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Research Article

On Certain Multivalent Functions

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Let $\mathcal{S}^*(p,\alpha)$ be the class of functions $f(z)$ which are analytic and p-valently starlike of order α in the open unit disk $\mathbb{E}$. The object of the present paper is to derive an interesting condition for $f(z)$ to be in the class $\mathcal{S}^*(p,\alpha)$.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1,2,3,\dots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{E} = \{z : z \in \mathbb{C},\ |z| < 1\}$. A function $f(z) \in \mathcal{A}(p)$ is said to be p-valently starlike of order $\alpha(0 \leqq \alpha < p)$ in $\mathbb{E}$ if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha \quad (z \in \mathbb{E}). \tag{1.2}$$

We denote by $\mathcal{S}^*(p,\alpha)$ the subclass of $\mathcal{A}(p)$ consisting of functions which are p-valently starlike of order α in $\mathbb{E}$. We only call a function $f(z) \in \mathcal{S}^*(p,0)$ to be p-valently starlike in $\mathbb{E}$. Further, a function $f(z) \in \mathcal{A}(p)$ is said to be p-valently convex of order $\alpha(0 \leqq \alpha < p)$ in $\mathbb{E}$ if and only if

$$1 + \operatorname{Re}\frac{zf''(z)}{f'(z)} > \alpha \quad (z \in \mathbb{E}). \tag{1.3}$$

We denote by $\mathscr{C}(p,\alpha)$ the subclass of $\mathscr{A}(p)$ consisting of all p-valently convex functions of order α in $\mathbb{E}$. We also call a function $f(z) \in \mathscr{C}(p,0)$ p-valently convex function. From the definition, it is trivial that if $f(z)$ is a p-valently convex function, then $zf'(z)$ is p-valently starlike in $\mathbb{E}$.

2. Preliminaries

In this paper, we need the following lemmas.

Lemma 2.1. *If* $M(z) = z^p + \sum_{n=p+k}^{\infty} a_n z^n$ $(1 \leqq p$ *and* $1 \leqq k)$ *and* $N(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ *are analytic in* $\mathbb{E}$ *and* $N(z)$ *satisfies* $\operatorname{Re}(N(z)/zN'(z)) > \delta$ $(0 \leqq \delta < 1/p)$, *then*

$$\operatorname{Re}\left(\frac{M'(z)}{N'(z)}\right) > \beta \quad \textit{implies} \ \operatorname{Re}\left(\frac{M(z)}{N(z)}\right) > \frac{2\beta + k\delta}{2 + k\delta}. \tag{2.1}$$

Remark 2.2. This lemma holds to be true for $N(z)$ which is multivalently starlike in $\mathbb{E}$.

We owe the above lemma to Ponnusamy and Karunakaran [1].

Lemma 2.3. *Let* $f(z) \in \mathscr{S}^*(p,0)$. *Then*

$$\frac{F(z)}{p+1} = \int_0^z f(t)dt \in S^*(p+1,0) \tag{2.2}$$

or

$$\operatorname{Re}\frac{zF'(z)}{F(z)} > 0 \quad (z \in \mathbb{E}). \tag{2.3}$$

The proof of this result can be found in [2].

3. Main result

Theorem 3.1. *If* $f(z) \in \mathscr{A}(p)$ *satisfies the following condition:*

$$\operatorname{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > \alpha \quad (z \in \mathbb{E}) \tag{3.1}$$

for $\alpha(0 \leqq \alpha < 1)$, *then* $f(z) \in \mathscr{S}^*(p,\alpha+p-1)$ *or*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha + p - 1 \quad (z \in \mathbb{E}). \tag{3.2}$$

Proof. From assumption (3.1), $f^{(p-1)}(z)$ is univalently starlike of order α and it is trivial that

$$\left(zf^{(p-1)}(z) - f^{(p-2)}(z)\right)' = zf^{(p)}(z). \tag{3.3}$$

Note that $f^{(p-2)}(z)$ is starlike in $\mathbb{E}$ by Lemma 2.3. Therefore, applying Lemma 2.1 and (3.3), we have

$$\operatorname{Re}\frac{zf^{(p-1)}(z) - f^{(p-2)}(z)}{f^{(p-2)}(z)} > \alpha \quad (z \in \mathbb{E}). \tag{3.4}$$

From Lemma 2.3, $f^{(p-2)}(z)$ is 2-valently starlike in $\mathbb{E}$. Now then, it is trivial that

$$(zf^{(p-2)}(z) - 2f^{(p-3)}(z))' = zf^{(p-1)}(z) - f^{(p-2)}(z). \tag{3.5}$$

Then, from Lemma 2.1, 2-valently starlikeness of $f^{(p-2)}(z)$, and (3.5), we have

$$\operatorname{Re}\frac{zf^{(p-2)}(z) - 2f^{(p-3)}(z)}{f^{(p-3)}(z)} > \alpha \quad (z \in \mathbb{E}). \tag{3.6}$$

Further, it is trivial that

$$(zf^{(p-3)}(z) - 3f^{(p-4)}(z))' = zf^{(p-2)}(z) - 2f^{(p-3)}(z), \tag{3.7}$$

and applying the same method and reason as above, we have

$$\operatorname{Re}\frac{zf^{(p-3)}(z) - 3f^{(p-4)}(z)}{f^{(p-4)}(z)} > \alpha \quad (z \in \mathbb{E}), \tag{3.8}$$

where $f^{(p-3)}(z)$ is 3-valently starlike in $\mathbb{E}$. Applying the mathematical induction, we have

$$\operatorname{Re}\frac{zf'(z) - (p-1)f(z)}{f(z)} > \alpha \quad (z \in \mathbb{E}), \tag{3.9}$$

where $f(z)$ is p-valently starlike in $\mathbb{E}$. This shows that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha + p - 1 \quad (z \in \mathbb{E}), \tag{3.10}$$

or $f(z) \in \mathcal{S}^*(p, \alpha + p - 1)$. □

Our main result shows the following.

Corollary 3.2. *Let $f(z) \in \mathcal{A}(p)$, $2 \leqq p$, $0 \leqq \alpha < 1$,*

$$\operatorname{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > \alpha \quad (z \in \mathbb{E}), \tag{3.11}$$

and put $f(z) = z^{p-1}f_1(z)$ where

$$f_1(z) = z + \sum_{n=p+1}^{\infty} a_n z^{n-p+1}. \tag{3.12}$$

Then $f_1(z)$ is univalently starlike of order α in $\mathbb{E}$.

Proof. From the definition of $f_1(z)$ and Theorem 3.1, it follows that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = p - 1 + \operatorname{Re}\frac{zf_1'(z)}{f_1(z)} > \alpha + p - 1. \tag{3.13}$$

This completes the proof. □

Corollary 3.3. *Let* $f(z) \in \mathcal{A}(p)$, $2 \leqq p$, $0 \leqq \alpha < 1$, *and*

$$1 + \operatorname{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)} > \alpha \quad (z \in \mathbb{E}). \tag{3.14}$$

Then one has

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \beta(\alpha) + p - 1 \quad (z \in \mathbb{E}), \tag{3.15}$$

where

$$\beta(\alpha) = \begin{cases} \dfrac{1-2\alpha}{2^{2-2\alpha}\left[1-2^{2\alpha-1}\right]} & \text{if } \alpha \neq \dfrac{1}{2}, \\ \dfrac{1}{2\log 2} & \text{if } \alpha = \dfrac{1}{2}. \end{cases} \tag{3.16}$$

Proof. Putting

$$g(z) = \frac{f^{(p-1)}(z)}{p!} = z + \sum_{n=2}^{\infty} b_n z^n, \tag{3.17}$$

then from assumption (3.14), $g(z)$ is univalently convex of order α, and therefore from Wilken-Feng result [3] and Theorem 3.1, we have

$$\operatorname{Re}\frac{zg'(z)}{g(z)} = \operatorname{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > \beta(\alpha) \quad (z \in \mathbb{E}), \tag{3.18}$$

and therefore it follows that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \beta(\alpha) + p - 1 \quad (z \in \mathbb{E}). \tag{3.19}$$

□

We will give here an open problem.

Problem 3.4. Let $f(z) \in \mathcal{S}^*(p,\alpha)$ and $0 \leqq \alpha < p$.
Then

$$\frac{F(z)}{p+1} = \int_0^z f(t)dt \in S^*(p+1, \beta(p,\alpha)). \tag{3.20}$$

What is the best $\beta(p,\alpha)$?

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