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*Research Article*

# On $P_3$-Paracompact Spaces

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Mashhour et al. [1] introduced the notions of $P_1$-paracompactness and $P_2$-paracom pactness of topological spaces in terms of preopen sets. In this paper, we introduce and investigate a weaker form of paracompactness which is called $P_3$-paracompact. We obtain various characterizations, properties, examples, and counterexamples concerning it and its relationships with other types of spaces. In particular, we show that if a space $(X,T)$ is quasi-submaximal, then $(X,T)$ is paracompact if it is $P_3$-paracompact.

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## 1. Introduction

Mashhour et al. [1] used preopen sets to define $P_1$-paracompact and $P_2$-paracompact spaces. In [2], Ganster and Reilly studied more properties of such spaces and they proved that these two notions coincide for the class of $T_1$-spaces.

In Section 2 of the present work, we introduce the notion of $p$-locally finite collections and study their properties which are used in Section 3 to define the class of $P_3$-paracompact spaces. We study these notions in the spaces $(X,T)$, $(X,T^{\alpha})$, $(X,T^{\gamma})$, and $(X,T_s)$. Finally, in Section 4, we deal with subspaces, sum, product, images, and the inverse images of $P_3$-paracompact spaces.

Throughout this work a space will always mean a topological space on which no separation axiom is assumed unless explicitly stated. Let $(X,T)$ be a space and let $A$ be a subset of $X$. The closure of $A$, the interior of $A$, and the relative topology on $A$ in $(X,T)$ will be denoted by $\mathrm{cl}(A)$, $\mathrm{int}(A)$, and $T_A$, respectively. $A$ is called a preopen subset (see [3]) of $(X,T)$ if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$. The complement of a preopen set is called a preclosed set. The preclosure of $A$, denoted by $\mathrm{pcl}(A)$, is the smallest preclosed set that contains $A$. $A$ is called regular open (resp., regular closed, semiopen [4], $\alpha$-set [5]) if $A = \mathrm{int}(\mathrm{cl}(A))$

(resp., $A = \mathrm{cl}(\mathrm{int}(A))$, $A \subseteq \mathrm{cl}(\mathrm{int}(A))$, $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$). The family of all subsets of a space $(X,T)$ which are preopen (resp., preclosed, regular open, regular closed, semiopen) is denoted by $\mathrm{PO}(X,T)$ (resp., $\mathrm{PC}(X,T)$, $\mathrm{RO}(X,T)$, $\mathrm{RC}(X,T)$, $\mathrm{SO}(X,T)$). It is known that for a space $(X,T)$ the collection $\mathrm{RO}(X,T)$ is a base for a topology $T_s$ on $X$ such that $T_s \subseteq T \subseteq \mathrm{PO}(X,T) \subseteq \mathrm{PO}(X,T_s)$ and that the collection of all $\alpha$-sets of $(X,T)$ forms a topology on $X$, denoted by $T^\alpha$, finer than $T$ and $\mathrm{PO}(X,T) = \mathrm{PO}(X,T^\alpha)$.

A space $(X,T)$ is called submaximal [6] if every dense subset of $(X,T)$ is open. It is known that $(X,T)$ is submaximal if and only if $T = \mathrm{PO}(X,T)$. In [7], Al-Nashef introduced the notion of quasi-submaximal spaces where a space $(X,T)$ is quasi-submaximal if $\mathrm{cl}(D) - D$ is nowhere dense subset for each dense subset $D$ of $(X,T)$. This is equivalent to saying that $\mathrm{int}(D)$ is dense for each dense subset $D$ of $(X,T)$ (see [7, Proposition 3.3]). Several characterizations of quasi-submaximal spaces are obtained by Ganster [8, Theorems 3.2 and 4.1] and others; see [9, 10]. A space $(X,T)$ is called locally indiscrete [11] if every open subset of $X$ is closed.

For a nonempty set $X$, $T_{\text{indisc.}}$ and $T_{\text{dis.}}$ will denote, respectively, the indiscrete and the discrete topologies on $X$.

The following lemmas will be used in the sequel.

Lemma 1.1 (see [11]). *For a space $(X,T)$, the following items are equivalent:*

(a) *$(X,T)$ is locally indiscrete;*
(b) *every subset of $X$ is preopen;*
(c) *every singleton in $X$ is preopen;*
(d) *every closed subset of $X$ is preopen;*

Lemma 1.2 (see [10]). *Let $A$ and $B$ be subsets of a space $(X,T)$.*

(a) *If $A \in \mathrm{PO}(X,T)$ and $B \in \mathrm{SO}(X,T)$, then $A \cap B \in \mathrm{PO}(B,T_B)$.*
(b) *If $A \in \mathrm{PO}(B,T_B)$ and $B \in \mathrm{PO}(X,T)$, then $A \in \mathrm{PO}(X,T)$.*

Lemma 1.3 (see [11]). *Let $A$ and $B$ be subsets of a space $(X,T)$. If $A \in \mathrm{PO}(X,T)$ and $B \in T$, then $A \cap B \in \mathrm{PO}(X,T)$.*

## 2. $p$-locally finite collections

A collection $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ of subsets of a space $(X,T)$ is called locally finite (resp., strongly locally finite [12]) if for each $x \in X$, there exists $U_x \in T$ (resp., $U_x \in \mathrm{RO}(X,T)$) containing $x$ and $U_x$ intersects at most finitely many members of $\mathcal{F}$.

*Definition 2.1.* A collection $\mathcal{P} = \{P_\alpha : \alpha \in I\}$ of subsets of a space $(X,T)$ is called $p$-locally finite if for each $x \in X$, there exists a preopen set $W_x$ in $(X,T)$ containing $x$ and $W_x$ intersects at most finitely many members of $\mathcal{P}$.

It follows from the definition and Lemma 1.1, if $(X,T)$ is locally indiscrete, then the collection $\mathcal{P} = \{\{x\} : x \in X\}$ is $p$-locally finite.

The following implications follow directly from the definitions:

Strongly locally finite$\Rightarrow$locally finite$\Rightarrow$ $p$-locally finite.

*Application 2.2.* Let $X$ be an infinite set and $p \in X$.

(a) The space $(X, T_{\text{indisc.}})$ is locally indiscrete and so $\mathcal{P} = \{\{x\} : x \in X\}$ is a $p$-locally finite collection of preopen subsets but not locally finite.

(b) The space $(X, T)$where $T = \{\emptyset, X, \{p\}\}$ is quasi-submaximal and the collection $\mathcal{P} = \{\{x\} : x \in X\}$ is $p$-locally finite but not locally finite.

Note that if $(X, T)$ is a submaximal space, then every $p$-locally finite collection of $(X, T)$ is locally finite. However, the converse does not hold in general. Let $X = \{a, b, c\}$ with the topology $T = \{\emptyset, X, \{a, b\}\}$. It is clear that $(X, T)$ satisfies the above property while $(X, T)$ is not submaximal.

*Question 2.3.* What is the property P on a space $(X, T)$ such that $(X, T)$ satisfies P if and only if every $p$-locally finite collection in $(X, T)$ is locally finite?

Theorem 2.4. *Let $\mathcal{P} = \{P_\alpha : \alpha \in I\}$ be a collection of semiopen subsets of a space $(X, T)$. Then $\mathcal{P}$ is p-locally finite if and only if it is strongly locally finite.*

*Proof.* We need to prove the necessity part only. Let $x \in X$ and let $W_x$ be a preopen subset of $(X, T)$ such that $x \in W_x$ and $W_x$ intersects at most finitely many members of $\mathcal{P}$, say $P_{\alpha 1}, P_{\alpha_2}, \dots, P_{\alpha_n}$. Put $R_x = \text{int}(\text{cl}(W_x))$. Then $R_x \in \text{RO}(X, T)$ and $x \in R_x$. We show for every $\alpha \in I - \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, $P_\alpha \cap R_x = \emptyset$. For each $\alpha \in I$, choose $U_\alpha \in T$ such that $U_\alpha \subseteq P_\alpha \subseteq \text{cl}(U_\alpha)$. Now, if $P_\alpha \cap R_x \neq \emptyset$, then $\text{cl}(U_\alpha) \cap R_x \neq \emptyset$ and so $U_\alpha \cap \text{int}(\text{cl}(W_x)) \neq \emptyset$ which implies that $\emptyset \neq U_\alpha \cap W_x \subseteq P_\alpha \cap W_x$. Thus, $\alpha \in I - \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and the result follows. $\square$

Corollary 2.5. *Every p-locally finite collection of open sets (α-sets, regular closed sets) is locally finite.*

Corollary 2.6. (1) *A space $(X, T)$ is paracompact (resp., nearly paracompact [13], rc-paracompact [14], S-paracompact [15]) if and only if every open (resp., regular open, regular closed, open) cover $\mathcal{U}$ of X has a p-locally finite open (resp., open, regular closed, semiopen) refinement.*

(2) *A space $(X, T)$ is expandable if and only if for every locally finite collection $\mathcal{F} = \{F_\alpha : \alpha \in I\}$ of subsets of X there exists a p-locally finite collection $\mathcal{P} = \{P_\alpha : \alpha \in I\}$ of open subsets of X such that $F_\alpha \subseteq P_\alpha$ for each $\alpha \in I$.*

Theorem 2.7. *Let $\mathcal{P} = \{P_\alpha : \alpha \in I\}$ be a collection of subsets of a space $(X, T)$. Then*

(a) *$\mathcal{P}$ is p-locally finite if and only if $\{\text{pcl}(P_\alpha) : \alpha \in I\}$ is p-locally finite;*

(b) *if $\mathcal{P}$ is p-locally finite, then $\bigcup_{\alpha\in I} \text{pcl}(P_\alpha) = \text{pcl}(\bigcup_{\alpha\in I} P_\alpha)$;*

(c) *$\mathcal{P}$ is locally finite if and only if the collection $\{\text{pcl}(P_\alpha) : \alpha \in I\}$ is locally finite.*

*The easy proof is left to the reader.*

Corollary 2.8. *Every p-locally finite collection $\mathcal{P} = \{P_\alpha : \alpha \in I\}$ of preopen subsets of a quasi-submaximal space $(X, T)$ is locally finite.*

*Proof.* By Theorem 2.7, the collection $\{\text{pcl}(P_\alpha) : \alpha \in I\}$ is $p$-locally finite. Since $(X, T)$ is quasi-submaximal, $\text{pcl}(P_\alpha) = \text{cl}(P_\alpha)$ for every $\alpha \in I$. The result now follows from Corollary 2.5 and the fact that $\text{cl}(P) \in \text{RC}(X, T)$whenever $P \in \text{PO}(X, T)$.

Recall that a function $f:(X,T)\to(Y,M)$ is called

(a) preirresolute [16] if and only if $f^{-1}(A)\in \mathrm{PO}(X,T)$ for each $A\in \mathrm{PO}(Y,M)$;

(b) strongly preclosed [17] if $f(A)\in \mathrm{PC}(Y,M)$ for each $A\in \mathrm{PC}(X,T)$.

It is not difficult to see that a function $f:(X,T)\to(Y,M)$ is strongly preclosed if and only if for every $y\in Y$ and every $P\in \mathrm{PO}(X,T)$ which contains $f^{-1}(y)$, there is a $V\in \mathrm{PO}(Y,M)$ such that $y\in V$ and $f^{-1}(V)\subseteq P$. □

Theorem 2.9. *Let $f:(X,T)\to(Y,M)$ be a preirresolute function. If $\mathcal{P}=\{P_\alpha:\alpha\in I\}$ is a $p$-locally finite collection in $(Y,M)$, then $\{f^{-1}(P_\alpha):\alpha\in I\}$ is a $p$-locally finite collection in $(X,T)$.*

The proof is obvious.

Recall that a subset $A$ of a space $(X,T)$ is called strongly compact relative to $(X,T)$ (see [1]) if every cover of $A$ by preopen sets of $X$ has a finite subcover.

Theorem 2.10. *Let $f:(X,T)\to(Y,M)$ be a strongly preclosed function such that $f^{-1}(y)$ is strongly compact relative to $(X,T)$ for every $y\in Y$. If $\mathcal{P}=\{P_\alpha:\alpha\in I\}$ is a $p$-locally finite collection of subsets of $(X,T)$, then $f(\mathcal{P})=\{f(P_\alpha):\alpha\in I\}$ is a $p$-locally finite collection in $(Y,M)$.*

*Proof.* Let $y\in Y$. For each $x\in f^{-1}(y)$ choose $P_x\in \mathrm{PO}(X,T)$ such that $x\in P_x$ and $P_x$ intersects at most finitely many members of $\mathcal{P}$. The collection $\{P_x:x\in f^{-1}(y)\}$ is a preopen cover of $f^{-1}(y)$ and so there exists a finite number of points $x_1,\dots,x_n$ of $f^{-1}(y)$ such that $f^{-1}(y)\subseteq\bigcup_{i=1}^{n}P_{x_n}=P$. Since $f$ is strongly preclosed, there is a $V\in \mathrm{PO}(Y,M)$ such that $y\in V$ and $f^{-1}(V)\subseteq P$. Therefore, $V$ intersects at most finitely many members of $f(\mathcal{P})$ and thus $f(\mathcal{P})$ is $p$-locally finite in $(Y,M)$. □

## 3. $P_3$-paracompact spaces

Recall that a space $(X,T)$ is called $P_1$-paracompact [1] if every preopen cover of $X$ has a locally finite open refinement and it is called $P_2$-paracompact [1] if every preopen cover of $X$ has a locally finite preopen refinement.

*Definition 3.1.* A space $(X,T)$ is called $P_3$-paracompact if every open cover of $X$ has a $p$-locally finite preopen refinement.

The following diagram follows immediately from the definitions in which none of the these implications is reversible:

$$\begin{tikzcd}
P_1\text{-paracompact} \arrow[r] \arrow[d] & \text{paracompact} \arrow[r] & P_3\text{-paracompact} \\
P_2\text{-paracompact} \arrow[ur] & &
\end{tikzcd} \tag{3.1}$$

*Application 3.2.* Let $N$ be the set of all positive integers and let $N^-$ be the set of all negative integers. Let $X=N\bigcup N^-$ with the topology $T=\{U\subseteq X:N\subseteq U\}\bigcup\{\varnothing\}$. Then $(X,T)$ is not paracompact since the collection $\{N \bigcup \{x\} : x \in N^-\}$ is an open cover of $X$ which admits no locally finite open refinement. On the other hand, $(X,T)$ is $P_3$-paracompact since $\mathrm{PO}(X,T) = \{A \subseteq X : A \cap N \neq \varnothing\}$ and so the collection $\{\{x\} : x \in N\} \bigcup \{\{x, -x\} : x \in N^-\}$ is a $p$-locally finite preopen cover of $X$.

Proposition 3.3. *Let $(X,T)$ be a quasi-submaximal space. Then $(X,T)$ is paracompact if and only if it is $P_3$-paracompact.*

*The proof follows from Corollary 2.8 and the fact that a space $(X,T)$ is paracompact if and only if every open cover of $X$ has a locally finite preopen refinement [1].*

Example 3.2 shows that the condition $(X,T)$ is quasi-submaximal in the above proposition is essential.

Recall that a space $(X,T)$ is called countably $P$-compact [18], if every countable preopen cover of $(X,T)$ has a finite subcover. It is clear that every $p$-locally finite collection of countably $P$-compact space is finite (see [19, proof of Theorem 3.10.3]).

Proposition 3.4. *Let $(X,T)$ be a countably $P$-compact space. Then $(X,T)$ is $P_3$-paracompact if and only if it is compact.*

*A space $(X,T)$ is called $p$-regular [20] if for each closed set $F$ and each point $x \notin F$, there exist disjoint preopen sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$. Note that a space $(X,T)$ is $p$-regular if and only if for each $U \in T$ and for each $x \in U$, there exists $P \in \mathrm{PO}(X,T)$ such that $x \in P \subseteq \mathrm{pcl}(P) \subseteq U$.*

*It is clear that every regular space is $p$-regular. However, the converse is not true in general as the following example shows.*

*Application 3.5.* Let $X$ be an infinite set and fix $p \in X$. Then $T = \{U \subseteq X : p \in U$ and $X - U$ is finite $\bigcup\{\varnothing\}$ is a topology on $X$such that $\mathrm{PO}(X,T) = \{S \subseteq X : p \in S$ or$S$ is infinite$\} \bigcup \{\varnothing\}$ (see [21, Example 1, page 118]). Note that for every open set $U \in T$ containing $p$, $\mathrm{cl}(U) = X$ and so $(X,T)$ is neither quasi-submaximal nor regular. On the other hand, for every $x \in X$, the set $\{p, x\}$ is both preopen and preclosed and hence $(X,T)$ is $p$-regular

Proposition 3.6. *Every $p$-regular quasi-submaximal space is regular.*

*Proof.* Let $(X,T)$ be a $p$-regular quasi-submaximal space. Let $U \in T$ and $x \in U$. Since $(X,T)$ is $p$-regular, there exists $V \in \mathrm{PO}(X,T)$ such that $x \in V \subseteq \mathrm{pcl}(V) \subseteq U$. Since $(X,T)$ is quasi-submaximal and $V \in \mathrm{PO}(X,T)$, $\mathrm{cl}(V) = \mathrm{pcl}(V)$. Therefore, $x \in \mathrm{int}(\mathrm{cl}(V)) \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(V))) = \mathrm{cl}(V) = \mathrm{pcl}(V) \subseteq U$ and hence $(X,T)$ is regular. □

Proposition 3.7. *Every $P_3$-paracompact $T_2$-space $(X,T)$ is $p$-regular.*

*Proof.* Let $A$ be a closed subset of $(X,T)$ and let $x \notin A$. For each $y \in A$, choose an open set $U_y$such that $y \in U_y$ and $x \notin \mathrm{cl}(U_y)$. Therefore, the family $\mathcal{U} = \{U_y : y \in A\} \bigcup \{X - A\}$ is an open cover of $(X,T)$ and so it has a $p$-locally finite preopen refinement $\mathcal{H}$. Put $V = \bigcup\{H \in \mathcal{H} : H \cap A \neq \varnothing\}$. Then $V$ is a preopen set containing $A$ and $\mathrm{pcl}(V) = \bigcup\{\mathrm{pcl}(H) : H \in \mathcal{H}$ and $H \cap A \neq \varnothing\}$ (see Theorem 2.7(b)). Therefore, $U = X - \mathrm{pcl}(V)$ is a preopen set containing $x$ such that $U \cap V = \varnothing$. Thus $(X,T)$ is $p$-regular. □

Theorem 3.8. *Let $(X,T)$ be a regular space. Then $(X,T)$ is $P_3$-paracompact if and only if every open cover $\mathcal{U}$ of $X$ has a $p$-locally finite preclosed refinement $\mathcal{V}$ (that is $V \in PC(X,T)$ for every $V \in \mathcal{V}$).*

*Proof.* To prove necessity, let $\mathcal{U}$ be an open cover of $X$. For each $x \in X$ we choose a member $U_x \in \mathcal{U}$ and, by the regularity of $(X,T)$, an open subset $V_x \in T$ such that $x \in V_x \subseteq \operatorname{cl}(V_x) \subseteq U_x$. Therefore, $\mathcal{V} = \{V_x : x \in X\}$ is an open cover of $X$ and so by assumption, it has a $p$-locally finite preopen refinement, say $\mathcal{W} = \{W_\beta : \beta \in B\}$. Now, consider the collection $\operatorname{pcl}(\mathcal{W}) = \{\operatorname{pcl}(W_\beta) : \beta \in B\}$. It is easy to see that $\operatorname{pcl}(\mathcal{W})$ is a $p$-locally finite collection (Theorem 2.7(a)) of preclosed subsets of $(X,T)$ such that for every $\beta \in B$, $\operatorname{pcl}(W_\beta) \subseteq \operatorname{pcl}(V_x) \subseteq \operatorname{cl}(V_x) \subseteq U_x$ for some $U_x \in \mathcal{U}$, that is, $\operatorname{pcl}(\mathcal{W})$ is a refinement of $\mathcal{U}$.

To prove sufficiency, let $\mathcal{U}$ be an open cover of $X$ and let $\mathcal{V}$ be a $p$-locally finite preclosed refinement of $\mathcal{U}$. For each $x \in X$, choose $W_x \in \mathrm{PO}(X,T)$ such that $x \in W_x$ and $W_x$ intersect at most finitely many members of $\mathcal{V}$. Let $\mathcal{H}$ be a preclosed $p$-locally finite refinement of $\mathcal{W} = \{W_x : x \in X\}$. For each $V \in \mathcal{V}$, let $V' = X - \{H \in \mathcal{H} : H \cap V = \varnothing\}$. Then $\{V' : V \in \mathcal{V}\}$ is a preopen cover of $X$. Finally, for each $V \in \mathcal{V}$, choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$.Therefore, the collection $\{U_V \cap V' : V \in \mathcal{V}\}$ is a locally finite preopen (Lemma 1.3) refinement of $\mathcal{U}$ and thus $(X,T)$ is $P_3$-paracompact.

Observe that to prove the sufficiency part of Theorem 3.8, there is no need for the space $(X,T)$ to be regular. However, the condition $(X,T)$ that is regular in the necessity part cannot be replaced by $p$-regular as Example 3.5 shows.

Recall that a subset $A$ of a space $(X,T)$ is called a $\gamma$-set [22] if $A \cap P \in \mathrm{PO}(X,T)$ for each $P \in \mathrm{PO}(X,T)$. It is well known that the collection of all $\gamma$-sets in a space $(X,T)$, denoted by $T^\gamma$, is a topology on $X$ satisfying $T \subseteq T^\alpha \subseteq T^\gamma \subseteq \mathrm{PO}(X,T^\gamma) \subseteq \mathrm{PO}(X,T^\alpha) = \mathrm{PO}(X,T)$.

Ganster [8, Theorems 3.2 and 4.1] shows that if $(X,T)$ is quasi-submaximal, then $T^\alpha = \mathrm{PO}(X,T)$. □

Proposition 3.9. *Let $(X,T)$ be a space.*

(1) *If $(X,T)$ is $P_1$-paracompact, then $(X,T^\alpha)$ is $P_1$-paracompact.*
(2) *$(X,T)$ is $P_2$-paracompact if and only if $(X,T^\alpha)$ is $P_2$-paracompact.*
(3) *If $(X,T^\alpha)$ is $P_3$-paracompact, then $(X,T)$ is $P_3$-paracompact.*

*Proof.* The proofs of (1) and (3) are clear. To prove (2), we need to prove the sufficiency part only. Let $\mathcal{U}$ be a preopen cover of $(X,T)$. Since $\mathrm{PO}(X,T^\alpha) = \mathrm{PO}(X,T)$, $\mathcal{U}$ is a preopen cover of the $P_2$-paracompact space $(X,T^\alpha)$ and so it has a locally finite preopen refinement $\mathcal{V}$ in $(X,T^\alpha)$. As in the proof of [15, Theorem 2.10], $\mathcal{V}$ is locally finite in $(X,T)$. Therefore, $\mathcal{V}$ is a locally finite preopen refinement of $\mathcal{U}$ in $(X,T)$ and thus $(X,T)$ is $P_2$-paracompact.

The following examples show that the converses of part (1) and part (3) of Proposition 3.9 are not true in general. □

*Application 3.10.* (a) Let $(X,T)$ be as in Example 2.2(b). Then $(X,T)$ is $P_3$-paracompact but $(X,T^\alpha)$ is not since $T^\alpha = \mathrm{PO}(X,T) = \mathrm{PO}(X,T^\alpha) = \{\varnothing\} \bigcup \{U \subseteq X : p \in U\}$ and the collection $\{\{p,x\} : x \in X\}$ is an open (preopen) cover of $(X,T^\alpha)$ which admits no locally finite preopen refinement in $(X,T^\alpha)$.

(b) Let $X = \{a,b,c\}$ with the topology $T = \{\varnothing,X,\{a\}\}$. Then $T^\alpha = \mathrm{PO}(X,T) = \mathrm{PO}(X,T^\alpha) = \{\varnothing,X,\{a\},\{a,b\},\{a,c\}\}$ and so $(X,T^\alpha)$ is $P_1$-paracompact but $(X,T)$ is not.

Proposition 3.11. *Let $(X,T)$ be a space. Then*

(1) *if $(X,T)$ is $P_1$-paracompact, then $(X,T^\gamma)$ is $P_1$-paracompact;*
(2) *if $(X,T)$ is $P_2$-paracompact, then $(X,T^\gamma)$ is $P_2$-paracompact;*
(3) *if $(X,T^\gamma)$ is $P_3$-paracompact, then $(X,T)$ is $P_3$-paracompact.*

*Proof.* The proofs of (1) and (3) are clear. For (2), let $\mathcal{U}$ be a preopen cover of $(X,T^\gamma)$. Since $\mathrm{PO}(X,T^\gamma) \subseteq \mathrm{PO}(X,T)$, $\mathcal{U}$ is a preopen cover of the $P_2$-paracompact space $(X,T)$ and so it has a locally finite preopen refinement $\mathcal{V}$ in $(X,T)$. Now, for every $V \in \mathcal{V}$, choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. Then, by Lemma 1.3, the collection $\{\mathrm{int}(\mathrm{cl}(V) \cap U_V : V \in \mathcal{V}\}$ is a locally finite preopen refinement of $\mathcal{U}$ in $(X,T^\gamma)$.

The converse of each part of Proposition 3.11 is not true in general. For (1), see Example 3.10(a) and for (2), see Example 2.2(a). Finally, for (3), we consider the space $(X,T)$ given in Example 3.10(a). Then $(X,T)$ is a quasi-submaximal semi-$T_D$ space (see [2]) and so $T^\alpha = T^\gamma = \mathrm{PO}(X,T) = \mathrm{PO}(X,T^\gamma)$ (see [21, page 117]). Therefore, $(X,T)$ is $P_3$-paracompact but $(X,T^\gamma)$ is not. □

Proposition 3.12. *Let $(X,T)$ be a space. Then*

(1) *if $(X,T_s)$ is $P_1$-paracompact, then $(X,T)$ is $P_1$-paracompact;*
(2) *if $(X,T_s)$ is $P_2$-paracompact, then $(X,T)$ is $P_2$-paracompact;*
(3) *if $(X,T)$ is $P_3$-paracompact, then $(X,T_s)$ is $P_3$-paracompact.*

*Proof.* The proofs of (1) and (3) are clear. The proof of (2) is similar to the proof of Proposition 3.11(2).

To see that the converse of (1) is not true in general, we consider the space $(X,T)$, where $X = \{a,b\}$ and $T = \{\varnothing,X,\{a\}\}$. Then $T = \mathrm{PO}(X,T)$ and so $(X,T)$ is $P_1$-paracompact while $(X,T_s)$ is not since $T_s = T_{\text{indisc.}}$.

To see that the converse of (3) is not true in general, we consider the following example. □

*Application 3.13.* Let $X$ be an infinite set and let $p \in X$ with the topology $T = \{U \subseteq X : p \in U\} \bigcup \{\varnothing\}$. Then $T_s = T_{\text{indisc.}}$ and so $(X,T_s)$ is $P_3$-paracompact. However, the space $(X,T)$ is not since the collection $\{\{p,x\} : x \in X\}$ is an open (preopen) cover of $(X,T)$ which admits no $p$-locally finite preopen refinement in $(X,T)$.

## 4. Operations

*Definition 4.1.* A subset $A$ of a space $(X,T)$ is called $P_3$-paracompact relative to $(X,T)$ if every cover of $A$ by open subsets of $(X,T)$ has a $p$-locally finite preopen refinement in $(X,T)$.

Recall that a subset $A$ of a space $(X,T)$ is called $g$-closed [23] (resp., pg-closed [24]) if $\mathrm{cl}(A) \subseteq U$ (resp., $\mathrm{pcl}(A) \subseteq U$), whenever $A \subseteq U$ and $U \in T$ (resp., $U \in \mathrm{PO}(X,T)$) and it is called $\theta$-open [25] (resp., pre-$\theta$-open [10]) if for every $x \in A$, there exists $U \in T$

(resp., $U \in \mathrm{PO}(X,T)$) such that $x \in U \subseteq \mathrm{cl}(U) \subseteq A$ (resp., $x \in U \subseteq \mathrm{pcl}(U) \subseteq A$). The complement of a $\theta$-open (resp., pre-$\theta$-open) set is called $\theta$ -closed (resp., pre-$\theta$-closed).

It is clear that every $\theta$-open is pre-$\theta$-open. However, the converse is not true in general. Since in $(X, T_{\text{indisc.}})$, every subset of $(X, T_{\text{indisc.}})$ is pre-$\theta$-open .

Lemma 4.2. *Let $A$ be a subset of a quasi-submaximal space $(X,T)$. Then $A$ is pre-$\theta$-open if and only if it is $\theta$-open.*

*Proof.* We need to prove the necessity part only. Let $x \in A$. Choose $U \in \mathrm{PO}(X,T)$ such that $x \in U \subseteq \mathrm{pcl}(U) \subseteq A$. Since $(X,T)$ is quasi-submaximal $\mathrm{pcl}(U) = \mathrm{cl}(U)$ and $\mathrm{cl}(U) \in \mathrm{RC}(X,T)$. Therefore, $x \in \mathrm{int}(\mathrm{cl}(U)) \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))) = \mathrm{cl}(U) = \mathrm{pcl}(U) \subseteq A$ and thus $A$ is $\theta$-open. □

Theorem 4.3. *Let $A$ be a $P_3$-paracompact relative subset of a $T_2$-space $(X,T)$. Then $A$ is pre-$\theta$-closed.*

*Proof.* Let $x \notin A$. For each $y \in A$, there exists $U_y \in T$ such that $y \in U_y$ and $x \notin \mathrm{cl}(U_y)$. Therefore, the family $\mathcal{U} = \{U_y : y \in A\}$ is an open cover of $A$ in $(X,T)$. Since $A$ is $P_3$-paracompact relative to $(X,T)$, $\mathcal{U}$ has a $p$-locally finite preopen refinement in $(X,T)$, say $\mathcal{V}$. Put $W = \bigcup\{V : V \in \mathcal{V}\}$ and $W' = X - \mathrm{pcl}(W)$. Then $W' \in \mathrm{PO}(X,T)$ and since $T \subseteq \mathrm{PO}(X,T)$, $x \in W'$. Moreover, by Theorem 2.7(b), $\mathrm{pcl}(W') \cap A = \varnothing$, that is, $x \in W' \subseteq \mathrm{pcl}(W') \subseteq X - A$ . This shows that $X - A$ is pre-$\theta$-open and hence $A$ is pre-$\theta$-closed. □

Theorem 4.4. *Let $A$ be a $g$-closed subset of a $P_3$-paracompact space $(X,T)$. Then $A$ is $P_3$-paracompact relative to $(X,T)$.*

*Proof.* Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be a cover of $A$ by open subsets of $(X,T)$. Since $A \subseteq \bigcup\{U_\alpha : \alpha \in I\}$ and $A$ is $g$-closed, $\mathrm{cl}(A) \subseteq \bigcup\{U_\alpha : \alpha \in I\}$. For each $x \notin \mathrm{cl}(A)$, there exists an open set $W_x$ of $(X,T)$ such that $A \cap W_x = \varnothing$. Now, put $\mathcal{U}' = \{U_\alpha : \alpha \in I\} \bigcup \{W_x : x \notin \mathrm{cl}(A)\}$. Then $\mathcal{U}'$ is an open cover of the $P_3$-paracompact space $(X,T)$. Let $\mathcal{H} = \{H_\beta : \beta \in B\}$ be a $p$-locally finite preopen refinement of $\mathcal{U}'$ and put $\mathcal{H}_{\mathcal{U}} = \{H_\beta : H_\beta \subseteq U_{\alpha(\beta)}, \beta \in B$ and $\alpha(\beta) \in I\}$. Then $\mathcal{H}_{\mathcal{U}}$ is a $p$-locally finite preopen refinement of $\mathcal{U}$. Therefore, $A$ is $P_3$-paracompact relative to $(X,T)$. □

Corollary 4.5. *Let $(X,T)$ be a quasi-submaximal $P_3$-paracompact $T_2$-space and $A$ any subset of $X$. Then, the following items are equivalent:*

(a) *$A$ is $P_3$-paracompact relative to $(X,T)$;*
(b) *$A$ is pre-$\theta$-closed;*
(c) *$A$ is $\theta$-closed;*
(d) *$A$ is closed;*
(e) *$A$ is $g$-closed.*

*At this point it is important to give an example of a quasi-submaximal $T_2$-space which is not submaximal.*

*Application 4.6.* Let $X = R$ be the set of all real numbers and let $D$ be a subset of $X$ such that $D$ and $X - D$ are dense in $(R, T_u)$. Let $T^*$ be the topology generated by $T_u$ and the collection $\mathcal{B} = \{\{x\} : x \in D\}$ (see [26, Example 71]). Then $(X, T^*)$ is a $T_5$-space. Note that if $H$ is dense in $(X, T^*)$, then $D \subseteq H$ and thus $(X, T^*)$ is quasi-submaximal. On the other hand, $D^* = D\bigcup\{x\}$, $x \notin D$ is a nonopen dense subset of $(X,T^*)$ and therefore $(X,T^*)$ is not submaximal.

Theorem 4.7. *Let $A$ be a regular closed subset of a $P_3$-paracompact space $(X,T)$. Then $(A,T_A)$ is $P_3$-paracompact.*

*Proof.* Let $\mathcal{V} = \{V_\alpha : \alpha \in I\}$ be an open cover of $A$ in $(A,T_A)$. For each $\alpha \in I$, choose $U_\alpha \in T$ such that $V_\alpha = A \cap U_\alpha$. Then the collection $\mathcal{U} = \{U_\alpha : \alpha \in I\}\bigcup\{X - A\}$ is an open cover of the $P_3$-paracompact space $(X,T)$ and so it has a $p$-locally finite preopen refinement, say $\mathcal{W} = \{W_\beta : \beta \in J\}$. Then, by Lemma 1.2(a) and the fact that $\mathrm{RC}(X,T) \subseteq \mathrm{SO}(X,T)$, the collection $\{A \cap W_\beta : \beta \in J\}$ is a $p$-locally finite preopen refinement of $\mathcal{V}$ in $(A,T_A)$. This completes the proof. □

Theorem 4.8. *Let $A$ and $B$ be subsets of a space $(X,T)$ such that $A \subseteq B$.*
(a) *If $B \in \mathrm{PO}(X,T)$ and $A$ is $P_3$-paracompact relative to $(B,T_B)$, then $A$ is $P_3$-paracompact relative to $(X,T)$.*
(b) *If $B \in \mathrm{SO}(X,T)$ and $A$ is $P_3$-paracompact relative to $(X,T)$, then $A$ is $P_3$-paracompact relative to $(B,T_B)$.*

*Proof.* (a) Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of $A$ in $(X,T)$. Then the collection $\mathcal{U}_B = \{B \cap U_\alpha : \alpha \in I\}$ is an open cover of $A$ in $(B,T_B)$. Since $A$ is $P_3$-paracompact relative to $(B,T_B)$, $\mathcal{U}_B$ has a $p$-locally finite preopen refinement $\mathcal{V}_{\mathcal{B}}$ in $(B,T_B)$. By Lemma 1.2(b), the collection $\mathcal{V}_B$ is a $p$-locally finite preopen refinement of in $(X,T)$. Therefore, $A$ is $P_3$-paracompact relative to $(X,T)$.

(b) Let $\mathcal{V} = \{V_\alpha : \alpha \in I\}$ be an open cover of $A$ in $(B,T_B)$. For every $\alpha \in I$, choose $U_\alpha \in T$ such that $V_\alpha = B \cap U_\alpha$. Therefore, the collection $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ is an open cover of $A$ in $(X,T)$ and so it has a $p$-locally finite preopen refinement of $\mathcal{U}$ in $(X,T)$, say $\mathcal{P}$. Then, by Lemma 1.2(a), the collection $\mathcal{P}_B = \{P \cap B : P \in \mathcal{P}\}$ is a $p$-locally finite preopen refinement of $\mathcal{U}$ in $(B,T_B)$. □

Corollary 4.9. *Let $A$ a subset of a space $(X,T)$.*
(a) *If $A \in \mathrm{PO}(X,T)$ and the subspace $(A,T_A)$ is $P_3$-paracompact, then $A$ is $P_3$-paracompact relative to $(X,T)$.*
(b) *If $A \in \mathrm{SO}(X,T)$ and $A$ is $P_3$-paracompact relative to $(X,T)$, then the subspace $(A,T_A)$ is $P_3$-paracompact.*
(c) *If $A \in T^\alpha$, then $A$ is $P_3$-paracompact relative to $(X,T)$ if and only if the subspace $(A,T_A)$ is $P_3$-paracompact.*

In Example 3.2, if we put $A = N^-\bigcup\{1\}$. Then $A \notin \mathrm{SO}(X,T)$ and $T_A = \{B\bigcup\{1\} : B \subseteq N^-\}\bigcup\{\varnothing\}$. Since the collection $\{\{1\}\bigcup\{\{n,-n\} : n \in N\}$ is a $p$-locally finite preopen cover of $A$ in $(X,T)$, $A$ is $P_3$-paracompact relative to $(X,T)$. On the other hand, for every $n \in N$, $\{1,-n\} \in T_A$ and $\{-n\} \notin \mathrm{PO}(A,T_A)$ and so the subspace $(A,T_A)$ is not $P_3$-paracompact. Therefore, in Corollary 4.9(a), the condition $A$ that is semiopen cannot be dropped.

To see that the condition $A$ is preopen in Corollary 4.9(a) is essential, we consider the space $(X,T)$ given in Example 3.13. Now, put $A = X - \{p\}$. Then $A \notin \mathrm{PO}(X,T)$ and $T_A = T_{\mathrm{disc.}}$. Therefore, $(A,T_A)$ is $P_3$-paracompact. However, $A$ is not $P_3$-paracompact relative to $(X,T)$ since the collection $\{\{a,p\} : a \in A\}$ is an open cover of $A$ in $(X,T)$ which admits no $p$-locally finite preopen refinement in $(X,T)$.

Proposition 4.10. *Let $A$ be a pg-closed subset of a space $(X,T)$ and let $B$ be any subset of $X$. If $A$ is $P_3$-paracompact relative to $(X,T)$ and $A \subseteq B \subseteq \mathrm{pcl}(A)$, then $B$ is $P_3$-paracompact relative to $(X,T)$.*

The easy proof is left to the reader.

Recall that a function $f : (X,T) \to (Y,M)$ is called M-preopen [1] if $f(A) \in \mathrm{PO}(Y,M)$ for each $A \in \mathrm{PO}(X,T)$.

It is clear that every continuous open function is preirresolute and M-preopen (see [19, Exercise 1.4.C]).

Theorem 4.11. *Let $f : (X,T) \to (Y,M)$ be a continuous, open, and strongly preclosed surjective function such that $f^{-1}(y)$ is strongly compact relative to $(X,T)$ for every $y \in Y$. If $(X,T)$ is $P_3$-paracompact, then so is $(Y,M)$.*

*Proof.* Let $\mathcal{V} = \{V_\alpha : \alpha \in I\}$ be an open cover of $(Y,M)$. Since $f$ is continuous, the collection $\mathcal{U} = f^{-1}(\mathcal{V}) = \{f^{-1}(V_\alpha) : \alpha \in I\}$ is an open cover of the $P_3$-paracompact $(X,T)$ space and so it has a $p$-locally finite preopen refinement, say $\mathcal{P}$. Since $f$ is continuous and open, then $f$ is M-preopen and so by Theorem 2.10, the collection $f(\mathcal{P})$ is a $p$-locally finite preopen refinement of $\mathcal{V}$. The result now follows. □

Theorem 4.12. *Let $f : (X,T) \to (Y,M)$ be a closed preirresolute surjective function such that $f^{-1}(y)$ is compact in $(X,T)$ for every $y \in Y$. If $(Y,M)$ is $P_3$-paracompact, then so is $(X,T)$.*

*Proof.* Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of space $(X,T)$. For each $y \in Y$, $\mathcal{U}$ is an open cover of the compact subspace $f^{-1}(y)$ and so there exists a finite subset $I(y)$ of $I$ such that $f^{-1}(y) \subseteq \bigcup_{\alpha \in I(y)} U_\alpha = U_y$ and $U_y$ is open in $(X,T)$. As $f$ is a closed function, for each $y \in Y$ we find an open subset $V_y$ of $Y$ such that $y \in V_y$ and $f^{-1}(y) \subseteq U_y$. Then the collection $\mathcal{V} = \{V_y : y \in Y\}$ is an open cover of the $P_3$-paracompact space $(Y,M)$ and so it has a $p$-locally finite preopen refinement, say $\mathcal{W} = \{W_\beta : \beta \in B\}$. Since $f$ is preirresolute, the family $f^{-1}(\mathcal{W}) = \{f^{-1}(W_\beta) : \beta \in B\}$ is a preopen $p$-locally finite cover of $(X,T)$ such that for each $\beta \in B$, $f^{-1}(W_\beta) \subseteq U_y$ for some $y \in Y$. Finally, the collection $\{f^{-1}(W_\beta) \cap U_{\alpha(x)} : \beta \in B,\ \alpha(x) \in I(y)\}$ is a $p$-locally finite preopen refinement of $\mathcal{U}$. Therefore, $(X,T)$ is $P_3$-paracompact. □

Corollary 4.13. *The product space of a compact space and a $P_3$-paracompact space is $P_3$-paracompact.*

Theorem 4.14. *The topological sum $\oplus_{\alpha \in I} X_\alpha$ is $P_3$-paracompact if and only if the space $(X_\alpha, T_\alpha)$ is $P_3$-paracompact, for each $\alpha \in I$.*

*Proof.* Necessity follows from Theorem 4.7, since $(X_\alpha, T_\alpha)$ is a regular closed subspace of the space $\oplus_{\alpha \in I} X_\alpha$, for each $\alpha \in I$. To prove sufficiency, let $\mathcal{U}$ be an open cover of $\oplus_{\alpha \in I} X_\alpha$. For each $\alpha \in I$ the family $\mathcal{U}_\alpha = \{U \cap X_\alpha : U \in \mathcal{U}\}$ is an open cover of the $P_3$-paracompact space $(X_\alpha, T_\alpha)$. Therefore, $\mathcal{U}_\alpha$ has a $p$-locally finite preopen refinement $\mathcal{V}_\alpha$ in $(X_\alpha, T_\alpha)$.

Put $\mathcal{V} = \bigcup_{\alpha \in I} \mathcal{V}_\alpha$. It is clear that $\mathcal{V}$ is a $p$-locally finite refinement of $\mathcal{U}$ such that $V \in \mathrm{PO}(X,T)$ for every $V \in \mathcal{V}$ (Lemma 1.2(b)). Thus $\oplus_{\alpha \in I} X_\alpha$ is $P_3$-paracompact. $\square$

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