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Research Article

Conditional Expectations for Unbounded Operator Algebras

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Two conditional expectations in unbounded operator algebras (O^*-algebras) are discussed. One is a vector conditional expectation defined by a linear map of an O^*-algebra into the Hilbert space on which the O^*-algebra acts. This has the usual properties of conditional expectations. This was defined by Gudder and Hudson. Another is an unbounded conditional expectation which is a positive linear map $\mathscr{E}$ of an O^*-algebra $\mathcal{M}$ onto a given O^*-subalgebra $\mathcal{N}$ of $\mathcal{M}$. Here the domain $D(\mathscr{E})$ of $\mathscr{E}$ does not equal to $\mathcal{M}$ in general, and so such a conditional expectation is called unbounded.

1. Introduction

In probability theory, conditional expectations play a fundamental role. A noncommutative analogue of conditional expectations in von Neumann algebras has been studied in [2–4]. A typical feature of probability in von Neumann algebras is that the observables permitted are usually bounded and some finiteness is imposed. But, unbounded observables occur naturally in quantum mechanics and quantum probability theory [1, 5–8] and so it is natural to consider conditional expectations in algebras of unbounded observables (O^*-algebras). The first study of conditional expectations in O^*-algebras was done by Gudder and Hudson [1]. Let $\mathcal{M}$ be an O^*-algebra on a dense subspace $\mathscr{D}$ in a Hilbert space $\mathscr{H}$ with a strongly cyclic and separating vector ξ_0 and $\mathcal{N}$ an O^*-subalgebra of $\mathcal{M}$. These notions are defined in Section 2. Gudder and Hudson have defined a conditional expectation given by $(\mathcal{N},\xi_0)$ by the map $A \mapsto P_{\mathcal{N}}A\xi_0$ of $\mathcal{M}$ into the closed subspace $\mathscr{H}_{\mathcal{N}} \equiv \overline{\mathcal{N}\xi_0}$ of $\mathscr{H}$, which has the usual properties of a conditional expectation, where $P_{\mathcal{N}}$ is the projection of $\mathscr{H}$ onto $\mathscr{H}_{\mathcal{N}}$. We call this the *vector conditional expectation* given by $(\mathcal{N},\xi_0)$. On the other hand, it is natural to consider when a conditional expectation of the O*-algebra $\mathcal{M}$ onto the O*-subalgebra $\mathcal{N}$ exists. Such a conditional expectation does not necessarily exist even for von Neumann algebras. In fact, Takesaki [2] has shown that there exists a conditional expectation of the von Neumann algebra $\mathcal{M}$ onto the von Neumann subalgebra $\mathcal{N}$ if and only if $\Delta_{\xi_0}^{it}\mathcal{N}\Delta_{\xi_0}^{-it} = \mathcal{N}$ for all $t \in \mathbb{R}$, where Δ_{ξ_0} is the modular operator of the left Hilbert algebra $\mathcal{M}\xi_0$. Here we consider a map $\mathcal{E}(\cdot \mid \mathcal{N}) : A \to P_{\mathcal{N}}A\lceil\mathcal{N}\xi_0$ of $\mathcal{M}$ into the partial O*-algebra $\mathcal{L}^\dagger(\mathcal{N}\xi_0, \mathcal{H}_{\mathcal{N}})$. We will show that $\mathcal{E}(\cdot \mid \mathcal{N})$ has properties similar to those of conditional expectations, so it will be called a *weak conditional-expectation* of $\mathcal{M}$ with respect to $(\mathcal{N}, \xi_0)$. Unfortunately, the range $\mathcal{E}(\mathcal{M}\lceil\mathcal{N})$ of the weak conditional-expectation $\mathcal{E}(\cdot \mid \mathcal{N})$ is not necessarily contained in $\mathcal{N}$, and so we define an *unbounded conditional-expectation* $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ξ_0 as follows: $\mathcal{E}$ is a map of $\mathcal{M}$ onto $\mathcal{N}$ satisfying

(i) the domain $D(\mathcal{E})$ of $\mathcal{E}$ is a $\dagger$-invariant subspace of $\mathcal{M}$ containing $\mathcal{N}$ such that $\mathcal{N}D(\mathcal{E}) \subset D(\mathcal{E})$;

(ii) $\mathcal{E}(A)^\dagger = \mathcal{E}(A^\dagger)$, for all $A \in D(\mathcal{E})$ and $\mathcal{E}(X) = X$, for all $X \in \mathcal{N}$;

(iii) $\mathcal{E}(AX) = \mathcal{E}(A)X$ and $\mathcal{E}(XA) = X\mathcal{E}(A)$, for all $A \in D(\mathcal{E})$, for all $X \in \mathcal{N}$;

(iv) $\omega_{\xi_0}(\mathcal{E}(A)) = \omega_{\xi_0}(A)$, for all $A \in D(\mathcal{E})$,

where ω_{ξ_0} is a positive linear functional on $\mathcal{M}$ defined by $\omega_{\xi_0}(A) = (A\xi_0 \mid \xi_0)$, $A \in \mathcal{M}$.

By restriction of the weak conditional-expectation $\mathcal{E}(\cdot \mid \mathcal{N})$, we will show that there exists a maximal unbounded conditional expectation $\mathcal{E}_{\mathcal{N}}$ of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ξ_0. Furthermore, we will investigate unbounded conditional-expectations in case that $\mathcal{M}$ and $\mathcal{N}$ are generalized von Neumann algebras which are unbounded generalization of von Neumann algebras and that the von Neumann algebra $(\mathcal{N}'_{\mathrm{w}})'$ (the usual commutant of the weak commutant $\mathcal{N}'_{\mathrm{w}}$ of $\mathcal{N}$) satisfies the Takesaki condition. As an application of vector conditional expectations we will establish the existence of coarse graining for absolutely continuous positive linear functionals.

2. Preliminaries

In this section we introduce the basic definitions and properties of (partial) O*-algebras. We refer to [6–9] for O*-algebras and to [10] for partial O*-algebras.

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot \mid \cdot)$ and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}(\mathcal{D}, \mathcal{H})$ the set of all linear operators X in $\mathcal{H}$ such that $\mathcal{D}(X)$ (the domain of X) $= \mathcal{D}$, and

$$\begin{aligned} \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) &= \{X \in \mathcal{L}(\mathcal{D}, \mathcal{H});\ \mathcal{D}(X^*) \supset \mathcal{D}\}, \\ \mathcal{L}^\dagger(\mathcal{D}) &= \{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H});\ X\mathcal{D} \subset \mathcal{D},\ X^*\mathcal{D} \subset \mathcal{D}\}. \end{aligned} \tag{2.1}$$

Then $\mathcal{L}(\mathcal{D}, \mathcal{H})$ is a vector space with the usual operations $X + Y$ and λX, and $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is equipped with the following operations and involution:

(i) the sum $X + Y$;

(ii) the scalar multiplication λX;

(iii) the involution $X \mapsto X^\dagger \equiv X^*\lceil\mathcal{D}$, that is, $(X + \lambda Y)^\dagger = X^\dagger + \bar{\lambda}Y^\dagger$, $X^{\dagger\dagger} = X$;

(iv) the weak partial multiplication $X \square Y = X^{\dagger *}Y$, defined whenever X is a left multiplier of Y, ($X \in L^{\mathrm{w}}(Y)$ or $Y \in R^{\mathrm{w}}(X)$), that is, if and only if $Y\mathcal{D} \subset \mathcal{D}(X^{\dagger *})$ and $X^\dagger\mathcal{D} \subset \mathcal{D}(Y^*)$.

Then $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a *partial $*$-algebra*, that is, the following hold:

(i) $X \in L^{\mathrm{w}}(Y)$ if and only if $Y^\dagger \in L^{\mathrm{w}}(X^\dagger)$ and then $(X\Box Y)^\dagger = Y^\dagger \Box X^\dagger$;

(ii) if $X \in L^{\mathrm{w}}(Y)$ and $X \in L^{\mathrm{w}}(Z)$, then $X \in L^{\mathrm{w}}(\lambda Y + \mu Z)$ for all $\lambda,\mu \in \mathbb{C}$ and $X\Box(\lambda Y + \mu Z) = \lambda(X\Box Y) + \mu(X\Box Z)$.

$\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra with the usual multiplication XY (which coincides with the weak partial multiplication $X\Box Y$) and the involution $X \mapsto X^\dagger$. A partial $*$-subalgebra of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is called a *partial O^*-algebra* on $\mathcal{D}$, and a $*$-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ is called an *O^*-algebra* on $\mathcal{D}$. Here we assume that a (partial) O*-algebra contains the identity operator I.

In analogy with the notion of a closed symmetric (selfadjoint) operator, we define the notion of a closed O*-algebra (a selfadjoint O*-algebra). Let $\mathcal{M}$ be an O*-algebra on $\mathcal{D}$. We define a natural graph topology on $\mathcal{D}$. This topology $t_{\mathcal{M}}$ is a locally convex topology defined by a family $\{\|\cdot\|_X;\ X \in \mathcal{M}\}$ of seminorms $\|\xi\|_X \equiv \|\xi\| + \|X\xi\|,\ (\xi \in \mathcal{D})$, and it is called the *graph* (or *induced*) topology on $\mathcal{D}$. If the locally convex space $\mathcal{D}[t_{\mathcal{M}}]$ is complete, then $\mathcal{M}$ is said to be *closed*. We denote by $\widetilde{\mathcal{D}}(\mathcal{M})$ the completion of the locally convex space $\mathcal{D}[t_{\mathcal{M}}]$ and put

$$\begin{aligned} \widetilde{X} &= \overline{X}\lceil\widetilde{\mathcal{D}}(\mathcal{M}), \quad X \in \mathcal{M}; \\ \widetilde{\mathcal{M}} &= \{\widetilde{X};\ X \in \mathcal{M}\}. \end{aligned} \tag{2.2}$$

Then $\widetilde{\mathcal{M}}$ is a closed O*-algebra on $\widetilde{\mathcal{D}}(\mathcal{M})$ in $\mathcal{H}$ which is the smallest closed extension of $\mathcal{M}$, and $\widetilde{\mathcal{D}}(\mathcal{M}) \equiv \bigcap_{X\in\mathcal{M}} \mathcal{D}(\overline{X})$. $\widetilde{\mathcal{M}}$ is called the *closure* of $\mathcal{M}$.

We next define the notion of selfadjointness of $\mathcal{M}$. If $\mathcal{D} = \mathcal{D}^*(\mathcal{M}) \equiv \bigcap_{X\in\mathcal{M}} \mathcal{D}(X^*)$, then $\mathcal{M}$ is said to be *selfadjoint*. If $\widetilde{\mathcal{D}}(\mathcal{M}) = \mathcal{D}^*(\mathcal{M})$, then $\mathcal{M}$ is said to be *essentially selfadjoint*. It is clear that

$$\begin{aligned} \mathcal{D} \subset \widetilde{\mathcal{D}}(\mathcal{M}) &\subset \mathcal{D}^*(\mathcal{M}), \\ X \subset \widetilde{X} &\subset X^*, \quad \forall X \in \mathcal{M}. \end{aligned} \tag{2.3}$$

We define commutants and bicommutants of $\mathcal{M}$. The *weak commutant* $\mathcal{M}'_{\mathrm{w}}$ of $\mathcal{M}$ is defined by

$$\mathcal{M}'_{\mathrm{w}} = \{C \in \mathcal{B}(\mathcal{H});\ (CX\xi \mid \eta) = (C\xi \mid X^\dagger\eta),\ \forall X \in \mathcal{M},\ \forall \xi,\eta \in \mathcal{D}\}, \tag{2.4}$$

where $\mathcal{B}(\mathcal{H})$ is a $*$-algebra of all bounded linear operators on $\mathcal{H}$. Then $\mathcal{M}'_{\mathrm{w}}$ is a weakly closed $*$-invariant subspace of $\mathcal{B}(\mathcal{H})$ such that $(\widetilde{\mathcal{M}})'_{\mathrm{w}} = \mathcal{M}'_{\mathrm{w}}$. If $\mathcal{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$, then $\mathcal{M}'_{\mathrm{w}}$ is a von Neumann algebra; in particular, if $\mathcal{M}$ is selfadjoint, then $\mathcal{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. The unbounded commutants and unbounded bicommutants of $\mathcal{M}$ are defined by

$$\begin{aligned} \mathcal{M}'_{\delta} &= \{S \in \mathcal{L}(\mathcal{D},\mathcal{H});\ (SX\xi \mid \eta) = (S\xi \mid X^\dagger\eta),\ \forall X \in \mathcal{M},\ \forall \xi,\eta \in \mathcal{D}\}; \\ \mathcal{M}'_{\sigma} &= \mathcal{M}'_{\delta} \cap \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}); \\ \mathcal{M}'_{\mathrm{c}} &= \mathcal{M}'_{\sigma} \cap \mathcal{L}^\dagger(\mathcal{D}); \\ \mathcal{M}''_{\mathrm{w}\sigma} &= \{X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H});\ (CX\xi \mid \eta) = (C\xi \mid X^\dagger\eta),\ \forall C \in \mathcal{M}'_{\mathrm{w}},\ \forall \xi,\eta \in \mathcal{D}\}; \\ \mathcal{M}''_{\mathrm{wc}} &= \mathcal{M}''_{\mathrm{w}\sigma} \cap \mathcal{L}^\dagger(\mathcal{D}). \end{aligned} \tag{2.5}$$

Then the following hold.

(i) $\mathcal{M}'_\delta$ is a subspace of $\mathcal{L}(\mathcal{D},\mathcal{H})$.

(ii) $\mathcal{M}'_\sigma$ is a $\dagger$-invariant subspace of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ and $(\mathcal{M}'_\sigma)_b \equiv \{S \in \mathcal{M}'_\sigma;\ \overline{S} \in \mathcal{B}(\mathcal{H})\} = \mathcal{M}'_{\mathrm{w}}\lceil\mathcal{D}$.

(iii) $\mathcal{M}'_{\mathrm{c}}$ is a subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$.

(iv) $\mathcal{M}''_{\mathrm{w}\sigma}$ is a $\dagger$-invariant τ_{s^*}-closed subspace of $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ containing $\mathcal{M}$, where the strong* topology τ_{s^*} is defined by the family $\{p^*_\xi(\cdot);\ \xi\in\mathcal{D}\}$ of seminorms

$$p^*_\xi(X) \equiv \|X\xi\| + \|X^\dagger\xi\|, \quad X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H}). \tag{2.6}$$

(iiv) $\mathcal{M}''_{\mathrm{wc}}$ is a τ_{s^*}-closed O*-algebra on $\mathcal{D}$ such that $\mathcal{M}\subset\mathcal{M}''_{\mathrm{wc}}$ and $(\mathcal{M}''_{\mathrm{wc}})'_{\mathrm{w}} = \mathcal{M}'_{\mathrm{w}}$.

(iiiv) If $\mathcal{M}'_{\mathrm{w}}\mathcal{D}\subset\mathcal{D}$, then $\mathcal{M}''_{\mathrm{w}\sigma}$ is a partial O*-algebra on $\mathcal{D}$ such that

$$\begin{aligned}\mathcal{M}''_{\mathrm{w}\sigma} &= \{X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H});\ \overline{X} \text{ is affiliated with } (\mathcal{M}'_{\mathrm{w}})'\}\\ &= \overline{(\mathcal{M}'_{\mathrm{w}})'\lceil\mathcal{D}}^{\tau_{s^*}} \qquad (\text{the } \tau_{s^*}\text{-closure of } (\mathcal{M}'_{\mathrm{w}})'\lceil\mathcal{D} \text{ in } \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})),\end{aligned} \tag{2.7}$$

and $\mathcal{M}''_{\mathrm{wc}}$ is an O*-algebra on $\mathcal{D}$ such that

$$\begin{aligned}\mathcal{M}''_{\mathrm{wc}} &= \{X \in \mathcal{L}^\dagger(\mathcal{D});\ \overline{X} \text{ is affiliated with } (\mathcal{M}'_{\mathrm{w}})'\}\\ &= \overline{(\mathcal{M}'_{\mathrm{w}})'\lceil\mathcal{D}}^{\tau_{s^*}} \cap \mathcal{L}^\dagger(\mathcal{D}).\end{aligned} \tag{2.8}$$

We introduce the notions of generalized von Neumann algebras and extended W*-algebras which are unbounded generalizations of von Neumann algebras. If $\mathcal{M}'_{\mathrm{w}}\mathcal{D}\subset\mathcal{D}$ and $\mathcal{M} = \mathcal{M}''_{\mathrm{wc}}$, then $\mathcal{M}$ is said to be a *generalized von Neumann algebra* (or a *GW*-algebra*) on $\mathcal{D}$. A closed O*-algebra $\mathcal{M}$ on $\mathcal{D}$ is said to be an *extended W*-algebra* (simply, an *EW*-algebra*) if $(I+X^\dagger X)^{-1}$ exists in $\mathcal{M}_b \equiv \{A\in\mathcal{M};\ \overline{A}\in\mathcal{B}(\mathcal{H})\}$ for all $X\in\mathcal{M}$ and $\overline{\mathcal{M}_b} \equiv \{\overline{A};\ A\in\mathcal{M}_b\}$ is a von Neumann algebra on $\mathcal{H}$.

We define the notion of strongly cyclic vectors for a closed O*-algebra $\mathcal{M}$ on $\mathcal{D}$ in $\mathcal{H}$. We denote by $\overline{\mathcal{K}}$ the closure of a subset $\mathcal{K}$ of $\mathcal{H}$ with respect to the Hilbert space norm and denote by $\overline{\mathfrak{M}}^{t_{\mathcal{M}}}$ the closure of a subset $\mathfrak{M}$ of $\mathcal{D}$ with respect to the graph topology $t_{\mathcal{M}}$. Let $\mathfrak{M}$ be an $\mathcal{M}$-invariant subspace of $\mathcal{D}$. Then $\mathcal{M}\lceil\mathfrak{M} \equiv \{X\lceil\mathfrak{M};\ X\in\mathcal{M}\}$ is an O*-algebra on $\mathfrak{M}$ and its closure $\mathcal{M}_{\mathfrak{M}} (\equiv (\mathcal{M}\lceil\mathfrak{M})^{\sim})$ is a closed O*-algebra on $\overline{\mathfrak{M}}^{t_{\mathcal{M}}}$ in $\overline{\mathfrak{M}}$. If $\mathfrak{M}$ is *essentially selfadjoint*, that is, $\mathcal{M}_{\mathfrak{M}}$ is selfadjoint, then the projection $P_{\overline{\mathfrak{M}}}$ of $\mathcal{H}$ onto $\overline{\mathfrak{M}}$ belongs to $\mathcal{M}'_{\mathrm{w}}$, $P_{\overline{\mathfrak{M}}}\mathcal{D}^*(\mathcal{M}) = \overline{\mathfrak{M}}^{t_{\mathcal{M}}} \subset\mathcal{D}$ and $\mathcal{M}_{\mathfrak{M}} = \mathcal{M}_{P_{\overline{\mathfrak{M}}}} \equiv \{X_{P_{\overline{\mathfrak{M}}}};\ X\in\mathcal{M}\}$, where $X_{P_{\overline{\mathfrak{M}}}}P_{\overline{\mathfrak{M}}}\xi = P_{\overline{\mathfrak{M}}}X\xi$ for $X\in\mathcal{M}$ and $\xi\in\mathcal{D}$. A vector ξ_0 in $\mathcal{D}$ is said to be *strongly cyclic* if $\overline{\mathcal{M}\xi_0}^{t_{\mathcal{M}}} = \mathcal{D}$, and ξ_0 is said to be *separating* if $\overline{\mathcal{M}'_{\mathrm{w}}\xi_0} = \mathcal{H}$.

We define the notions of (unbounded) $*$-representations of $*$-algebras. Let $\mathcal{A}$ be an $*$-algebra with identity 1. A ($*$-)homomorphism π of $\mathcal{A}$ into an O*-algebra $\mathcal{L}^\dagger(\mathcal{D})$ with $\pi(1) = I$ is said to be a ($*$-)*representation* of $\mathcal{A}$. In this case, $\mathcal{D}$ and $\mathcal{H}$ are denoted, respectively, by $\mathcal{D}(\pi)$ and $\mathcal{H}_\pi$.

Let π_1 and π_2 be $(*$-)representations of $\mathscr{A}$ on the same Hilbert space. If $\pi_1(x) \subset \pi_2(x)$ for each $x \in \mathscr{A}$, then π_2 is said to be an *extension* of π_1 and it is denoted by $\pi_1 \subset \pi_2$. Let π be a $(*$-)representation of $\mathscr{A}$. We put

$$\widetilde{\mathscr{D}}(\pi) = \bigcap_{x \in \mathscr{A}} \mathscr{D}(\overline{\pi(x)}), \quad \widetilde{\pi}(x) = \overline{\pi(x)} \lceil \widetilde{\mathscr{D}}(\pi), \quad x \in A. \tag{2.9}$$

If $\pi = \widetilde{\pi}$, then π is said to be *closed*; $\widetilde{\pi}$ is a closed $(*$-)representation of $\mathscr{A}$ which is the smallest closed extension of π and it is called the *closure* of π. Let π be a $*$-representation of $\mathscr{A}$. We put

$$\mathscr{D}^*(\pi) = \bigcap_{x \in \mathscr{A}} \mathscr{D}(\pi(x)^*), \quad \pi^*(x) = \pi(x^*)^* \lceil \mathscr{D}^*(\pi), \quad x \in \mathscr{A}. \tag{2.10}$$

Then π^* is a closed representation of $\mathscr{A}$ such that $\pi \subset \widetilde{\pi} \subset \pi^*$ and is called the *adjoint* of π. If $\pi = \pi^*$, then π is said to be *selfadjoint*. We remark that π is closed (resp., selfadjoint) if and only if the O*-algebra $\pi(\mathscr{A})$ is closed (resp., selfadjoint).

3. Vector conditional expectations

Let $\mathscr{M}$ be a closed O*-algebra on $\mathscr{D}$ in $\mathscr{H}$, $\xi_0 \in \mathscr{D}$ a strongly cyclic and separating vector for $\mathscr{M}$, and $\mathscr{N}$ an O*-subalgebra of $\mathscr{M}$. Then

$$\begin{array}{ccccccc} \mathscr{N}\xi_0 \subset \overline{\mathscr{N}\xi_0}^{t_{\mathscr{M}}} & \subset & \overline{\mathscr{N}\xi_0}^{t_{\mathscr{N}}} & \subset & \overline{\mathscr{N}\xi_0} & & \\ \cap & & & & & \subset \mathscr{H}. & \\ \overline{\mathscr{M}\xi_0}^{t_{\mathscr{M}}} = \mathscr{D} & & & & & & \end{array} \tag{3.1}$$

If $\mathscr{N}$ is closed, then $\mathscr{N}\xi_0 \subset \overline{\mathscr{N}\xi_0}^{t_{\mathscr{M}}} \subset \overline{\mathscr{N}\xi_0}^{t_{\mathscr{N}}} \subset \mathscr{D} = \overline{\mathscr{M}\xi_0}^{t_{\mathscr{M}}}$. The following is easily shown.

Lemma 3.1. *Put*

$$\begin{gathered} \mathscr{D}(\pi_{\mathscr{N}}) = \mathscr{N}\xi_0, \\ \pi_{\mathscr{N}}(X)Y\xi_0 = XY\xi_0, \quad \forall X, Y \in \mathscr{N}, \\ \mathscr{D}(\pi_{\mathscr{N}}^{\mathscr{M}}) = \overline{\mathscr{N}\xi_0}^{t_{\mathscr{M}}}, \\ \pi_{\mathscr{N}}^{\mathscr{M}}(X)\xi = X\xi, \quad \forall X \in \mathscr{N}, \ \forall \xi \in \mathscr{D}(\pi_{\mathscr{N}}^{\mathscr{M}}). \end{gathered} \tag{3.2}$$

Then $\pi_{\mathscr{N}}$ and $\pi_{\mathscr{N}}^{\mathscr{M}}$ are faithful $*$-representations of $\mathscr{N}$ in $\mathscr{H}_{\mathscr{N}} \equiv \overline{\mathscr{N}\xi_0}$ such that $\pi_{\mathscr{N}} \subset \pi_{\mathscr{N}}^{\mathscr{M}} \subset \widetilde{\pi_{\mathscr{N}}}$, and

$$\mathscr{D}(\pi_{\mathscr{N}}) \subset \mathscr{D}(\pi_{\mathscr{N}}^{\mathscr{M}}) \subset \mathscr{D}(\widetilde{\pi_{\mathscr{N}}}), \qquad \mathscr{D}^*(\pi_{\mathscr{N}}) = \mathscr{D}^*(\pi_{\mathscr{N}}^{\mathscr{M}}). \tag{3.3}$$

We denote by $P_{\mathscr{N}}$ the projection of $\mathscr{H}$ onto $\mathscr{H}_{\mathscr{N}} \equiv \overline{\mathscr{N}\xi_0}$. Then we have the following lemma.

Lemma 3.2. *$P_{\mathscr{N}}\mathscr{D}^*(\mathscr{M}) \subset \mathscr{D}^*(\pi_{\mathscr{N}})$ and $\pi_{\mathscr{N}}^*(X)P_{\mathscr{N}}\xi = P_{\mathscr{N}}X^{\dagger *}\xi$, for all $X \in \mathscr{N}$ and for all $\xi \in \mathscr{D}^*(\mathscr{M})$.*

Proof. Take arbitrary $X \in \mathcal{N}$ and $\xi \in \mathcal{D}^*(\mathcal{M})$. For any $Y \in \mathcal{N}$, we have

$$(X^\dagger Y\xi_0 \mid P_{\mathcal{N}}\xi) = (X^\dagger Y\xi_0 \mid \xi) = (Y\xi_0 \mid X^{\dagger *}\xi) = (Y\xi_0 \mid P_{\mathcal{N}}X^{\dagger *}\xi), \tag{3.4}$$

and so $P_{\mathcal{N}}\mathcal{D}^*(\mathcal{M}) \subset \mathcal{D}^*(\pi_{\mathcal{N}})$ and $\pi_{\mathcal{N}}^*(X)P_{\mathcal{N}}\xi = P_{\mathcal{N}}X^{\dagger *}\xi$. □

First we introduce the notion of a vector conditional expectation defined by Gudder and Hudson [1].

Definition 3.3. A map E of $\mathcal{M}$ into $\mathcal{D}^*(\pi_{\mathcal{N}})$ is said to be a *vector conditional expectation* of $\mathcal{M}$ given by $(\mathcal{N}, \xi_0)$ if the following hold.

(i) $E(XA) = \pi_{\mathcal{N}}^*(X)E(A)$, for all $A \in \mathcal{M}$, for all $X \in \mathcal{N}$.

(ii) $\omega_{\xi_0}(A) = (E(A) \mid \xi_0)$, for all $A \in \mathcal{M}$.

A map E satisfying the conditions of Definition 3.3 was called a conditional expectation of $\mathcal{M}$ given by $(\mathcal{N}, \xi_0)$ by Gudder and Hudson [1]. They gave the following theorem. We prove the theorem for the sake of completeness.

Theorem 3.4. *A vector conditional expectation E of $\mathcal{M}$ given by $(\mathcal{N}, \xi_0)$ exists uniquely, and*

$$E(A) = P_{\mathcal{N}}A\xi_0, \quad \forall A \in \mathcal{M}. \tag{3.5}$$

Denote by $E(A \mid \mathcal{N})$ the unique vector conditional expectation of $\mathcal{M}$ given by $(\mathcal{N}, \xi_0)$, that is,

$$E(A \mid \mathcal{N}) = P_{\mathcal{N}}A\xi_0, \quad \forall A \in \mathcal{M}. \tag{3.6}$$

Proof. We put

$$E(A) = P_{\mathcal{N}}A\xi_0, \quad A \in \mathcal{M}. \tag{3.7}$$

By Lemma 3.2 E is a map of $\mathcal{M}$ into $\mathcal{D}^*(\pi_{\mathcal{N}})$. It is clear that E is linear. For any $A \in \mathcal{M}$ and $X \in \mathcal{N}$ we have, by Lemma 3.2,

$$\begin{aligned} E(XA) &= P_{\mathcal{N}}XA\xi_0 = \pi_{\mathcal{N}}^*(X)P_{\mathcal{N}}A\xi_0 = \pi_{\mathcal{N}}^*(X)E(A), \\ \omega_{\xi_0}(XA) &= (A\xi_0 \mid X^\dagger\xi_0) = (P_{\mathcal{N}}A\xi_0 \mid X^\dagger\xi_0) = (\pi_{\mathcal{N}}^*(X)E(A) \mid \xi_0); \end{aligned} \tag{3.8}$$

in particular,

$$\omega_{\xi_0}(A) = (E(A) \mid \xi_0). \tag{3.9}$$

Hence E is a vector conditional expectation of $\mathcal{M}$ given by $(\mathcal{N}, \xi_0)$.

We show the uniqueness of vector conditional expectations. Let E' be any vector conditional expectation of $\mathcal{M}$ given by $(\mathcal{N}, \xi_0)$. For any $A \in \mathcal{M}$ and $X \in \mathcal{N}$ we have

$$\begin{aligned} (E'(A) \mid X\xi_0) &= (\pi_{\mathcal{N}}^*(X^\dagger)E'(A) \mid \xi_0) = (E'(X^\dagger A) \mid \xi_0) = \omega_{\xi_0}(X^\dagger A) \\ &= (A\xi_0 \mid X\xi_0) = (P_{\mathcal{N}}A\xi_0 \mid X\xi_0), \end{aligned} \tag{3.10}$$

which implies that

$$E'(A) = P_{\mathcal{N}}A\xi_0. \tag{3.11}$$

□

4. Unbounded conditional expectations for O*-algebras

We begin with the definition of unbounded conditional expectations of O*-algebras. In this section let $\mathcal{M}$ be a closed O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ with a strongly cyclic and separating vector ξ_0 and $\mathcal{N}$ an O*-subalgebra of $\mathcal{M}$.

Definition 4.1. A map $\mathcal{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ is said to be an *unbounded conditional expectation* of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ξ_0 if

(i) the domain $D(\mathcal{E})$ of $\mathcal{E}$ is a †-invariant subspace of $\mathcal{M}$ containing $\mathcal{N}$ such that $\mathcal{N}D(\mathcal{E}) \subset D(\mathcal{E})$;

(ii) $\mathcal{E}$ is a projection; that is, it is hermitian ($\mathcal{E}(A)^\dagger = \mathcal{E}(A^\dagger)$, for all $A \in D(\mathcal{E})$) and $\mathcal{E}(X) = X$, for all $X \in \mathcal{N}$;

(iii) $\mathcal{E}$ is $\mathcal{N}$-linear, that is,

$$\mathcal{E}(AX) = \mathcal{E}(A)X, \quad \mathcal{E}(XA) = X\mathcal{E}(A), \quad \forall A \in D(\mathcal{E}),\ \forall X \in \mathcal{N}; \tag{4.1}$$

(iv) $\omega_{\xi_0}(\mathcal{E}(A)) = \omega_{\xi_0}(A)$, for all $A \in D(\mathcal{E})$.

In particular, if $D(\mathcal{E}) = \mathcal{M}$, then $\mathcal{E}$ is said to be a *conditional expectation* of $\mathcal{M}$ onto $\mathcal{N}$.

For unbounded conditional expectations we have the following lemma.

Lemma 4.2. *Let $\mathcal{E}$ be an unbounded conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ξ_0. Then the following statements hold.*

(1) $\mathcal{E}(A)\xi_0 = E(A \mid \mathcal{N})$, *for all* $A \in D(\mathcal{E})$.

(2) $\mathcal{E}$ *is an* $\mathcal{N}$*-Schwarz map, that is,*

$$\mathcal{E}(A^\dagger)\mathcal{E}(A) \le \mathcal{E}(A^\dagger A) \quad \text{on } \mathcal{D}(\pi_{\mathcal{N}}^{\mathcal{M}}) \text{ whenever } A \in D(\mathcal{E}) \text{ s.t. } A^\dagger A \in D(\mathcal{E}). \tag{4.2}$$

Proof. (1) For all $A \in D(\mathcal{E})$ and $X \in \mathcal{N}$ we have

$$\begin{aligned}(\mathcal{E}(A)\xi_0 \mid X\xi_0) &= (X^\dagger\mathcal{E}(A)\xi_0 \mid \xi_0) = (\mathcal{E}(X^\dagger A)\xi_0 \mid \xi_0)\\ &= \omega_{\xi_0}(X^\dagger A) = (A\xi_0 \mid X\xi_0) = (P_{\mathcal{N}}A\xi_0 \mid X\xi_0),\end{aligned} \tag{4.3}$$

which implies

$$\mathcal{E}(A)\xi_0 = P_{\mathcal{N}}A\xi_0 = E(A \mid \mathcal{N}). \tag{4.4}$$

(2) Take an arbitrary $A \in D(\mathcal{E})$ s.t. $A^\dagger A \in D(\mathcal{E})$. Then we have

$$\begin{aligned}(\mathcal{E}(A^\dagger)\mathcal{E}(A)X\xi_0 \mid X\xi_0) &= \|\mathcal{E}(A)X\xi_0\|^2 = \|\mathcal{E}(AX)\xi_0\|^2\\ &= \|P_{\mathcal{N}}AX\xi_0\|^2 \le \|AX\xi_0\|^2 \quad (\text{by } (1)),\\ (\mathcal{E}(A^\dagger A)X\xi_0 \mid X\xi_0) &= (\mathcal{E}(X^\dagger A^\dagger AX)\xi_0 \mid \xi_0) = \omega_{\xi_0}(\mathcal{E}(X^\dagger A^\dagger AX))\\ &= \omega_{\xi_0}(X^\dagger A^\dagger AX) = \|AX\xi_0\|^2,\end{aligned} \tag{4.5}$$

for each $X \in \mathcal{N}$, which by $\mathcal{D}(\pi_{\mathcal{N}}^{\mathcal{M}}) = \overline{\mathcal{N}\xi_0}^{t_{\mathcal{M}}}$ implies that

$$\mathcal{E}(A^\dagger A) \le \mathcal{E}(A)^\dagger\mathcal{E}(A) \quad \text{on } \mathcal{D}(\pi_{\mathcal{N}}^{\mathcal{M}}). \tag{4.6}$$

□

Let $\mathfrak{E}$ be the set of all unbounded conditional expectations of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ξ_0. Then $\mathfrak{E}$ is an ordered set with the following order $\subset$.

$$\mathscr{E}_1 \subset \mathscr{E}_2 \quad \text{iff } D(\mathscr{E}_1) \subset \mathscr{D}(\mathscr{E}_2), \qquad \mathscr{E}_1(A) = \mathscr{E}_2(A), \quad \forall A \in D(\mathscr{E}_1). \tag{4.7}$$

In Theorem 4.6 we will show that there exists a maximal unbounded conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ξ_0.

Definition 4.3. A map $\mathscr{E}$ of $\mathcal{M}$ into the partial O*-algebra $\mathscr{L}^\dagger(\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}}), \mathscr{H}_{\mathcal{N}})$ is said to be a *weak conditional expectation* of $\mathcal{M}$ with respect to $(\mathcal{N}, \xi_0)$ if

(i) $\mathscr{E}$ is hermitian, that is, $\mathscr{E}(A)^\dagger = \mathscr{E}(A^\dagger)$, for all $A \in \mathcal{M}$;

(ii) $\mathscr{E}(\mathcal{M})\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}}) \subset \mathscr{D}^*(\pi_{\mathcal{N}}^{\mathcal{M}})$ and

$$\pi_{\mathcal{N}}^{\mathcal{M}}(X) \square \mathscr{E}(A) = \mathscr{E}(XA), \quad \forall X \in \mathcal{N}, \ \forall A \in \mathcal{M}; \tag{4.8}$$

(iii) $\omega_{\xi_0}(A) = (\mathscr{E}(A)\xi_0 \mid \xi_0)$, for all $A \in \mathcal{M}$.

For weak conditional expectations we have the following.

Theorem 4.4. *There exists a unique weak conditional-expectation $\mathscr{E}(\cdot \mid \mathcal{N})$ of $\mathcal{M}$ with respect to $(\mathcal{N}, \xi_0)$, and*

$$\mathscr{E}(A \mid \mathcal{N}) = P_{\mathcal{N}} A \lceil \mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}}), \quad \forall A \in \mathcal{M}. \tag{4.9}$$

Proof. We first show the existence: we put $\mathscr{E}(A \mid \mathcal{N}) = P_{\mathcal{N}} A \lceil \mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}})$, $A \in \mathcal{M}$. It follows from Lemma 3.2 that for any $A \in \mathcal{M}$, $\mathscr{E}(A \mid \mathcal{N})$ is a linear map of $\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}})$ into $\mathscr{D}^*(\pi_{\mathcal{N}}^{\mathcal{M}})$, and furthermore

$$\begin{aligned}(\mathscr{E}(A \mid \mathcal{N})\xi \mid \eta) &= (P_{\mathcal{N}} A\xi \mid \eta) = (A\xi \mid \eta) = (\xi \mid A^\dagger \eta) \\ &= (\xi \mid P_{\mathcal{N}} A^\dagger \eta) = (\xi \mid \mathscr{E}(A^\dagger \mid \mathcal{N})\eta)\end{aligned} \tag{4.10}$$

for each $\xi, \eta \in \mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}})$, which implies that $\mathscr{E}(A \mid \mathcal{N}) \in \mathscr{L}^\dagger(\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}}), \mathscr{H}_{\mathcal{N}})$ and $\mathscr{E}(A \mid \mathcal{N})^\dagger = \mathscr{E}(A^\dagger \mid \mathcal{N})$. Thus $\mathscr{E}(\cdot \mid \mathcal{N})$ satisfies the condition (i) in Definition 4.3. Furthermore, we show that it satisfies the conditions (ii) and (iii) in Definition 4.3.

(ii) Take arbitrary $X \in \mathcal{N}$ and $A \in \mathcal{M}$. Since $\mathscr{E}(A \mid \mathcal{N}) \in \mathscr{L}^\dagger(\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}}), \mathscr{H}_{\mathcal{N}})$ and $\mathscr{E}(A \mid \mathcal{N})\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}}) \subset \mathscr{D}^*(\pi_{\mathcal{N}}^{\mathcal{M}})$ as shown above, it follows that $\pi_{\mathcal{N}}^{\mathcal{M}}(X) \square \mathscr{E}(A \mid \mathcal{N})$ is well defined and

$$\begin{aligned}(\pi_{\mathcal{N}}^{\mathcal{M}}(X)^\dagger \square \mathscr{E}(A \mid \mathcal{N})^\dagger)\xi &= \pi_{\mathcal{N}}^{\mathcal{M}}(X)^* P_{\mathcal{N}} A^\dagger \xi = P_{\mathcal{N}} X^\dagger A^\dagger \xi \\ &= \mathscr{E}(X^\dagger A^\dagger \mid \mathcal{N})\xi \quad \text{(by Lemma 3.2)}\end{aligned} \tag{4.11}$$

for each $A \in \mathcal{M}$, $X \in \mathcal{N}$, and $\xi \in \mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}})$.

(iii) This follows from the equality

$$\omega_{\xi_0}(A) = (A\xi_0 \mid \xi_0) = (P_{\mathcal{N}} A\xi_0 \mid \xi_0) = (\mathscr{E}(A \mid \mathcal{N})\xi_0 \mid \xi_0) \tag{4.12}$$

for each $A \in \mathcal{M}$.

We next show the uniqueness: let $\mathscr{E}$ be any weak conditional expectation of $\mathcal{M}$ with respect to $(\mathcal{N},\xi_0)$. By (i) and (ii) in Definition 4.3 we have

$$\mathscr{E}(A)\Box\pi_{\mathcal{N}}^{\mathcal{M}}(X)=\mathscr{E}(AX),\quad \forall A\in\mathcal{M},\ \forall X\in\mathcal{N}, \tag{4.13}$$

which implies

$$\begin{aligned}(\mathscr{E}(A)X\xi_0\mid Y\xi_0)&=(\pi_{\mathcal{N}}^{\mathcal{M}}(Y)^*\mathscr{E}(AX)\xi_0\mid\xi_0)=(\mathscr{E}(Y^\dagger AX)\xi_0\mid\xi_0)=\omega_{\xi_0}(Y^\dagger AX)\\&=(AX\xi_0\mid Y\xi_0)=(P_{\mathcal{N}}AX\xi_0\mid Y\xi_0)\end{aligned} \tag{4.14}$$

for each $A\in\mathcal{M}$ and $X,Y\in\mathcal{N}$. Hence, we have

$$\mathscr{E}(A)X\xi_0=P_{\mathcal{N}}AX\xi_0,\quad \forall A\in\mathcal{M},\ \forall X\in\mathcal{N}, \tag{4.15}$$

which implies

$$\mathscr{E}(A)=P_{\mathcal{N}}A\lceil\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}}),\quad \forall A\in\mathcal{M}. \tag{4.16}$$

□

The weak conditional expectation of $\mathcal{M}$ with respect to $(\mathcal{N},\xi_0)$ has the following properties.

Proposition 4.5. *$\mathscr{E}(\cdot\mid\mathcal{N})$ is a map of $\mathcal{M}$ into the partial O^*-algebra $\mathscr{L}^\dagger(\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}}),\mathscr{H}_{\mathcal{N}})$ satisfying*

(i) $\mathscr{E}(A\mid\mathcal{N})\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}})\subset\mathscr{D}^*(\pi_{\mathcal{N}}^{\mathcal{M}})$, *for all* $A\in\mathcal{M}$*;*

(ii) $\mathscr{E}(\cdot\mid\mathcal{N})$ *is linear;*

(iii) $\mathscr{E}(A\mid\mathcal{N})^\dagger=\mathscr{E}(A^\dagger\mid\mathcal{N})$, *for all* $A\in\mathcal{M}$ $\mathscr{E}(X\mid\mathcal{N})=X\lceil\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}})$, *for all* $X\in\mathcal{N}$*;*

(iv) $\mathscr{E}(A^\dagger A\mid\mathcal{N})\geq 0$, *for all* $A\in\mathcal{M}$*;*

(v) $\mathscr{E}(A\mid\mathcal{N})^\dagger\Box\mathscr{E}(A\mid\mathcal{N})\leq\mathscr{E}(A^\dagger A\mid\mathcal{N})$ *whenever* $\mathscr{E}(A\mid\mathcal{N})^\dagger\in L^{\mathrm{w}}(\mathscr{E}(A\mid\mathcal{N}))$*;*

(vi) $\mathscr{E}(A\mid\mathcal{N})\Box\pi_{\mathcal{N}}^{\mathcal{M}}(X)$ *and* $\pi_{\mathcal{N}}^{\mathcal{M}}(X)\Box\mathscr{E}(A\mid\mathcal{N})$ *are well defined for each* $A\in\mathcal{M}$ *and* $X\in\mathcal{N}$*, and*

$$\mathscr{E}(A\mid\mathcal{N})\Box\pi_{\mathcal{N}}^{\mathcal{M}}(X)=\mathscr{E}(AX\mid\mathcal{N}),\qquad \pi_{\mathcal{N}}^{\mathcal{M}}(X)\Box\mathscr{E}(A\mid\mathcal{N})=\mathscr{E}(XA\mid\mathcal{N}); \tag{4.17}$$

(vii) $\omega_{\xi_0}(AX)=(\mathscr{E}(AX\mid\mathcal{N})\xi_0\mid\xi_0)$ *for each* $A\in\mathcal{M}$ *and* $X\in\mathcal{N}$.

Proof. The statements (i), (ii), (iii), and (vi) follow from Theorem 4.4.

(iv) This follows from the equality

$$(\mathscr{E}(A^\dagger A\mid\mathcal{N})\xi\mid\xi)=(P_{\mathcal{N}}A^\dagger A\xi\mid\xi)=(A^\dagger A\xi\mid\xi)=\|A\xi\|^2 \tag{4.18}$$

for each $A\in\mathcal{M}$ and $\xi\in\mathscr{D}(\pi_{\mathcal{N}}^{\mathcal{M}})$.

(v) Take an arbitrary $A\in\mathcal{M}$ s.t. $\mathscr{E}(A\mid\mathcal{N})^\dagger\in L^{\mathrm{w}}(\mathscr{E}(A\mid\mathcal{N}))$. Then we have

$$\begin{aligned}&((\mathscr{E}(A\mid\mathcal{N})^\dagger\Box\mathscr{E}(A\mid\mathcal{N}))\xi\mid\xi)\\&\quad=(\mathscr{E}(A\mid\mathcal{N})^*\mathscr{E}(A\mid\mathcal{N})\xi\mid\xi)=\|\mathscr{E}(A\mid\mathcal{N})\xi\|^2\\&\quad=\|P_{\mathcal{N}}A\xi\|^2\leq\|A\xi\|^2=(\mathscr{E}(A^\dagger A\mid\mathcal{N})\xi\mid\xi)\quad\text{(by (4.18))}\end{aligned} \tag{4.19}$$

for each $\xi \in \mathfrak{D}(\pi_{\mathcal{N}}^{\mathcal{M}})$, which implies that

$$\mathcal{E}(A \mid \mathcal{N})^{\dagger} \square \mathcal{E}(A \mid \mathcal{N}) \leq \mathcal{E}(A^{\dagger} A \mid \mathcal{N}). \tag{4.20}$$

(vii) This follows from

$$\omega_{\xi_0}(AX) = (AX\xi_0 \mid \xi_0) = (P_{\mathcal{N}} AX\xi_0 \mid \xi_0) = (\mathcal{E}(AX \mid \mathcal{N})\xi_0 \mid \xi_0) \tag{4.21}$$

for each $A \in \mathcal{M}$ and $X \in \mathcal{N}$. □

Here we put

$$D(\mathcal{E}_{\mathcal{N}}) = \{A \in \mathcal{M};\ \mathcal{E}(A \mid \mathcal{N}) \in \pi_{\mathcal{N}}^{\mathcal{M}}(\mathcal{N})\}. \tag{4.22}$$

Since $\pi_{\mathcal{N}}^{\mathcal{M}}$ is faithful, for any $A \in D(\mathcal{E}_{\mathcal{N}})$ there exists a unique element X_A of $\mathcal{N}$ such that $\mathcal{E}(A \mid \mathcal{N}) = \pi_{\mathcal{N}}^{\mathcal{M}}(X_A)$. Hence, the map $\mathcal{E}_{\mathcal{N}}$ from $D(\mathcal{E}_{\mathcal{N}})$ to $\mathcal{N}$ is defined by

$$\mathcal{E}_{\mathcal{N}}(A) = X_A, \quad A \in D(\mathcal{E}_{\mathcal{N}}). \tag{4.23}$$

Then we have the following.

Theorem 4.6. *$\mathcal{E}_{\mathcal{N}}$ is a maximal among unbounded conditional expectations of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ξ_0.*

Proof. We show that $D(\mathcal{E}_{\mathcal{N}})$ is a $\dagger$-invariant subspace of $\mathcal{M}$ containing $\mathcal{N}$ such that $\mathcal{N}D(\mathcal{E}_{\mathcal{N}}) \subset D(\mathcal{E}_{\mathcal{N}})$. In fact, it is clear that $D(\mathcal{E}_{\mathcal{N}})$ is a subspace of $\mathcal{M}$ containing $\mathcal{N}$. By Proposition 4.5(iii), $D(\mathcal{E}_{\mathcal{N}})$ is $\dagger$-invariant, and it follows from Proposition 4.5(vi) that $\mathcal{E}(XA \mid \mathcal{N}) = \pi_{\mathcal{N}}^{\mathcal{M}}(X) \square \mathcal{E}(A \mid \mathcal{N}) \in \pi_{\mathcal{N}}^{\mathcal{M}}(\mathcal{N})$ for each $X \in \mathcal{N}$ and $A \in D(\mathcal{E}_{\mathcal{N}})$, which implies that $\mathcal{N}D(\mathcal{E}_{\mathcal{N}}) \subset D(\mathcal{E}_{\mathcal{N}})$. It is easily shown that $\mathcal{E}_{\mathcal{N}}$ is a projection. Since

$$\begin{aligned}\pi_{\mathcal{N}}^{\mathcal{M}}(\mathcal{E}_{\mathcal{N}}(AX)) &= \mathcal{E}(AX \mid \mathcal{N}) = \mathcal{E}(A \mid \mathcal{N}) \square \pi_{\mathcal{N}}^{\mathcal{M}}(X) = \pi_{\mathcal{N}}^{\mathcal{M}}(\mathcal{E}_{\mathcal{N}}(A)) \square \pi_{\mathcal{N}}^{\mathcal{M}}(X) \\ &= \pi_{\mathcal{N}}^{\mathcal{M}}(\mathcal{E}_{\mathcal{N}}(A)X), \quad \text{(by Proposition 4.5(vi))}\end{aligned} \tag{4.24}$$

for each $A \in D(\mathcal{E}_{\mathcal{N}})$ and $X \in \mathcal{N}$, it follows that $\mathcal{E}_{\mathcal{N}}(AX) = \mathcal{E}_{\mathcal{N}}(A)X$. Similarly, $\mathcal{E}_{\mathcal{N}}(XA) = X\mathcal{E}_{\mathcal{N}}(A)$. Hence, $\mathcal{E}_{\mathcal{N}}$ is $\mathcal{N}$-linear. Furthermore, it follows from Proposition 4.5(vii) that $\omega_{\xi_0}(\mathcal{E}_{\mathcal{N}}(A)) = \omega_{\xi_0}(A)$ for each $A \in D(\mathcal{E}_{\mathcal{N}})$. Thus $\mathcal{E}_{\mathcal{N}}$ is an unbounded conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ξ_0. Finally we show that $\mathcal{E}_{\mathcal{N}}$ is maximal. Let $\mathcal{E}$ be any unbounded conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ξ_0. Take an arbitrary $A \in D(\mathcal{E})$. Then it follows from Lemma 4.2(1) that

$$P_{\mathcal{N}} AX\xi_0 = E(AX \mid \mathcal{N}) = \mathcal{E}(A)X\xi_0 \tag{4.25}$$

for each $X \in \mathcal{N}$, which implies that

$$P_{\mathcal{N}} A\xi = \mathcal{E}(A)\xi, \quad \forall \xi \in \mathfrak{D}(\pi_{\mathcal{N}}^{\mathcal{M}}). \tag{4.26}$$

Hence, we have

$$P_{\mathcal{N}} A \lceil \mathfrak{D}(\pi_{\mathcal{N}}^{\mathcal{M}}) = \mathcal{E}(A) \lceil \mathfrak{D}(\pi_{\mathcal{N}}^{\mathcal{M}}) \in \pi_{\mathcal{N}}^{\mathcal{M}}(\mathcal{N}), \tag{4.27}$$

which implies $A \in D(\mathcal{E}_{\mathcal{N}})$ and $\mathcal{E}_{\mathcal{N}}(A) = \mathcal{E}(A)$. Thus, $\mathcal{E} \subset \mathcal{E}_{\mathcal{N}}$. □

5. Unbounded conditional expectations for special O*-algebras

In this section we consider conditional expectations for special O*-algebras (EW*-algebras, generalized von Neumann algebras). For conditional expectations for von Neumann algebras Takesaki [2] has obtained the following.

Lemma 5.1. *Let $\mathcal{M}$ be a von Neumann algebra on a Hilbert space $\mathcal{H}$ with a separating and cyclic vector ξ_0 and $\mathcal{N}$ a von Neumann subalgebra of $\mathcal{M}$. Then $\mathcal{E}_{\mathcal{N}}$ is a conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to ξ_0 if and only if $\Delta_{\xi_0}^{it}\mathcal{N}\Delta_{\xi_0}^{-it}=\mathcal{N}$ for all $t\in\mathbb{R}$, where Δ_{ξ_0} is the modular operator of the left Hilbert algebra $\mathcal{M}\xi_0$.*

The following is our extension of Lemma 5.1 to generalized von Neumann algebras.

Lemma 5.2. *Let $\mathcal{M}$ be a closed O*-algebra on $\mathcal{D}$ in $\mathcal{H}$, $\xi_0\in\mathcal{D}$ a strongly cyclic and separating vector for $\mathcal{M}$ and $\mathcal{N}$ a closed O*-subalgebra of $\mathcal{M}$. Suppose*
(i) *$\mathcal{N}'_{\mathrm{w}}\mathcal{D}\subset\mathcal{D}$;*
(ii) *$\mathcal{N}\xi_0$ is essentially selfadjoint for $\mathcal{N}$.*
Put

$$\overline{\mathcal{E}}(A\mid\mathcal{N})=P_{\mathcal{N}}A\lceil P_{\mathcal{N}}\mathcal{D},\quad A\in\mathcal{M}''_{\mathrm{wc}}. \tag{5.1}$$

Then $\overline{\mathcal{E}}(\cdot\mid\mathcal{N})$ is a linear map of the generalized von Neumann algebra $\mathcal{M}''_{\mathrm{wc}}$ into the O-algebra $\mathcal{L}^\dagger(P_{\mathcal{N}}\mathcal{D})$ such that*
(a) $\overline{\mathcal{E}}(A\mid\mathcal{N})^\dagger=\overline{\mathcal{E}}(A^\dagger\mid\mathcal{N})$, *for all* $A\in\mathcal{M}''_{\mathrm{wc}}$; $\overline{\mathcal{E}}(X\mid\mathcal{N})=X\lceil P_{\mathcal{N}}\mathcal{D}$, *for all* $X\in\mathcal{N}''_{\mathrm{wc}}$;
(b) $\overline{\mathcal{E}}(A^\dagger A\mid\mathcal{N})\geq 0$, *for all* $A\in\mathcal{M}''_{\mathrm{wc}}$;
(c) $\overline{\mathcal{E}}(A\mid\mathcal{N})^\dagger\overline{\mathcal{E}}(A\mid\mathcal{N})\leq\overline{\mathcal{E}}(A^\dagger A\mid\mathcal{N})$, *for all* $A\in\mathcal{M}''_{\mathrm{wc}}$;
(d) $\overline{\mathcal{E}}(A\mid\mathcal{N})X=\overline{\mathcal{E}}(AX\mid\mathcal{N})$, $X\overline{\mathcal{E}}(A\mid\mathcal{N})=\overline{\mathcal{E}}(XA\mid\mathcal{N})$, *for all* $A\in\mathcal{M}''_{\mathrm{wc}}$, *for all* $X\in\mathcal{N}''_{\mathrm{wc}}$;
(e) $\omega_{\xi_0}(AX)=(\overline{\mathcal{E}}(AX\mid\mathcal{N})\xi_0\mid\xi_0)$, *for all* $A\in\mathcal{M}''_{\mathrm{wc}}$, *for all* $X\in\mathcal{N}''_{\mathrm{wc}}$.
Furthermore, suppose
(iii) $\Delta_{\xi_0}^{it}(\mathcal{N}'_{\mathrm{w}})'\Delta_{\xi_0}^{-it}=(\mathcal{N}'_{\mathrm{w}})'$, *for all* $t\in\mathbb{R}$,
where Δ_{ξ_0} is the modular operator of the left Hilbert algebra $(\mathcal{M}'_{\mathrm{w}})'\xi_0$. Then, $\overline{\mathcal{E}}(A\mid\mathcal{N})\in(\mathcal{N}_{P_{\mathcal{N}}})''_{\mathrm{wc}}$, for all $A\in\mathcal{M}''_{\mathrm{wc}}$.

Proof. By (i) we have $\mathcal{M}'_{\mathrm{w}}\mathcal{D}\subset\mathcal{D}$, and hence it follows from [6, Propositions 1.7.3, 1.7.5] that $\mathcal{M}''_{\mathrm{wc}}$ is a generalized von Neumann algebra on $\mathcal{D}$ and $\mathcal{N}''_{\mathrm{wc}}$ is a generalized von Neumann subalgebra of $\mathcal{M}''_{\mathrm{wc}}$. Since the $\mathcal{N}$-invariant subspace $\mathcal{N}\xi_0$ of $\mathcal{D}$ is essentially selfadjoint, it follows from [7, Theorem 4.7] that

$$P_{\mathcal{N}}\in\mathcal{N}'_{\mathrm{w}},\qquad P_{\mathcal{N}}\mathcal{D}=\overline{\mathcal{N}\xi_0}^{t_{\mathcal{N}}}\subset\mathcal{D}, \tag{5.2}$$

$$\overline{\mathcal{N}\xi_0}=\overline{(\mathcal{N}'_{\mathrm{w}})'\xi_0}. \tag{5.3}$$

By (5.2), $\overline{\mathcal{E}}(\cdot\mid\mathcal{N})$ is a linear map of $\mathcal{M}''_{\mathrm{wc}}$ into $\mathcal{L}^\dagger(P_{\mathcal{N}}\mathcal{D})$, and it is shown in a similar way to the proof of Proposition 4.5 that $\overline{\mathcal{E}}(\cdot\mid\mathcal{N})$ satisfies (a)–(e). Suppose (iii) holds. We show $\overline{\mathcal{E}}(A\mid\mathcal{N})\in(\mathcal{N}_{P_{\mathcal{N}}})''_{\mathrm{wc}}$, for all $A\in\mathcal{M}''_{\mathrm{wc}}$. By (5.3) we have $P_{\mathcal{N}}=P_{(\mathcal{N}'_{\mathrm{w}})'}$, and so by (iii) and by the Takesaki theorem [2] there exists a unique conditional expectation $\mathcal{E}''$ of the von Neumann algebra $(\mathcal{M}'_{\mathrm{w}})'$ onto the von Neumann algebra $(\mathcal{N}'_{\mathrm{w}})'$ with respect to ξ_0 such that $\mathcal{E}''(A)P_{\mathcal{N}} = P_{\mathcal{N}}AP_{\mathcal{N}}$ for each $A \in (\mathcal{M}'_{\mathrm{w}})'$. Take an arbitrary $A \in \mathcal{M}''_{\mathrm{wc}}$. Then there exists a net $\{A_\alpha\}$ in $(\mathcal{M}'_{\mathrm{w}})'$ which converges strongly* to A. From (5.2) it follows immediately that

$$(\mathcal{N}'_{\mathrm{w}})_{P_{\mathcal{N}}} = (\mathcal{N}_{P_{\mathcal{N}}})'_{\mathrm{w}}, \tag{5.4}$$

and by the basic theory of von Neumann algebras [11]

$$((\mathcal{N}_{P_{\mathcal{N}}})'_{\mathrm{w}})' = ((\mathcal{N}'_{\mathrm{w}})_{P_{\mathcal{N}}})' = ((\mathcal{N}'_{\mathrm{w}})')_{P_{\mathcal{N}}}. \tag{5.5}$$

Hence we have

$$\begin{aligned} &\mathcal{E}''(A_\alpha)_{P_{\mathcal{N}}} \in ((\mathcal{N}_{P_{\mathcal{N}}})'_{\mathrm{w}})', \\ &\mathcal{E}''(A_\alpha)_{P_{\mathcal{N}}} \xrightarrow[\tau_{s^*}]{} P_{\mathcal{N}}A\lceil P_{\mathcal{N}}\mathcal{D}, \end{aligned} \tag{5.6}$$

which implies that $P_{\mathcal{N}}A\lceil P_{\mathcal{N}}\mathcal{D} \in \overline{((\mathcal{N}_{P_{\mathcal{N}}})'_{\mathrm{w}})'}^{\tau_{s^*}} = (\mathcal{N}_{P_{\mathcal{N}}})''_{\mathrm{wc}}$. Hence we have

$$\overline{\mathcal{E}}(A \mid \mathcal{N}) = P_{\mathcal{N}}A\lceil P_{\mathcal{N}}\mathcal{D} \in (\mathcal{N}_{P_{\mathcal{N}}})''_{\mathrm{wc}}, \quad A \in \mathcal{M}''_{\mathrm{wc}}. \tag{5.7}$$

In a similar way to the proof of Theorem 4.4 one can show that $\overline{\mathcal{E}}(\cdot \mid \mathcal{N})$ is the unique weak conditional expectation of the generalized von Neumann algebra $\mathcal{M}''_{\mathrm{wc}}$ with respect to $((\mathcal{N}_{P_{\mathcal{N}}})''_{\mathrm{wc}}, \xi_0)$. □

Now we put

$$D(\overline{\mathcal{E}_{\mathcal{N}}}) = \{A \in \mathcal{M}''_{\mathrm{wc}};\ \overline{\mathcal{E}}(A \mid \mathcal{N}) \in (\mathcal{N}''_{\mathrm{wc}})_{P_{\mathcal{N}}}\}. \tag{5.8}$$

Then, for any $A \in D(\overline{\mathcal{E}_{\mathcal{N}}})$ there exists a unique element $\overline{\mathcal{E}_{\mathcal{N}}}(A)$ of $\mathcal{N}''_{\mathrm{wc}}$ such that $\overline{\mathcal{E}_{\mathcal{N}}}(A)\lceil P_{\mathcal{N}}\mathcal{D} = \overline{\mathcal{E}}(A \mid \mathcal{N})$, and in a similar way to the proof of Theorem 4.6 we can show the following.

Lemma 5.3. *$\overline{\mathcal{E}_{\mathcal{N}}}$ is an unbounded conditional expectation of the generalized von Neumann algebra $\mathcal{M}''_{\mathrm{wc}}$ onto the generalized von Neumann algebra $\mathcal{N}''_{\mathrm{wc}}$ with respect to ξ_0 which is an extension of $\mathcal{E}_{\mathcal{N}}$.*

By Lemmas 5.2 and 5.3 we have the following.

Theorem 5.4. *Let $\mathcal{M}$ be a generalized von Neumann algebra on $\mathcal{D}$ in $\mathcal{H}$, ξ_0 a strongly cyclic and separating vector for $\mathcal{M}$ and $\mathcal{N}$ a generalized von Neumann subalgebra of $\mathcal{M}$. Suppose*

(i) *$\mathcal{N}\xi_0$ is essentially selfadjoint for $\mathcal{N}$;*

(ii) *$\Delta_{\xi_0}^{it}(\mathcal{N}'_{\mathrm{w}})'\Delta_{\xi_0}^{-it} = (\mathcal{N}'_{\mathrm{w}})'$, for all $t \in \mathbb{R}$,*

where Δ_{ξ_0} is the modular operator of the left Hilbert algebra $(\mathcal{M}'_{\mathrm{w}})'\xi_0$. Then the following statements hold.

(1) *$\overline{\mathcal{E}}(A \mid \mathcal{N}) = \mathcal{E}(A \mid \mathcal{N})^\sim \in (\mathcal{N}_{P_{\mathcal{N}}})''_{\mathrm{wc}}$ for each $A \in \mathcal{M}$.*

(2) *$\overline{\mathcal{E}_{\mathcal{N}}} = \mathcal{E}_{\mathcal{N}}$.*

Proof. (1) Since $\mathcal{N}\xi_0 \subset \mathcal{D}(\pi_{\mathcal{N}}^{\mathcal{M}}) \subset \overline{\mathcal{N}\xi_0}^{t_{\mathcal{N}}} = P_{\mathcal{N}}\mathcal{D}$, it follows that $\overline{\mathcal{E}}(A \mid \mathcal{N}) = \mathcal{E}(A \mid \mathcal{N})^{\sim}$ for each $A \in \mathcal{M}$, and $\overline{\mathcal{E}}(A \mid \mathcal{N})$ is contained in $(\mathcal{N}_{P_{\mathcal{N}}})''_{\mathrm{wc}}$ by Lemma 5.2.

(2) This follows from (1). □

It is natural to consider the following question.

Question 1. Let $(\mathcal{M}, \xi_0, \mathcal{N})$ be as in Theorem 5.4. Does $D(\mathcal{E}_{\mathcal{N}})$ contain any elements of $\mathcal{M}$? For example, when is $\mathcal{M}_b \subset D(\mathcal{E}_{\mathcal{N}})$?

For this question we have the following.

Proposition 5.5. *Let $(\mathcal{M}, \xi_0, \mathcal{N})$ be as in Theorem 5.4. Suppose that $\mathcal{N}$ is an EW*-algebra on $\mathcal{D}$, that is, $(\mathcal{N}'_{\mathrm{w}})' = \overline{\mathcal{N}_b}$. Then $\mathcal{N}\mathcal{M}_b\mathcal{N} \subset D(\mathcal{E}_{\mathcal{N}})$.*

Proof. By Lemma 5.1 there exists a unique conditional expectation $\mathcal{E}''$ of the von Neumann algebra $(\mathcal{M}'_{\mathrm{w}})'$ onto the von Neumann algebra $(\mathcal{N}'_{\mathrm{w}})'$ such that

$$\mathcal{E}''(A)P_{\mathcal{N}} = P_{\mathcal{N}}AP_{\mathcal{N}}, \quad \forall A \in (\mathcal{M}'_{\mathrm{w}})'. \tag{5.9}$$

Take an arbitrary $A \in \mathcal{M}_b$. Then $\mathcal{E}''(\overline{A})\lceil\mathcal{D} \in \mathcal{N}_b$; it follows that

$$P_{\mathcal{N}}AP_{\mathcal{N}} = \mathcal{E}''(\overline{A})P_{\mathcal{N}} \in \mathcal{N}_bP_{\mathcal{N}}, \tag{5.10}$$

which implies $A \in D(\overline{\mathcal{E}_{\mathcal{N}}}) = D(\mathcal{E}_{\mathcal{N}})$. Thus, $\mathcal{M}_b \subset D(\mathcal{E}_{\mathcal{N}})$, which implies $\mathcal{N}\mathcal{M}_b\mathcal{N} \subset D(\mathcal{E}_{\mathcal{N}})$. □

Corollary 5.6. *Let $(\mathcal{M}, \xi_0, \mathcal{N})$ be as in Theorem 5.4. If one of the following conditions (i) and (ii) holds, then $\mathcal{N}\mathcal{M}_b\mathcal{N} \subset D(\mathcal{E}_{\mathcal{N}})$.*

(i) $(\mathcal{N}'_{\mathrm{w}})'$ *is commutative.*

(ii) $\mathcal{D} = \mathcal{D}^{\infty}(H') \equiv \bigcap_{n\in\mathbb{N}} \mathcal{D}(H'^n)$, *where H' is a selfadjoint operator in $\mathcal{H}$ affiliated with $\mathcal{N}'_{\mathrm{w}}$.*

Proof. Suppose (i) holds, that is, $(\mathcal{N}'_{\mathrm{w}})'$ is commutative. Then, $(\mathcal{N}'_{\mathrm{w}})'\mathcal{D} \subset \mathcal{N}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. Suppose (ii) holds. Then, $(\mathcal{N}'_{\mathrm{w}})'\mathcal{D} \subset \mathcal{D}$ clearly. Hence $\mathcal{N}$ is an EW*-algebra on $\mathcal{D}$ in either of the cases (i) and (ii), and so it follows from Proposition 5.5 that $\mathcal{N}\mathcal{M}_b\mathcal{N} \subset D(\mathcal{E}_{\mathcal{N}})$. □

6. Absolute continuity and coarse graining

In this section we define the notions of absolutely continuous positive linear functionals and investigate them.

Let $\mathcal{M}$ be an O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ with a strongly cyclic vector ξ_0. A linear functional F on $\mathcal{M}$ is called *hermitian* if $F(A^{\dagger}) = \overline{F(A)}$ for each $A \in \mathcal{M}$ and it is called *positive* (denoted by $F \geq 0$) if $F(A^{\dagger}A) \geq 0$ for each $A \in \mathcal{M}$. Since $\mathcal{M}$ contains the identity operator, it follows that if $F \geq 0$ then it is hermitian. The positive linear functional ω_{ξ_0} on $\mathcal{M}$ is defined by

$$\omega_{\xi_0}(A) = (A\xi_0 \mid \xi_0), \quad A \in \mathcal{M}. \tag{6.1}$$

We define the notion of ω_{ξ_0}-absolutely continuous linear functionals on $\mathcal{M}$ and investigate their properties.

Definition 6.1. Let F be a linear functional on $\mathcal{M}$. If for any $A \in \mathcal{M}$ there exists a constant $r_A > 0$ such that

$$|F(AX)|^2 \le r_A \omega_{\xi_0}(X^\dagger X), \quad \forall X \in \mathcal{M}, \tag{6.2}$$

then F is said to be ω_{ξ_0}-*absolutely continuous* and denoted by $F < \omega_{\xi_0}$. If there exists a constant $r > 0$ such that

$$F(X^\dagger X) \le r\omega_{\xi_0}(X^\dagger X), \quad \forall X \in \mathcal{M}, \tag{6.3}$$

then F is said to be ω_{ξ_0}-*dominated* and denoted by $F <_d \omega_{\xi_0}$. Denote by $\mathcal{M}^*(< \omega_{\xi_0})$ (resp., $\mathcal{M}_h^*(< \omega_{\xi_0})$, $\mathcal{M}_+^*(< \omega_{\xi_0})$) the set of all ω_{ξ_0}-absolutely continuous (resp., hermitian, positive) linear functionals on $\mathcal{M}$; and denote by $\mathcal{M}^*(<_d \omega_{\xi_0})$ (reps., $\mathcal{M}_h^*(<_d \omega_{\xi_0}), \mathcal{M}_+^*(<_d \omega_{\xi_0})$) the set of all ω_{ξ_0}-dominated (resp., hermitian, positive) linear functionals on $\mathcal{M}$.

Theorem 6.2. *Let F be a linear functional on $\mathcal{M}$.*

(1) *The following statements are equivalent.*

(i) $F \in \mathcal{M}^*(< \omega_{\xi_0})$ *(resp.,* $\mathcal{M}_h^*(< \omega_{\xi_0})$, $\mathcal{M}_+^*(< \omega_{\xi_0})$*).*

(ii) *There exists an element ξ of $\mathcal{D}^*(\mathcal{M})$ (resp., $\mathcal{D}^*(\mathcal{M})_h$, $\mathcal{D}^*(\mathcal{M})_+$) such that*

$$F(A) = F_\xi(A) \equiv (A\xi_0 \mid \xi), \quad \forall A \in \mathcal{M}, \tag{6.4}$$

where

$$\begin{aligned}\mathcal{D}^*(\mathcal{M})_h &= \{\xi \in \mathcal{D}^*(\mathcal{M});\ F_\xi \text{ is hermitian}\},\\ \mathcal{D}^*(\mathcal{M})_+ &= \{\xi \in \mathcal{D}^*(\mathcal{M});\ F_\xi \text{ is positive}\}.\end{aligned} \tag{6.5}$$

(iii) *There exists an element S of the unbounded commutant $(\mathcal{M}\lceil\mathcal{M}\xi_0)'_\delta$ (resp., $((\mathcal{M}\lceil\mathcal{M}\xi_0)'_\sigma)_h$, $((\mathcal{M}\lceil\mathcal{M}\xi_0)'_\sigma)_+$) of the O^*-algebra $\mathcal{M}\lceil\mathcal{M}\xi_0$ on $\mathcal{M}\xi_0$ such that*

$$F(A) = F_S(A) \equiv (A\xi_0 \mid S\xi_0), \quad \forall A \in \mathcal{M}. \tag{6.6}$$

The vector ξ in (ii) and the operator S in (iii) are unique. S is called the Radon-Nikodym derivative of F with respect to ω_{ξ_0} and denoted by $dF/d\omega_{\xi_0}$.

(2) $F \in \mathcal{M}^*(<_d \omega_{\xi_0})$ *(resp.,* $\mathcal{M}_h^*(<_d \omega_{\xi_0})$, $\mathcal{M}_+^*(<_d \omega_{\xi_0})$*) if and only if* $dF/d\omega_{\xi_0} \in \mathcal{M}'_{\mathrm{w}}$ *(resp.,* $(\mathcal{M}'_{\mathrm{w}})_h$, $(\mathcal{M}'_{\mathrm{w}})_+$*).*

Proof. (1) (i)$\Rightarrow$(ii). Let $F \in \mathcal{M}^*(< \omega_{\xi_0})$. Then we have

$$|F(A)|^2 \le r_I \|A\xi_0\|^2, \quad \forall A \in \mathcal{M}. \tag{6.7}$$

Since ξ_0 is strongly cyclic, that is, $\mathcal{M}\xi_0$ is dense in $\mathcal{D}[t_\mathcal{M}]$, and the graph topology $t_\mathcal{M}$ is finer than the topology defined by the Hilbert space norm $\|\cdot\|$, it follows that $\mathcal{M}\xi_0$ is dense in $\mathcal{H}$, which implies that the map $A\xi_0 \to F(A)$ can be extended to a continuous linear functional on $\mathcal{H}$ and by the Riesz theorem there exists a unique element ξ of $\mathcal{H}$ such that

$$F(A) = (A\xi_0 \mid \xi), \quad \forall A \in \mathcal{M}. \tag{6.8}$$

Furthermore, since $F < \omega_{\xi_0}$, it follows that

$$|(X^\dagger A\xi_0 \mid \xi)| = |F(X^\dagger A)| \le r_X \omega_{\xi_0}(A^\dagger A) = r_X \|A\xi_0\|^2 \tag{6.9}$$

for all $A, X \in \mathcal{M}$, which implies $\xi \in \mathcal{D}^*(\mathcal{M})$. In particular, it is clear that if $F \in \mathcal{M}_h^*(< \omega_{\xi_0})$ (resp., $\mathcal{M}_+^*(< \omega_{\xi_0})$) then $\xi \in \mathcal{D}^*(\mathcal{M})_h$ (resp., $\xi \in \mathcal{D}^*(\mathcal{M})_+$).

(ii)⇒(iii). We put

$$SA\xi_0 = \xi, \quad A \in \mathcal{M}. \tag{6.10}$$

Then since

$$(X\xi_0 \mid SAY\xi_0) = F((AY)^\dagger X) = F(Y^\dagger(A^\dagger X)) = (A^\dagger X\xi_0 \mid SY\xi_0) \tag{6.11}$$

for each $A, X, Y \in \mathcal{M}$, it follows that $S \in (\mathcal{M}\lceil\mathcal{M}\xi_0)'_\delta$ and $F = F_S$. It is clear that S is uniquely determined. Furthermore, it is easily shown that if F is hermitian (resp., positive) then S is hermitian (resp., positive).

(iii)⇒(i). This is trivial.

(2) This is shown in a similar way to (1).

The equivalence of (i) and (ii) in Theorem 6.2 follows from [1, Theorem 1]. □

The following schemes may serve as a sketch of Theorem 6.2:

$$\begin{tikzcd}
\begin{array}{c}\xi \in \mathcal{D}^*(\mathcal{M}) \\ (\text{resp., } \mathcal{D}^*(\mathcal{M})_h,\ \mathcal{D}^*(\mathcal{M})_+)\end{array} \arrow[r, "\text{bijection}"] \arrow[dr, "\text{bijection}"'] & \begin{array}{c}F_\xi \in \mathcal{M}^*(<\omega_{\xi_0}) \\ (\text{resp., } \mathcal{M}_h^*(<\omega_{\xi_0}),\ \mathcal{M}_+^*(<\omega_{\xi_0}))\end{array} \arrow[d, "\text{bijection}"] \\
 & \begin{array}{c}\dfrac{dF_\xi}{d\omega_{\xi_0}} \in (\mathcal{M}\lceil\mathcal{M}\xi_0)'_\delta \\ (\text{resp., } ((\mathcal{M}\lceil\mathcal{M}\xi_0)'_\sigma)_h,\ ((\mathcal{M}\lceil\mathcal{M}\xi_0)'_\sigma)_+)\end{array}
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c}F \in \mathcal{M}^*(<_d\omega_{\xi_0}) \\ (\text{resp., } \mathcal{M}_h^*(<_d\omega_{\xi_0}),\ \mathcal{M}_+^*(<_d\omega_{\xi_0}))\end{array} \arrow[r, "\text{bijection}"] & \begin{array}{c}\dfrac{dF}{d\omega_{\xi_0}} \in \mathcal{M}'_{\mathrm{w}} \\ (\text{resp., } (\mathcal{M}'_{\mathrm{w}})_h, (\mathcal{M}'_{\mathrm{w}})_+).\end{array}
\end{tikzcd} \tag{6.12}$$

The Radon-Nikodym theorems for O*-algebras can be found in [1, 6, 10, 12, 13].

Next we define and investigate the notion of coarse graining.

Let $\mathcal{M}$ be a closed O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ with a strongly cyclic (and separating) vector ξ_0 and $\mathcal{N}$ an O*-subalgebra of $\mathcal{M}$.

Let f be a positive linear functional on $\mathcal{N}$ such that $f < \omega_{\xi_0}\lceil\mathcal{N}$.

Definition 6.3. A positive linear functional F on $\mathcal{M}$ is said to be $(\mathcal{N}, \omega_{\xi_0})$ *coarse graining* of f, if the following conditions hold:

(i) $f \subset F$;

(ii) $F < \omega_{\xi_0}$;

(iii) $F(A) = F(E(A \mid \mathcal{N})) \equiv (E(A \mid \mathcal{N}) \mid (dF/d\omega_{\xi_0})\xi_0)$, for all $A \in \mathcal{M}$.

We put

$$F_c(A) = \left(E(A \mid \mathcal{N}) \mid \frac{df}{d\omega_{\xi_0} \lceil \mathcal{N}} \xi_0\right), \quad A \in \mathcal{M}. \tag{6.13}$$

Then F_c is a linear functional on $\mathcal{M}$ satisfying the following:

$$F_c \supset f. \tag{6.14}$$

In fact, (6.14) follows from

$$F_c(X) = \left(X\xi_0 \mid \frac{df}{d\omega_{\xi_0} \lceil \mathcal{N}} \xi_0\right) = f(X), \quad \forall X \in \mathcal{N}. \tag{6.15}$$

Question 2. Is F_c a $(\mathcal{N}, \omega_{\xi_0})$ coarse graining of f? That is, does F_c satisfy the following conditions?

$$F_c \text{ is positive,} \tag{6.16}$$

$$F_c < \omega_{\xi_0}. \tag{6.17}$$

We remark that if (6.17) holds, then

$$F_c(A) = F_c(E(A \mid \mathcal{N})), \quad \forall A \in \mathcal{M}. \tag{6.18}$$

In fact, since

$$\left(A\xi_0 \mid \frac{dF_c}{d\omega_{\xi_0}} \xi_0\right) = F_c(A) = \left(E(A \mid \mathcal{N}) \mid \frac{df}{d\omega_{\xi_0} \lceil \mathcal{N}} \xi_0\right) = \left(A\xi_0 \mid \frac{df}{d\omega_{\xi_0} \lceil \mathcal{N}} \xi_0\right) \tag{6.19}$$

for all $A \in \mathcal{M}$, it follows that $(dF_c/d\omega_{\xi_0})\xi_0 = (df/d\omega_{\xi_0} \lceil \mathcal{N})\xi_0$, which implies that $F_c(A) = F_c(E(A \mid \mathcal{N}))$ for all $A \in \mathcal{M}$.

Almost all the results of Theorem 6.4, Proposition 6.6, and Theorem 6.12 can be found in [1, Theorem 3], but it seems that they contain a few gaps. So, we introduce here these results and their proofs.

Theorem 6.4. *The following statements are equivalent.*

(i) *f is $(\mathcal{N}, \omega_{\xi_0})$ coarse grainable, that is, there exists a $(\mathcal{N}, \omega_{\xi_0})$ coarse graining of f.*

(ii) *There exists a positive linear functional F on $\mathcal{M}$ such that $F \supset f$, $F < \omega_{\xi_0}$ and $(dF/d\omega_{\xi_0})\xi_0 \in \overline{\mathcal{N}\xi_0}$.*

(iii) *There exists a positive operator S in $(\mathcal{M} \lceil \mathcal{M}\xi_0)'_\sigma$ such that $S\xi_0 = (df/d\omega_{\xi_0} \lceil \mathcal{N})\xi_0$.*

If this is true, then F_c is a unique $(\mathcal{N}, \omega_{\xi_0})$ coarse graining of f and the F in (ii) equals F_c.

Proof. (i)⇒(ii). Let F be a $(\mathcal{N},\omega_{\xi_0})$ coarse-graining of f. Then F is a positive linear functional on $\mathcal{M}$ such that $F \supset f$ and $F < \omega_{\xi_0}$. Furthermore, since

$$\left(A\xi_0 \mid \frac{dF}{d\omega_{\xi_0}}\xi_0\right) = F(A) = F(E(A \mid \mathcal{N})) = \left(P_{\mathcal{N}}A\xi_0 \mid \frac{dF}{d\omega_{\xi_0}}\xi_0\right) \tag{6.20}$$

for all $A \in \mathcal{M}$, we have

$$\frac{dF}{d\omega_{\xi_0}}\xi_0 = P_{\mathcal{N}}\frac{dF}{d\omega_{\xi_0}}\xi_0 \in \overline{\mathcal{N}\xi_0}. \tag{6.21}$$

(ii)⇒(iii). We put $S = dF/d\omega_{\xi_0}$. Then, $S \in ((\mathcal{M}\lceil\mathcal{M}\xi_0)'_\sigma)_+$ and $S\xi_0 \in \overline{\mathcal{N}\xi_0}$. Furthermore, we have

$$\begin{aligned}(A\xi_0 \mid S\xi_0) &= \left(P_{\mathcal{N}}A\xi_0 \mid \frac{dF}{d\omega_{\xi_0}}\xi_0\right) = \lim_{n\to\infty}\left(X_n\xi_0 \mid \frac{dF}{d\omega_{\xi_0}}\xi_0\right) = \lim_{n\to\infty}F(X_n)\\ &= \lim_{n\to\infty}f(X_n) = \lim_{n\to\infty}\left(X_n\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0\right) = \left(A\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0\right)\end{aligned} \tag{6.22}$$

for all $A \in \mathcal{M}$, where $\{X_n\}$ is a sequence in $\mathcal{N}$ such that $\lim_{n\to\infty}X_n\xi_0 = P_{\mathcal{N}}A\xi_0$, which implies that $S\xi_0 = (df/d\omega_{\xi_0}\lceil\mathcal{N})\xi_0$.

(iii)⇒(i). Since

$$F_c(B^\dagger A) = \left(P_{\mathcal{N}}B^\dagger A\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0\right) = (B^\dagger A\xi_0 \mid S\xi_0) = (A\xi_0 \mid SB\xi_0) \tag{6.23}$$

for each $A, B \in \mathcal{M}$, it follows that $F_c \geq 0$ and $F_c < \omega_{\xi_0}$. Hence F_c is a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f.

Finally we show that F_c is the unique $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f. Let F be a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f. By the proof of (ii)⇒(iii) we have

$$F(A) = \left(A\xi_0 \mid \frac{dF}{d\omega_{\xi_0}}\xi_0\right) = \left(A\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0\right) = F_c(A) \tag{6.24}$$

for all $A \in \mathcal{M}$. This completes the proof. □

Definition 6.5. $\mathcal{N}$ is said to be *positivity preserving* if for any $A \in \mathcal{M}$ there exists a sequence $\{X_n\}$ in $\mathcal{N}$ such that $\lim_{n\to\infty}X_n^\dagger X_n\xi_0 = P_{\mathcal{N}}A^\dagger A\xi_0$.

It is clear that if $\mathcal{N}$ is positivity preserving then F_c is positive. In fact, this follows from

$$\begin{aligned}F_c(A^\dagger A) &= \left(E(A^\dagger A \mid \mathcal{N}) \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0\right)\\ &= \lim_{n\to\infty}\left(X_n^\dagger X_n\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0\right)\\ &= \lim_{n\to\infty}\left(X_n\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}X_n\xi_0\right) \geq 0, \quad \forall A \in \mathcal{M}.\end{aligned} \tag{6.25}$$

Hence we have the following.

Proposition 6.6. *Suppose that $\mathcal{N}$ is positivity preserving. Then the following statements are equivalent.*

(i) *f is $(\mathcal{N},\omega_{\xi_0})$ coarse grainable.*
(ii) *F_c is a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f.*
(iii) *$F_c < \omega_{\xi_0}$.*
(iv) *$(df/d\omega_{\xi_0}\lceil\mathcal{N})\xi_0 \in \mathfrak{D}^*(\mathcal{M})$.*

Proof. (i)⇔(ii) This follows from Theorem 6.4.

(ii)⇒(iii). This is trivial.

(iii)⇒(ii). As shown above, F_c is automatically positive, and so (iii) implies (ii).

(ii)⇒(iv). This follows from

$$\frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0 = \frac{dF_c}{d\omega_{\xi_0}}\xi_0 \in \mathfrak{D}^*(\mathcal{M}). \tag{6.26}$$

(iv)⇒(ii). Let $\xi \equiv (df/d\omega_{\xi_0}\lceil\mathcal{N})\xi_0$. Then, $\xi \in \mathfrak{D}^*(\mathcal{M}) = \mathfrak{D}^*(\mathcal{M}\lceil\mathcal{M}\xi_0)$, and so by Theorem 6.2

$$\frac{dF_\xi}{d\omega_{\xi_0}} \in ((\mathcal{M}\lceil\mathcal{M}\xi_0)'_\sigma)_+, \qquad \frac{dF_\xi}{d\omega_{\xi_0}}\xi_0 = \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0, \tag{6.27}$$

which by Theorem 6.4 implies that F_c is a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f. □

Proposition 6.7. *Suppose that $\mathcal{N}$ is positivity preserving and $f <_d \omega_{\xi_0}\lceil\mathcal{N}$. Then F_c is a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f and $F_c <_d \omega_{\xi_0}$.*

Proof. Since

$$\begin{aligned} F_c(A^\dagger A) &= \left(P_{\mathcal{N}}A^\dagger A\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0\right) = \lim_{n\to\infty}\left(X_n^\dagger X_n\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0\right) \\ &= \lim_{n\to\infty}\left(X_n\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}X_n\xi_0\right) \le \left\|\frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\right\|\lim_{n\to\infty}\|X_n\xi_0\|^2 \\ &= \left\|\frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\right\|(P_{\mathcal{N}}A^\dagger A\xi_0 \mid \xi_0) \le \left\|\frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\right\|\omega_{\xi_0}(A^\dagger A) \end{aligned} \tag{6.28}$$

for all $A \in \mathcal{M}$, we have $F_c <_d \omega_{\xi_0}$, which from Proposition 6.6 implies that F_c is a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f. □

We characterize the coarse grainability of positive linear functionals on $\mathcal{N}$ by elements of $\mathfrak{D}^*(\mathcal{M})_+$.

Proposition 6.8. *Let f be a positive linear functional on $\mathcal{N}$. The following statements are equivalent.*

(i) *f is $(\mathcal{N},\omega_{\xi_0})$ coarse grainable.*
(ii) *There exists an element ξ of $\mathfrak{D}^*(\mathcal{M})_+$ such that $P_{\mathcal{N}}\xi = \xi$ and $f = F_\xi\lceil\mathcal{N}$.*

Proof. (i)⇒(ii). Let F be a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f. Then, $F<\omega_{\xi_0}$ implies $f<\omega_{\xi_0}\lceil\mathcal{N}$. By Theorem 6.4(ii) we have

$$\xi\equiv\frac{dF}{d\omega_{\xi_0}}\xi_0\in\mathcal{D}^*(\mathcal{M})_+,\qquad P_{\mathcal{N}}\xi=\xi,\qquad F_\xi=F. \tag{6.29}$$

(ii)⇒(i). It is easily shown that F_ξ is a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f. □

By Proposition 6.8 and Theorem 6.4 we have the following:

$$\begin{tikzcd} \begin{array}{c}\{\xi\in\mathcal{D}^*(\mathcal{M})_+;\\ P_{\mathcal{N}}\xi=\xi\}\in\xi\end{array} \arrow[r, "\text{bijection}"] \arrow[dr] & \begin{array}{c}\{f;\ (\mathcal{N},\omega_{\xi_0})\ \text{coarse-grainable positive}\\ \text{linear functionals}\}\in f_\xi\equiv F_\xi\lceil\mathcal{N}\end{array} \arrow[d, "\text{bijection}"] \\ & \dfrac{dF_\xi}{d\omega_{\xi_0}}\in\{S\in((\mathcal{M}\lceil\mathcal{M}\xi_0)'_\sigma)_+;\ S\xi_0\in\overline{\mathcal{N}\xi_0}\} \end{tikzcd} \tag{6.30}$$

Remark 6.9. Let $\xi\in\mathcal{D}^*(\mathcal{M})_+$. By Theorem 6.2, F_ξ is a ω_{ξ_0}-absolutely continuous positive linear functional on $\mathcal{M}$, but it is not a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of $f_\xi\equiv F_\xi\lceil\mathcal{N}$ in general. In fact,

$$F_\xi(E(A\mid\mathcal{N}))=(P_{\mathcal{N}}A\xi_0\mid\xi)=(A\xi_0\mid P_{\mathcal{N}}\xi)\neq F_\xi(A)\quad\text{in general.} \tag{6.31}$$

We consider whether $F_{P_{\mathcal{N}}\xi}$ is a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f_ξ. $F_{P_{\mathcal{N}}\xi}$ is a linear functional on $\mathcal{M}$ such that $F_{P_{\mathcal{N}}\xi}\supset f_\xi$ and $F_{P_{\mathcal{N}}\xi}(E(A\mid\mathcal{N}))=F_{P_{\mathcal{N}}\xi}(A)$, for all $A\in\mathcal{M}$, but it is not positive and not ω_{ξ_0}-absolutely continuous in general. Hence we consider when $F_{P_{\mathcal{N}}\xi}$ is positive and ω_{ξ_0}-absolutely continuous.

Proposition 6.10. *Suppose that $\mathcal{M}$ is essentially selfadjoint, $\mathcal{N}$ is positivity preserving, and the $\mathcal{N}$-invariant subspace $\mathcal{N}\xi_0$ is essentially selfadjoint. Then $P_{\mathcal{N}}\mathcal{D}_+=\{\xi\in\mathcal{D}_+;\ P_{\mathcal{N}}\xi=\xi\}$ and the map $P_{\mathcal{N}}\xi\to f_{P_{\mathcal{N}}\xi}=f_\xi$ is a bijection of $P_{\mathcal{N}}\mathcal{D}_+$ onto $\{f\in\mathcal{N}_+^*;\ (\mathcal{N},\omega_{\xi_0})$ coarse grainable$\}$, where $\mathcal{D}_+=\{\xi\in\mathcal{D};\ F_\xi\geq 0\}$. Hence, any ω_{ξ_0}-absolutely continuous positive linear functional F on $\mathcal{M}$ is a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of $F\lceil\mathcal{N}$.*

Proof. Since $\mathcal{M}$ is essentially selfadjoint and $\mathcal{N}\xi_0$ is essentially selfadjoint, we have $P_{\mathcal{N}}\mathcal{D}\subset\mathcal{D}$. Furthermore, since $\mathcal{N}$ is positivity preserving, we have $P_{\mathcal{N}}\mathcal{D}_+\subset\mathcal{D}_+$, which implies $P_{\mathcal{N}}\mathcal{D}_+=\{\xi\in\mathcal{D}_+;\ P_{\mathcal{N}}\xi=\xi\}$. Hence it follows from Proposition 6.8 that the map $P_{\mathcal{N}}\xi\to f_\xi$ is a bijection. Let F be any ω_{ξ_0}-absolutely continuous positive linear functional on $\mathcal{M}$. By Theorem 6.2 there exists an element ξ of $\mathcal{D}_+$ such that $F=F_\xi$. By the above $F_{P_{\mathcal{N}}\xi}$ is a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of $f_\xi=F\lceil\mathcal{N}$. □

Definition 6.11. A map $I_{\omega_{\xi_0}}$ of $\mathcal{M}_+^*(<\omega_{\xi_0})$ into $\mathbb{R}_+$ is said to be an *information measure* with respect to ω_{ξ_0} if

(i) $I_{\omega_{\xi_0}}(F)\geq 0$, for all $F\in\mathcal{M}_+^*(<\omega_{\xi_0})$ and $I_{\omega_{\xi_0}}(\omega_{\xi_0})=1$;

(ii) $I_{\omega_{\xi_0}}(F_1+F_2)=I_{\omega_{\xi_0}}(F_1)+I_{\omega_{\xi_0}}(F_2)$, whenever F_1 and F_2 are mutually singular, that is, $((dF_1/d\omega_{\xi_0})\xi_0\mid(dF_2/d\omega_{\xi_0})\xi_0)=0$. $I_{\omega_{\xi_0}}(F)$ is called the *information* of F.

Theorem 6.12. (1) *The information measure* $I_{\omega_{\xi_0}}$ *with respect to* ω_{ξ_0} *exists uniquely, and* $I_{\omega_{\xi_0}}(F) = \|(dF/d\omega_{\xi_0})\xi_0\|^2$ *for each* $F \in \mathcal{M}_+^*(<\omega_{\xi_0})$.

(2) *If* f *is a* $(\mathcal{N}, \omega_{\xi_0})$ *coarse-grainable positive linear functional on* $\mathcal{N}$, *then* F_c *is the* ω_{ξ_0}-*absolutely continuous extension of* f *with minimal information with respect to* ω_{ξ_0}, *that is,* $I_{\omega_{\xi_0}}(F_c) \leq I_{\omega_{\xi_0}}(F)$ *for each* $F \in \mathcal{M}_+^*(<\omega_{\xi_0})$ *s.t.* $F\lceil\mathcal{N} = f$.

Proof. (1) Let $I_{\omega_{\xi_0}}$ be an information measure with respect to ω_{ξ_0}. By Theorem 6.2 the map $F \to (dF/d\omega_{\xi_0})\xi_0$ is a bijection of $\mathcal{M}_+^*(<\omega_{\xi_0})$ onto $\mathfrak{D}^*(\mathcal{M})_+$, and so we may define a map φ of $\mathfrak{D}^*(\mathcal{M})_+$ into $\mathbb{R}_+$ by

$$\varphi\left(\frac{dF}{d\omega_{\xi_0}}\xi_0\right) = I_{\omega_{\xi_0}}(F), \quad F \in \mathcal{M}_+^*(<\omega_{\xi_0}). \tag{6.32}$$

Take arbitrary $F_1, F_2 \in \mathcal{M}_+^*(<\omega_{\xi_0})$ s.t. $((dF_1/d\omega_{\xi_0})\xi_0 \mid (dF_2/d\omega_{\xi_0})\xi_0) = 0$. Then we have

$$\begin{aligned}\varphi\left(\frac{dF_1}{d\omega_{\xi_0}}\xi_0 + \frac{dF_2}{d\omega_{\xi_0}}\xi_0\right) &= \varphi\left(\frac{d(F_1+F_2)}{d\omega_{\xi_0}}\xi_0\right) = I_{\omega_{\xi_0}}(F_1+F_2)\\ &= I_{\omega_{\xi_0}}(F_1) + I_{\omega_{\xi_0}}(F_2) = \varphi\left(\frac{dF_1}{d\omega_{\xi_0}}\xi_0\right) + \varphi\left(\frac{dF_2}{d\omega_{\xi_0}}\xi_0\right),\end{aligned} \tag{6.33}$$

which by [14, Corollary 2.3] implies that

$$\varphi\left(\frac{dF}{d\omega_{\xi_0}}\xi_0\right) = r\left\|\frac{dF}{d\omega_{\xi_0}}\xi_0\right\|^2, \quad \forall F \in \mathcal{M}_+^*(<\omega_{\xi_0}) \tag{6.34}$$

for some $r > 0$. Since $\varphi(\xi_0) = I_{\omega_{\xi_0}}(\omega_{\xi_0}) = 1$, we have $r = 1$ and

$$I_{\omega_{\xi_0}}(F) = \left\|\frac{dF}{d\omega_{\xi_0}}\xi_0\right\|^2, \quad \forall F \in \mathcal{M}_+^*(<\omega_{\xi_0}). \tag{6.35}$$

(2) Take an arbitrary $F \in \mathcal{M}_+^*(<\omega_{\xi_0})$ s.t. $F\lceil\mathcal{N} = f$. Then, by Theorem 6.4, we have

$$\left(X\xi_0 \mid \frac{dF_c}{d\omega_{\xi_0}}\xi_0\right) = \left(X\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0\right) = f(X) = \left(X\xi_0 \mid \frac{dF}{d\omega_{\xi_0}}\xi_0\right) \tag{6.36}$$

for each $X \in \mathcal{N}$, which implies that $P_{\mathcal{N}}(dF/d\omega_{\xi_0})\xi_0 = (dF_c/d\omega_{\xi_0})\xi_0$. Hence we have

$$I_{\omega_{\xi_0}}(F) = \left\|\frac{dF}{d\omega_{\xi_0}}\xi_0\right\|^2 \geq \left\|\frac{dF_c}{d\omega_{\xi_0}}\xi_0\right\|^2 = I_{\omega_{\xi_0}}(F_c). \tag{6.37}$$

□

Let $\mathcal{N}$ and $\mathcal{N}_1$ be O*-subalgebras of $\mathcal{M}$ such that $\mathcal{N}_1 \subset \mathcal{N}$ and f a $\omega_{\xi_0}\lceil\mathcal{N}$-absolutely continuous positive linear functional on $\mathcal{N}$. Then $f_1 \equiv f\lceil\mathcal{N}_1$ is a $\omega_{\xi_0}\lceil\mathcal{N}_1$-absolutely continuous positive linear functional on $\mathcal{N}_1$. We consider the following questions.

(1) Does the $(\mathcal{N}, \omega_{\xi_0})$ coarse grainabilitiy of f imply the $(\mathcal{N}_1, \omega_{\xi_0})$ coarse grainability of f_1? Conversely, does the $(\mathcal{N}_1, \omega_{\xi_0})$ coarse grainability of f_1 imply the $(\mathcal{N}, \omega_{\xi_0})$ coarse grainability of f?

(2) When f is $(\mathcal{N}, \omega_{\xi_0})$ coarse-grainable and f_1 is $(\mathcal{N}_1, \omega_{\xi_0})$ coarse-grainable, $F_c = (F_1)_c$?

Since

$$F_c(A) = \left(A\xi_0 \mid \frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0\right), \quad (F_1)_c(A) = \left(A\xi_0 \mid \frac{df_1}{d\omega_{\xi_0}\lceil\mathcal{N}_1}\xi_0\right), \quad A \in \mathcal{M}, \tag{6.38}$$

we have

$$F_c(X) = f(X) = f_1(X) = (F_1)_c(X) \tag{6.39}$$

for each $X \in \mathcal{N}_1$, and so

$$\frac{df_1}{d\omega_{\xi_0}\lceil\mathcal{N}_1}\xi_0 = P_{\mathcal{N}_1}\frac{df}{d\omega_{\xi_0}\lceil\mathcal{N}}\xi_0. \tag{6.40}$$

For the above question we have the following.

Proposition 6.13. *The following statements are equivalent.*

(i) f *is* $(\mathcal{N},\omega_{\xi_0})$ *coarse grainable,* f_1 *is* $(\mathcal{N}_1,\omega_{\xi_0})$ *coarse grainable and* $F_c = (F_1)_c$.

(ii) f *is* $(\mathcal{N},\omega_{\xi_0})$ *coarse grainable,* f_1 *is* $(\mathcal{N}_1,\omega_{\xi_0})$ *coarse grainable, and* $I_{\omega_{\xi_0}}(F_c) = I_{\omega_{\xi_0}}((F_1)_c)$.

(iii) f *is* $(\mathcal{N},\omega_{\xi_0})$ *coarse grainable and* $(df/d\omega_{\xi_0}\lceil\mathcal{N})\xi_0 \in \overline{\mathcal{N}_1\xi_0}$.

(iv) f_1 *is* $(\mathcal{N}_1,\omega_{\xi_0})$ *coarse grainable and* $(F_1)_c \supset f$.

Proof. (i)⇒(ii). This is trivial.

(ii)⇒(iii). Since $\|(dF_c/d\omega_{\xi_0})\xi_0\| = \|(d(F_1)_c/d\omega_{\xi_0})\xi_0\|$, it follows that $(df/d\omega_{\xi_0}\lceil\mathcal{N})\xi_0 = (dF_c/d\omega_{\xi_0})\xi_0 \in \overline{\mathcal{N}_1\xi_0}$.

(iii)⇒(iv). By Theorem 6.4 it is shown that F_c is a unique $(\mathcal{N}_1,\omega_{\xi_0})$ coarse graining of f_1, and so f_1 is $(\mathcal{N}_1,\omega_{\xi_0})$ coarse grainable and $(F_1)_c = F_c \supset f$.

(iv)⇒(i). By Theorem 6.2 it is shown that $(F_1)_c$ is a $(\mathcal{N},\omega_{\xi_0})$ coarse graining of f, and so f is $(\mathcal{N},\omega_{\xi_0})$ coarse grainable and $F_c = (F_1)_c$. □

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