposition 5.2. *Hall's condition is necessary for the existence of a proper (L,λ)-coloring of G.*

The proof is much like that of Proposition 4.2, with $\varphi(v)$ and $L(v)$ here replacing $\text{char}_{\varphi(v)}$ and $\text{char}_{L(v)}$ there. At a crucial point in the proof, with φ a proper (L,λ)-coloring and $x \in X$,

$$\sum_{v\in V(H)} \varphi(v)(x) = \max_{\substack{S\subseteq V(H)\\ S \text{ independent}}} \sum_{v\in S} \varphi(v)(x) \tag{5.4}$$

because $0 \leq \varphi(v)(x)$ for all v and x, and $\{v \in V(H) \mid \varphi(v)(x) > 0\}$ is independent.

Again we face the question: when is Hall's condition sufficient for a proper coloring, in this new context? The cases in which $L(v)$ is a function from X into $[0,1]$, for each $v \in V(G)$, are of particular interest, perhaps for reasons of fashion alone; such a function is called a *fuzzy subset* of X. Thus, if μ is the counting measure on X, the sufficiency of HC for a proper coloring whenever, in addition, G is a clique and $L(v)$ is a fuzzy subset of X for each $v \in V(G)$, would be a wonderful thing, Hall's theorem (or, more properly, a Hall-Rado-Halmos-Vaughan theorem) for fuzzy sets.

But call off the paparazzi—the dream is murdered by Figure 4.1—note the applicability of Lemma 5.1. It is even worse: if the measure space contains an atom of finite measure and G contains an edge, then there exists a fuzzy-set list assignment L to $V(G)$ and a function $\lambda : V(G) \to [0,\infty)$ such that G, L, and λ satisfy HC, yet no proper (L,λ)-coloring of G is possible.

But when $(X,\mathcal{M},\mu)$ is atomless and semifinite, we are confident that Theorem HJ holds, as stated (but with a different meaning in the words), with no restrictions on $L : V(G) \to \mathcal{F}(X)$. Indeed, the proof in [17] seems readily adaptable to the special case when $L(v)$ is a *bounded* measurable function on X, for each $v \in V(G)$. But we will leave this question hanging for now, except for the easy part of the proposed analogue of Theorem HJ.

Proposition 5.3. *Suppose that $(X,\mathcal{M},\mu)$ is atomless and semifinite with $\mu(X) > 0$ and that there is a proper functional (L,λ)-coloring of G for every $L : V(G) \to \mathcal{F}(X)$ and every $\lambda : V(G) \to [0,\infty)$ such that G, L, and λ satisfy HC. Then G is a CGL graph.*

Proof. The conclusion is an immediate consequence of Lemma 5.1 and the "only if" assertion of Theorem HJ. □

6. List multicolorings of independence systems

We say that $M = (V, \mathcal{I})$ is an *independence system* (i.s., for short) if V is a finite set and $\mathcal{I}$ is a set of subsets of V (called the *independent sets* of M) satisfying

(1) $\varnothing \in \mathcal{I}$ and

(2) $A \subseteq B \in \mathcal{I} \to A \in \mathcal{I}$.

For instance, the set V of vertices of a finite simple graph with the family $\mathcal{I}$ of independent sets of vertices (sets of mutually nonadjacent vertices) of the graph form an i.s.

If $M = (V, \mathcal{I})$ is an i.s. and $S \subseteq V$, a largest independent subset of S is called a *basis* of S, and its size is called the *rank* of S in M, denoted $\alpha_M(S)$. For instance, if $V = V(G)$ is the set of vertices of a graph and $\mathcal{I} = \mathcal{I}(G)$ is the family of independent sets of vertices of G, and $S \subseteq V$, then $\alpha_M(S) = \alpha(\langle S \rangle_G)$, the vertex independence number of the subgraph of G induced by S.

As before, let C be an infinite set (of "colors") and $\mathcal{F}$ the family of C's finite subsets. A list assignment or color supply function on an i.s. $M = (V, \mathcal{I})$ is a function $L : V \to \mathcal{F}$; a color demand function on M is a function $\kappa : V \to \mathbb{N} = \{0, 1, \dots\}$. (Note that we are allowing κ to take the value 0, unlike before, in Section 3. The difference is not really significant.) If L and κ are color supply and demand functions, respectively, on M, a *proper* (L,κ)*-coloring* of M is a function $\varphi : V \to \mathcal{F}$ satisfying, for each $v \in V$, $\sigma \in C$

(i) $\varphi(v) \subseteq L(v)$,

(ii) $\operatorname{supp}(\sigma, \varphi) = \{v \in V \mid \sigma \in \varphi(v)\} \in \mathcal{I}$, and

(iii) $|\varphi(v)| = \kappa(v)$.

With L and κ as above and, for each $\sigma \in C$, $\operatorname{supp}(\sigma, L) = \{v \in V \mid \sigma \in L(v)\}$, we will say that M, L, and κ *satisfy Hall's condition* (HC) if and only if, for each $S \subseteq V$,

$$\sum_{\sigma \in C} \alpha_M(S \cap \operatorname{supp}(\sigma, L)) \geq \sum_{v \in S} \kappa(v). \tag{$*$ i.s.}$$

It is straightforward that these definitions generalize those of Section 3; the following is a generalization of the generalization of Proposition 2.3 alluded to in Section 3, in the paragraph following the formulation of Hall's condition for ordinary multicolorings.

Proposition 6.1. *Hall's condition is necessary for the existence of a proper (L,κ)-coloring of M.*

Proof. Suppose that $\varphi : V \to \mathcal{F}$ is a proper (L,κ)-coloring of M. Suppose that $S \subseteq V$. We verify ($*$ i.s.):

$$\begin{aligned} \sum_{v \in S} \kappa(v) &= \sum_{v \in S} |\varphi(v)| = \sum_{v \in S} \sum_{\sigma \in C} \operatorname{char}_{\varphi(v)}(\sigma) = \sum_{\sigma \in C} \sum_{v \in S} \operatorname{char}_{\varphi(v)}(\sigma) \\ &= \sum_{\sigma \in C} |\operatorname{supp}(\sigma, \varphi) \cap S| \leq \sum_{\sigma \in C} \alpha_M(\operatorname{supp}(\sigma, L) \cap S), \end{aligned} \tag{6.1}$$

because $\operatorname{supp}(\sigma, \varphi) \cap S$ is an independent subset of $\operatorname{supp}(\sigma, L) \cap S$; this last assertion follows from $\varphi(v) \subseteq L(v)$ for all $v \in V$ and $\operatorname{supp}(\sigma, \varphi)$ is independent, for each $\sigma \in C$. □

As before, the question that most easily arises is: for which independence systems M is HC sufficient for the existence of a proper (L,κ)-coloring? And what if κ is confined to the case $\kappa \equiv 1$? We know the answers when $V = V(G)$, $\mathcal{I} = \mathcal{I}(G)$ for some graph G. We do not know the answers for independence systems in general, but we do know something.

An i.s. $M = (V,\mathcal{I})$ is a *matroid* if and only if whenever $I \subseteq S \subseteq V$ and $I \in \mathcal{I}$, there is a basis B of S with $I \subseteq B$; that is, every independent subset of any $S \subseteq V$ can be "extended" to an independent set of maximum size in S. It is straightforward to see that if $M(G) = (V(G),\mathcal{I}(G))$ is the i.s. associated with a finite simple graph G, as discussed above, then $M(G)$ is a matroid if and only if every component of G is a clique (when not every component of G is clique, take S to be a set of 3 vertices that induce a path). Thus the following is a generalization of Theorem RHV (reframing that theorem as a statement about list multicolourings of complete graphs).

Theorem 6.2. *If $M = (V,\mathcal{I})$ is a matroid, then for every color supply function $L: V \to \mathcal{F}$ and every color demand function $\kappa: V \to \mathbb{N}$ such that M, L, and κ satisfy HC there is a proper (L,κ)-coloring of M.*

We will give the proof of Theorem 6.2 in easy stages. It turns out that Theorem 6.2 is, in somewhat disguised form, a restatement of the following classic theorem about matroids.

Theorem 6.3 (Theorem EF [18]). *Let D and V be finite sets, and suppose that for each $\sigma \in D$, $M_\sigma = (V,\mathcal{I}_\sigma)$ is a matroid on V with rank function α_σ. Then there is a family $F = \{I_\sigma \mid \sigma \in D\}$ such that $I_\sigma \in \mathcal{I}_\sigma$ for each $\sigma \in D$ and*

$$\bigcup_{\sigma\in D} I_\sigma = V \tag{6.2}$$

if and only if, for each $S \subseteq V$,

$$\sum_{\sigma\in D} \alpha_\sigma(S) \geq |S|. \tag{$*$ EF}$$

We will see that the "if" assertion of Theorem EF implies Theorem 6.2. It will be evident to the careful reader that the two theorems are essentially "equivalent," in the sense that each is easily derivable from the other.

Lemma 6.4. *Suppose that $M = (V,\mathcal{I})$ is an i.s. and $L: V \to \mathcal{F}$ and $\kappa: V \to \mathbb{N}$ are color supply and demand functions, respectively. Let $[K(v) \mid v \in V]$ be an indexed collection of pairwise disjoint sets such that $|K(v)| = \kappa(v)$ for each $v \in V$. Let $V' = \bigcup_{v\in V} K(v)$; let $\pi: V' \to V$ be defined by $\pi(x) = v$ if and only if $x \in K(v)$. Let $\mathcal{I}' = \{I' \subseteq V' \mid |I' \cap K(v)| \leq 1$ for each $v \in V$ and $\pi(I') \in \mathcal{I}\}$. (Here $\pi(I') = \{\pi(x) \mid x \in I'\}$.) Then $M' = (V',\mathcal{I}')$ is an i.s., with rank function $\alpha_{M'} = \alpha_M \circ \pi$, and is a matroid if M is. Further, if $L': V' \to \mathcal{F}$ is defined by $L'(x) = L(\pi(x))$ and $\kappa' \equiv 1$ on V', then M', L', and κ' satisfy HC if and only if M, L, and κ satisfy HC, and there is a proper (L',κ')-coloring of M' if and only if there is a proper (L,κ)-coloring of M.*

Proof. The proof is an exercise in following one's nose through the definitions, and we are going to skip most of it. It helps to realize that, for each $\sigma \in C$, $\mathrm{supp}(\sigma, L') = \bigcup_{v\in \mathrm{supp}(\sigma,L)} K(v)$. Also, note that if φ' is a proper $(L',1)$-coloring of M', so that

$\operatorname{supp}(\sigma,\varphi') \in \mathcal{I}'$ for each $\sigma \in C$, then, for each $v \in V$, by the definition of $\mathcal{I}'$, σ can appear at most once as a φ' color on $K(v)$; otherwise, $|\operatorname{supp}(\sigma,\varphi') \cap K(v)| \geq 2$, so $\operatorname{supp}(\sigma,\varphi') \notin \mathcal{I}'$. Therefore, $\{\varphi'(x) \mid x \in K(v)\}$ is a set of $\kappa(v)$ colors, and a subset of $L(v)$, for each $v \in V$. It should be clear how to go from φ' to a proper (L,κ)-coloring of M, and once that is understood, going the other way is straightforward.

We explicitly verify that if M, L, and κ satisfy HC, then so do M', L', and $\kappa' \equiv 1$. (This is the implication that we will be using in the proof of Theorem 6.2. The reverse implication, which is part of the claim of this lemma, is left to the reader.) Suppose that $S' \subseteq V'$. Since $\kappa' \equiv 1$,

$$\begin{aligned}\sum_{x\in S'} \kappa'(x) = |S'| \leq \sum_{v\in\pi(S')} \kappa(v) &\leq \sum_{\sigma\in C} \alpha_M(\pi(S') \cap \operatorname{supp}(\sigma,L)) \\ &\quad [\text{this is } (*\text{ i.s.}) \text{ applied with } S = \pi(S')] \\ &= \sum_{\sigma\in C} \alpha_M(\pi(S' \cap \operatorname{supp}(\sigma,L'))) = \sum_{\sigma\in C} \alpha_{M'}(S' \cap \operatorname{supp}(\sigma,L')).\end{aligned} \tag{6.3}$$

□

Proof of Theorem 6.2. Suppose that $M = (V,\mathcal{I})$ is a matroid, L and κ are color supply and demand functions, respectively, on M, and M, L, and κ satisfy HC. We want to show that there is a proper (L,κ)-coloring of M. By Lemma 6.4 we may assume that $\kappa \equiv 1$.

For each $\sigma \in D = \bigcup_{v\in V} L(v)$, let $\mathcal{I}_\sigma = \{I \in \mathcal{I} \mid I \subseteq \operatorname{supp}(\sigma,L)\}$. Then $M_\sigma = (V,\mathcal{I}_\sigma)$ is a matroid on V; let α_σ denote the rank function on M_σ. For any $S \subseteq V$, clearly $\alpha_\sigma(S) = \alpha_M(S \cap \operatorname{supp}(\sigma,L))$, and, therefore, $(*$ i.s.$)$ implies $(*$ EF$)$:

$$|S| = \sum_{v\in S} \kappa(v) \leq \sum_{\sigma\in C} \alpha_M(S \cap \operatorname{supp}(\sigma,L)) = \sum_{\sigma\in D} \alpha_\sigma(S). \tag{6.4}$$

By Theorem EF, there is a family $F = \{I_\sigma \mid \sigma \in D\}$ such that $I_\sigma \in \mathcal{I}_\sigma$ for each $\sigma \in D$ and $\bigcup_{\sigma\in D} I_\sigma = V$. Color each $v \in V$ with some $\sigma \in D$ such that $v \in I_\sigma$ (in case there is more than one such σ, choose one). Call the coloring φ. Since $I_\sigma \in \mathcal{I}_\sigma$ for each $\sigma \in D$, for each $v \in V$, if $\sigma = \varphi(v)$ then $v \in I_\sigma \subseteq \operatorname{supp}(\sigma,L)$, so $\varphi(v) = \sigma \in L(v)$. Further, for $\sigma \in D$, $\operatorname{supp}(\sigma,\varphi) \subseteq I_\sigma \in \mathcal{I}_\sigma \subseteq \mathcal{I}$. Thus φ is a proper (L,κ)-coloring (recall $\kappa \equiv 1$) of M. □

Obviously there is a lot of territory yet to be explored in the province of list multicolorings of independence systems. Are there succinct extensions of Theorems HJW and CGL, or possible CGL′, to this new frontier? And what about colorings by measurable sets from a measure space, from measurable "lists," or colorings by measurable nonnegative functions on a measure space, as in Section 5? We await developments.

References

[1] V. G. Vizing, "Coloring the vertices of a graph in prescribed colors," *Metody Diskretnogo Analiza*, no. 29, pp. 3–10, 101, 1976 (Russian).

[2] P. Erdös, A. L. Rubin, and H. Taylor, "Choosability in graphs," in *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979)*, vol. 26 of *Congressus Numerantium*, pp. 125–157, Utilitas Math., Winnipeg, Man, Canada, 1980.

[3] F. Galvin, "The list chromatic index of a bipartite multigraph," *Journal of Combinatorial Theory. Series B*, vol. 63, no. 1, pp. 153–158, 1995.

[4] P. Hall, "On representatives of subsets," *Journal of the London Mathematical Society*, vol. 10, no. 37, pp. 26–30, 1935.

[5] A. J. W. Hilton and P. D. Johnson Jr., "A variation of Ryser's theorem and a necessary condition for the list-colouring problem," in *Graph Colourings (Milton Keynes, 1988)*, R. Nelson and R. Wilson, Eds., vol. 218 of *Pitman Research Notes in Mathematics*, chapter 10, pp. 135–143, John Wiley & Sons, New York, NY, USA, 1990.

[6] A. J. W. Hilton and P. D. Johnson Jr., "Extending Hall's theorem," in *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*, pp. 359–371, Physica, Heidelberg, Germany, 1990.

[7] A. J. W. Hilton, P. D. Johnson Jr., and E. B. Wantland, "The Hall number of a simple graph," *Congressus Numerantium*, vol. 121, pp. 161–182, 1996.

[8] C. Eslahchi and M. Johnson, "Characterization of graphs with Hall number 2," *Journal of Graph Theory*, vol. 45, no. 2, pp. 81–100, 2004.

[9] P. D. Johnson Jr., "Hall's condition for list-coloring, and the Hall parameters: recent developments," *Discrete Mathematics*, vol. 249, no. 1–3, pp. 135–147, 2002.

[10] R. Rado, "A theorem on general measure functions," *Proceedings of the London Mathematical Society*, vol. 44, no. 1, pp. 61–91, 1938.

[11] P. R. Halmos and H. E. Vaughan, "The marriage problem," *American Journal of Mathematics*, vol. 72, no. 1, pp. 214–215, 1950.

[12] B. Bollobás and N. Th. Varopoulos, "Representation of systems of measurable sets," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 78, no. 2, pp. 323–325, 1975.

[13] S. Stahl, "n-tuple colorings and associated graphs," *Journal of Combinatorial Theory. Series B*, vol. 20, no. 2, pp. 185–203, 1976.

[14] M. M. Cropper, "Hall's condition yields less for multicolorings," *The Australasian Journal of Combinatorics*, vol. 18, pp. 263–266, 1998.

[15] M. Cropper, A. Gyárfás, and J. Lehel, "Edge list multicoloring trees: an extension of Hall's theorem," *Journal of Graph Theory*, vol. 42, no. 3, pp. 246–255, 2003.

[16] P. D. Johnson Jr. and E. B. Wantland, "Two problems concerning Hall's condition," *Congressus Numerantium*, vol. 152, pp. 189–196, 2001.

[17] A. J. W. Hilton and P. D. Johnson Jr., "List multicolorings of graphs with measurable sets," *Journal of Graph Theory*, vol. 54, no. 3, pp. 179–193, 2007.

[18] J. Edmonds and D. R. Fulkerson, "Transversals and matroid partition," *Journal of Research of the National Bureau of Standards*, vol. 69B, pp. 147–153, 1965.

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