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Research Article

Symmetry Conditions on the Coincidence of Some Notions of Quasi-Uniform Completeness

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We generalize the notions of quietness and semisymmetry defined by Doitchinov (1991) and Deák (1991) and we study the role of these extended notions on the coincidence of some well-known quasi-uniform completeness. In particular, it is shown that the bicompletion coincides (up to quasi-uniformism) with the standard D-completion in quiet $\star$-weakly pair Cauchy bounded quasi-uniform spaces and it coincides with $\star$-half-completion defined by Romaguera and Sánchez-Granero (2002), in T_0-weakly quiet quasi-uniform spaces.

1. Introduction

A quasi-uniformity on a (nonempty) set X is a filter $\mathcal{U}$ on $X \times X$ such that (a) each member of $\mathcal{U}$ contains $\Delta(X)$, where $\Delta(X)$ is the diagonal of X, and (b) if $U \in \mathcal{U}$, then $V \circ V \subseteq U$ for some $V \in \mathcal{U}$. The pair $(X,\mathcal{U})$ is called a *quasi-uniform space*. If $\mathcal{U}$ is a quasi-uniformity on a set X, then $\mathcal{U}^{-1} = \{U^{-1} \mid U \in \mathcal{U}\}$ is also a quasi-uniformity on X called the *conjugate* of $\mathcal{U}$. The uniformity $\mathcal{U} \bigvee \mathcal{U}^{-1}$ will be denoted by $\mathcal{U}^\star$. If $U \in \mathcal{U}$, the entourage $U \cap U^{-1}$ of $\mathcal{U}^\star$ will be denoted by $U^\star$. Each quasi-uniformity $\mathcal{U}$ on X induces a topology $\tau(\mathcal{U})$ on X, defined as follows:

$$\tau(\mathcal{U}) = \{A \subseteq X \mid \text{ for each } x \in A \text{ there is } U \in \mathcal{U} \text{ such that } U(x) \subseteq A\}. \tag{1.1}$$

For quasi-uniform spaces, the notion of completeness presents serious difficulties. There are various attempts to define a notion of completeness and completion, but none of these is able to handle all spaces in a satisfying manner. By definition, the notion of completeness as well as the construction of the completion depends on the choice of the definition of Cauchy net or Cauchy filter (often nets and filters lead to equivalent theories). The reason for the difficulty to develop a satisfactory theory of completeness for the class of all quasi-uniform spaces is due to the difficulty to approach the notion of Cauchy net (filter) in an appropriate way from the case of uniform spaces to the case of quasi-uniform spaces. More precisely, since uniform spaces belong to the class of quasi-uniform spaces, according to Doitchinov [1], a notion of Cauchy net in any quasi-uniform space has to be defined in such a manner that this definition provides the properties that convergent nets are Cauchy, and it agrees with the usual definition for uniform spaces. Moreover, the suggested completion must be a monotone operator with respect to inclusion and give rise to the usual uniform completion in the uniform case.

Fletcher and Lindgren in [4, 5] give two equivalent theories of completeness, the pair completeness or bicompleteness. The first one uses pair of filters $(\mathcal{F},\mathcal{G})$ for which $\mathcal{F}\cup\mathcal{G}$ is a filter base and the second is modelled upon the presentations of the completion of a uniform space given by Bourbaki [6]. It turns out that this concept coincides (for quasi-uniform spaces) with that of double completeness introduced by Császár in [7]. The authors proved that the bicompletion has a very nice behavior: every T_0 quasi-uniform space can be quasi-uniformly embedded in a unique (i.e., up to quasi-uniformism) T_0 quasi-uniform space; for uniform spaces the construction coincides with the usual one. In [1], Doitchinov has introduced the notion of D-completeness and he proved that a certain category of quasi-uniform T_2-spaces, the so-called *quiet spaces*, has a nice behavior: every quiet T_2-space can be quasi-uniformly embedded in a unique D-complete quiet T_2-space; for uniform spaces (which are always quiet) the construction coincides with the usual one. D-completeness is founded on a concept of a Cauchy net which can be done satisfying some most natural basic requirements. The Doitchinov completion provides another sophisticated completion that is in general larger than the bicompletion. (Consider for instance the set of the rationals equipped with the Sorgenfrey quasimetric. The space is bicomplete and only its Doitchinov completion yields the expected completion.)

The definition of Cauchy net or filter in quasi-uniform space which has been proposed in bicompleteness and D-completeness gives a completing theory which satisfies the above requirements given by Doitchinov.

Császár in [8, page 228] and Deák in [2, page 412] have introduced the notion of half-completeness in a quasi-uniform space which generalizes the well-known notions of bicompleteness and D-completeness.

Romaguera and Sánchez-Granero in [3] introduced the notions of *-half-completion and *-compactification of a T_1 quasi-uniform space $(X,\mathcal{U})$. The authors use the notion of *-half-completion to show that if a T_1 quasi-uniform space has a *-compactification, then it is unique (up to quasi-uniformism).

In this paper, we introduce the notions of weakly quiet and *-weakly pair Cauchy bounded quasi-uniform spaces which generalize the notions of quiet and semisymmetric quasi-uniform spaces defined by Doitchinov [1] and Deák [2], respectively. We discuss the relationship between the notions of half-completeness, bicompleteness, and D-completeness and characterize those quasi-uniform spaces in which the bicompletion, constructed by Lindgren and Fletcher in [4], coincides (up to quasi-uniformism) with the standard D-completion constructed by Doitchinov in [1]. Finally, we prove that the bicompletion of a T_0 weakly quiet quasi-uniform space is the T_1 *-half-completion of the given space which is unique up to quasi-uniformism.

2. Definitions and preliminary results

Throughout the paper, the letter $\mathbb{R}$ will denote the set of real numbers. If (X,τ) is a topological space and A is a subset of X, $cl_\tau A$ will denote the closure of A in (X,τ), and if $\mathcal{F}$ (resp., $(x_a)_{a\in A}$) is a filter (resp., net) on X that converges with respect to the topology τ, we will write "$\mathcal{F}$ (resp., $(x_a)_{a\in A}$) is τ-convergent."

We recall some definitions from [1, 5, 2, 9].

Definitions 2.1. Let $(X,\mathcal{U})$ be a quasi-uniform space.

(1) A filter base $\mathcal{B}$ is called *$\mathcal{U}^\star$-Cauchy* if and only if for each $U\in\mathcal{U}$, there exists $B\in\mathcal{B}$ such that $B\times B\subseteq U$ (see [5, page 48]).

(2) A net $(x_a)_{a\in A}$ is called *$\mathcal{U}^\star$-Cauchy net* if for each $U\in\mathcal{U}$, there exists an $a_U\in A$ such that $(x_a,x_{a'})\in U$ whenever $a\geq a_U$, $a'\geq a_U$.

(3) A net $(x_a)_{a\in A}$ is called *D-Cauchy net* if there exists a net $(y_\beta)_{\beta\in B}$ in $(X,\mathcal{U})$ with the property; for any $U\in\mathcal{U}$, there are $a_U\in A$ and $\beta_U\in B$ such that $(y_\beta,x_a)\in U$ whenever $a\geq a_U$, $\beta\geq\beta_U$. The net $(y_\beta)_{\beta\in B}$ is called a *conet* of the net $(x_a)_{a\in A}$. In this case, we write $(y_\beta,x_a)\to 0$ (see [1, Definition 1]).

(4) The space $(X,\mathcal{U})$ is called (i) *bicomplete*(equivalent to the double completeness of [7]) if each $\mathcal{U}^\star$-Cauchy filter (net) is $\tau(\mathcal{U}^\star)$-covergent, (ii) *half-complete* if each $\mathcal{U}^\star$-Cauchy filter (net) is $\tau(\mathcal{U})$-convergent, and (iii) *D-complete* if each D-Cauchy filter (net) is $\tau(\mathcal{U})$-convergent.

(5) The space $(X,\mathcal{U})$ is called *quiet*, provided that for each $U\in\mathcal{U}$, there is a $V\in\mathcal{U}$ such that if $(x_a)_{a\in A}$, $(y_\beta)_{\beta\in B}$ are nets in X and x,y are points of X such that $(x,x_a)\in V$, $(y_\beta,y)\in V$ and $(y_\beta,x_a)\to 0$, then $(x,y)\in U$ (see [1, page 207]).

(6) The space $(X,\mathcal{U})$ is called *uniformly weakly regular* if for each $U\in\mathcal{U}$, there is $V\in\mathcal{U}$ such that for all $x\in X$, $cl_{\tau(\mathcal{U})}(V^{-1}\cap V)(x)\subseteq U(x)$ (see [9, page 753]).

(7) Given a net $(x_a)_{a\in A}$ in $(X,\mathcal{U})$, let $T_a=\{x_{a'}\mid a'\geq a\}$ for each $a\in A$. This set is called the *tail of* $(x_a)_{a\in A}$ *beginning at* a. The sets $\{T_a\mid a\in A\}$ form a filter base. The filter $\mathcal{F}((x_a)_{a\in A})$ generated by this filter base is called the *filter of tails of the net* $(x_a)_{a\in A}$ or *the elementary filter in* X *generated by the net* $(x_a)_{a\in A}$.

Definition 2.2. A quasi-uniform space $(X,\mathcal{U})$ is *weakly quiet* provided that for each $U\in\mathcal{U}$, there is $V\in\mathcal{U}$ such that if $(y_\beta)_{\beta\in B}$ and $(x_a)_{a\in A}$ are nets on X with $(y_\beta,x_a)\to 0$ and x and y are points of X such that $(x,x_a)\in V^\star$ for $a\in A$ and $(y_\beta,y)\in V$ for $\beta\in B$, then $(x,y)\in U$. If V satisfies the above condition, we say that V is *weakly quiet* for U.

Proposition 2.3. *Every weakly quiet quasi-uniform space is uniformly weakly regular.*

Proof. Let $(X,\mathcal{U})$ be a weakly quiet quasi-uniform space and let $U\in\mathcal{U}$. Pick a $V\in\mathcal{U}$ which is weakly quiet to U. We show that $cl_{\tau(\mathcal{U})}V^{-1}\cap V(x)\subseteq U(x)$. Indeed, if $y\in cl_{\tau(\mathcal{U})}V^{-1}\cap V(x)$, then there exists a net $(x_a)_{a\in A}$ such that $x_a\in V^{-1}\cap V(x)$ for each $a\in A$ and $x_a\to y$. Then from $(x,x_a)\in V^\star$ for $a\in A$, $(y,y)\in V$ and $(y,x_a)\to 0$, it follows that $(x,y)\in U$ or equivalently $y\in U(x)$. □

Definition 2.4 (see [2, page 412]). A quasi-uniform space $(X,\mathcal{U})$ is *semisymmetric* provided that if A is $\tau(\mathcal{U})$-closed, B $\tau(\mathcal{U})$-open, $A \subset B$, and $U^{-1}(A) \subset B$ for some $U \in \mathcal{U}$, then there is a $V \in \mathcal{U}$ with $V(A) \subset B$.

Definition 2.5. Let $(X,\mathcal{U})$ be a quasi-uniform space and let $((y_\beta)_{\beta\in B},(x_a)_{a\in A})$ be an ordered pair of nets on X. We define $((y_\beta)_{\beta\in B},(x_a)_{a\in A})$ to be a *Cauchy pair of nets* if for each $U \in \mathcal{U}$, there exist some $a_U \in A$ and $\beta_U \in B$ such that $(y_\beta,x_a) \in U$ for $a \geq a_U$, $\beta \geq \beta_U$.

Definition 2.6 (see [10, Definition 2.3]). A quasiuniform space $(X,\mathcal{U})$ is **-pair Cauchy bounded* provided that for each Cauchy pair of nets $((y_\beta)_{\beta\in B},(x_a)_{a\in A})$ on X, there is a $\mathcal{U}^\star$-Cauchy net $(z_\gamma)_{\gamma\in\Gamma}$ on X, subnet of $(y_\beta)_{\beta\in B}$, such that $(z_\gamma,x_a) \to 0$.

We give a more general definition the previous one.

Definition 2.7. A quasi-uniform space $(X,\mathcal{U})$ is **-weakly pair Cauchy bounded* provided that for each Cauchy pair of nets $((y_\beta)_{\beta\in B},(x_a)_{a\in A})$ on X, there is a $\mathcal{U}^\star$-Cauchy net $(z_\gamma)_{\gamma\in\Gamma}$ on X, $cl_{\tau(\mathcal{U})}\mathcal{F}((y_\beta)_{\beta\in B}) \subseteq \mathcal{F}((x_\gamma)_{\gamma\in\Gamma})$, such that $(z_\gamma,x_a) \to 0$.

Proposition 2.8. *Every semisymmetric quasi-uniform space is* **-weakly pair Cauchy bounded.*

Proof. It is an immediate consequence of [10, Proposition 2.4]. □

Proposition 2.9. *Let* $(X,\mathcal{U})$ *be a* **-weakly pair Cauchy bounded, uniformly weakly regular, and half-complete quasi-uniform space. Then* $(X,\mathcal{U})$ *is weakly quiet.*

Proof. Let $U,V,W \in \mathcal{U}$ be such that $cl_{\tau(\mathcal{U})}(W^{-1} \cap W)(x) \subseteq V(x)$ and $V \circ V \circ V \subseteq U$. We prove that W is weakly quiet to U. Suppose that $x',x'' \in X$, $(x_a)_{a\in A}$ and $(y_\beta)_{\beta\in B}$ are nets on X such that $(x',x_a) \in W^\star$ for $a \in A$, $(y_\beta,x'') \in W$ for $\beta \in B$ and $(y_\beta,x_a) \to 0$. Let now $\mathcal{F}((y_\beta)_{\beta\in B})$ and $\mathcal{F}((x_a)_{a\in A})$ be the filters of tails of the nets $(y_\beta)_{\beta\in B}$ and $(x_a)_{a\in A}$, respectively. By [11, Proposition 2.4], $(cl_{\tau(\mathcal{U})}\mathcal{F}((y_\beta)_{\beta\in B}),\mathcal{F}((x_a)_{a\in A}))$ is a Cauchy pair of filters. Since $(X,\mathcal{U})$ is *-weakly pair Cauchy bounded, there exists a $\mathcal{U}^\star$-Cauchy net $(z_\gamma)_{\gamma\in\Gamma}$ with $cl_{\tau(\mathcal{U})}\mathcal{F}((y_\beta)_{\beta\in B}) \subseteq \mathcal{F}((x_\gamma)_{\gamma\in\Gamma})$ such that $(z_\gamma,x_a) \to 0$. Thus, since $(X,\mathcal{U})$ is half-complete, there exists $t \in X$ such that $(z_\gamma)_{\gamma\in\Gamma}$ is $\tau(\mathcal{U})$-convergent to t. It follows that $(x_a)_{a\in A}$ converges to t with respect to $\tau(\mathcal{U})$ as well as t is a $\tau(\mathcal{U})$-cluster point of $(y_\beta)_{\beta\in B}$. Thus from $t \in cl_{\tau(\mathcal{U})}(W^{-1} \cap W)(x') \subseteq V(x')$, $(t,y_\beta) \in V$ and $(y_\beta,x'') \in V$ from some index β_V onwards, we conclude that $(x',x'') \in V \circ V \circ V \subseteq U$. Thus $(X,\mathcal{U})$ is weakly quiet. □

Proposition 2.10. *Every* T_0 *weakly quiet quasi-uniform space is* T_1.

Proof. Let $(X,\mathcal{U})$ be a T_0 weakly quiet quasi-uniform space and let $(x,y) \in \cap\{W \mid W \in \mathcal{U}\}$ for $x,y \in X$. Let $W \in \mathcal{U}$. Pick a $V \in \mathcal{U}$ which is weakly quiet to W. Then from $(y,y) \in V^\star$, $(x,x) \in V$, and $(x,y) \to 0$, we conclude that $(y,x) \in W$. Hence $(x,y) \in \cap\{W \cap W^{-1} \mid W \in \mathcal{U}\}$. Thus $x = y$. □

In what follows, we show that the notion of half-completeness generalizes those of bicompleteness and D-completeness.

The following proposition is obvious.

Proposition 2.11. *Each bicomplete quasi-uniform space is half-complete.*

The next example shows that the converse of the previous proposition is in general not true.

Example 2.12. Let $A = \{(x,y) \mid x,y \in \mathbb{R},\ y > 0\}$ be the open upper half-plane with the Euclidean topology τ. We generate a topology σ on $X = A \cup \mathbb{R}$ by adding to τ all sets of the form $\{x\} \cup (A \cap \Gamma)$ where $x \in \mathbb{R}$, and Γ is an Euclidean neighborhood of x in the plane. Let $\mathcal{U}$ be a quasi-uniformity on X which is compatible with σ. It is easy to observe that the space $(X,\mathcal{U})$ is half-complete and nonbicomplete (every $\tau(\mathcal{U})$-convergent net to a point $x \in \mathbb{R}$ is a $\mathcal{U}^\star$-Cauchy net which does not $\tau(\mathcal{U}^\star)$-converge to x).

Proposition 2.13. *Each weakly quiet half-complete quasi-uniform space is bicomplete.*

Proof. Let $(X,\mathcal{U})$ be a weakly quiet half-complete quasi-uniform space and let $(x_a)_{a\in A}$ be a $\mathcal{U}^\star$-Cauchy net on X which $\tau(\mathcal{U})$-converges to a point $x \in X$. Suppose that $U,V \in \mathcal{U}$ such that V is weakly quiet to U. Then from $(x_a,x_{a'}) \in V^\star$, $(x,x) \in V$, and $(x,x_{a'}) \to 0$, we conclude that $(x_a,x) \in U$. Hence $(x_a)_{a\in A}$ $\tau(\mathcal{U}^\star)$-converges to x. □

Proposition 2.14. *Each D-complete quasi-uniform space is half-complete.*

Proof. It is an immediate consequence of the basic theorem in [2, page 412]. □

The example given below shows that the converse of the previous proposition is in general not true.

Example 2.15 (see [2, page 412]). Let $X = \mathbb{R} \setminus \{0\}$ and let d be the quasimetric on X defined by

$$d(x,y) = \begin{cases} \min\{y-x,1\} & \text{if } x < 0 < y, \\ 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases} \tag{2.1}$$

Denote by $\mathcal{U}_d$ the quasi-uniformity on X induced by d. Then $(X,\mathcal{U}_d)$ is half-complete, quiet, but not D-complete. Indeed, $\mathcal{U}_d$ is discrete, therefore is half-complete. On the other hand, if $\mathcal{F}$ is a filter generated by $\{(-\epsilon,0) \mid \epsilon > 0\}$ and $\mathcal{G}$ is a filter generated by $\{(0,\epsilon) \mid \epsilon > 0\}$, then $(\mathcal{F},\mathcal{G})$ is Cauchy, but $\mathcal{G}$ is not $\tau(\mathcal{U})$-convergent.

Proposition 2.16. *Each *-weakly pair Cauchy bounded, half-complete quasi-uniform space is D-complete.*

Proof. Let $(X,\mathcal{U})$ be a *-weakly pair Cauchy bounded, half-complete quasi-uniform space and let $(x_a)_{a\in A}$ be a D-Cauchy net on X. Then there is a net $(y_\beta)_{\beta\in B}$ on X such that $(y_\beta,x_a) \to 0$. Since $(X,\mathcal{U})$ is *-weakly pair Cauchy bounded, there is a $\mathcal{U}^\star$-Cauchy net $(z_\gamma)_{\gamma\in\Gamma}$ on X such that $(z_\gamma,x_a) \to 0$. Since the space is half-complete, $(z_\gamma)_{\gamma\in\Gamma}$ $\tau(\mathcal{U})$-converges to a point $x \in X$. Hence $(x_a)_{a\in A}$ $\tau(\mathcal{U})$-converges to x and thus $(X,\mathcal{U})$ is D-complete. □

Deák in [2] proves that the notions of bicompleteness, half-completeness, and D-completeness coincide in semisymmetric uniformly regular spaces. The author in [10] extends this result to a class of uniformly weakly regular spaces (this class includes interesting examples of nonregular spaces).

Proposition 2.17 (see [10, Theorem 2.5]). *The notions of half-completeness, pair completeness, and D-completeness coincide in $\star$-pair Cauchy bounded uniformly weakly regular spaces.*

2.1. Bicompletion and standard D-completion. We proceed characterizing those quasi-uniform spaces for which the bicompletion constructed by Fletcher and Lindgren in [4] coincides (up to quasi-uniformism) with the standard D-completion constructed by Doitchinov in [1].

Recall that a $\mathcal{U}^\star$-Cauchy filter on a quasi-uniform space $(X,\mathcal{U})$ is *minimal* provided that it contains no $\mathcal{U}^\star$-Cauchy filter other than itself.

Lemma 2.18 (see [5, page 63]). *Let $\mathcal{F}$ be a $\mathcal{U}^\star$-Cauchy filter on a quasi-uniform space $(X,\mathcal{U})$. Then there is exactly one minimal $\mathcal{U}^\star$-Cauchy filter coarser than $\mathcal{F}$. Furthermore, if $\mathcal{F}_0$ is any base for $\mathcal{F}$, then $\{U(F_0) \mid F_0 \in \mathcal{F}_0$ and U is a symmetric member of $\mathcal{U}^\star\}$ is a base for the minimal $\mathcal{U}^\star$-Cauchy filter $\widetilde{\mathcal{F}}$ coarser than $\mathcal{F}$.*

Lemma 2.19 (see [5, page 65]). *Let $(X,\mathcal{U})$ be a T_0 quasi-uniform space and let $\widetilde{X} = \{\mathcal{F} \mid \mathcal{F}$ is minimal $\mathcal{U}^\star$-Cauchy filter on $X\}$. For each $U \in \mathcal{U}$, let $\widetilde{U} = \{(\mathcal{F},\mathcal{G}) \in \widetilde{X} \times \widetilde{X}$: there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U\}$, and let $\widetilde{\mathcal{U}}$ be the filter generated by $\{\widetilde{U} \mid U \in \mathcal{U}\}$. Then the following statements hold.*

(a) *$(\widetilde{X},\widetilde{\mathcal{U}})$ is bicomplete T_0 quasi-uniform space and $(X,\mathcal{U})$ is quasi-uniformly embedded as a $\tau(\widetilde{\mathcal{U}^\star})$-dense subspace of $(\widetilde{X},\widetilde{\mathcal{U}})$ by the mapping $i : X \to \widetilde{X}$ such that, for each $x \in X$, $i(x)$ is the $\tau(\mathcal{U}^\star)$-neighborhood filter at x.*

(b) *Any T_0 bicompletion of $(X,\mathcal{U})$ is quasi-isomorphic to $(\widetilde{X},\widetilde{\mathcal{U}})$.*

We call $(\widetilde{X},\widetilde{\mathcal{U}})$ the bicompletion of $(X,\mathcal{U})$.

We now give some details for the D-completion.

In a quasi-uniform space $(X,\mathcal{U})$, two D-Cauchy nets are called *equivalent* if every conet of the first is a conet of the second and vice versa. We symbolize by $\overline{X}_D$ the collection of all equivalence classes of Cauchy nets in the given quasi-uniform space $(X,\mathcal{U})$. To every $\xi \in \overline{X}_D$, we correspond a pair $(\mathcal{A}_\xi,\mathcal{B}_\xi)$, where $\mathcal{A}_\xi$ contains all the D-Cauchy nets of ξ and $\mathcal{B}_\xi$ contains all the conets of all the nets which are elements of $\mathcal{A}_\xi$. We will often say the elements of $\mathcal{A}_\xi$ and $\mathcal{B}_\xi$ *nets and conets*, respectively, *of* ξ.

Given $U \in \mathcal{U}$ we define $\overline{U}_D$ as the set of all pairs (ξ',ξ''), where $\xi',\xi'' \in \overline{X}_D$ having the property: if $(y'_\beta)_{\beta\in B}$ is a conet of ξ' and $(x''_a)_{a\in A}$ is a net of ξ'', there exist some a_0 and β_0 such that $(y''_\beta,x'_a) \in U$ for $a \geq a_0$ and $\beta \geq \beta_0$. Then the family $\{\overline{U}_D \mid U \in \mathcal{U}_0\}$ is a base for a quasi-uniformity $\overline{\mathcal{U}}_D$ on $\overline{X}_D$.

In the D-completion, we define a map $\phi : (X,\mathcal{U}) \to (\overline{X}_D,\overline{\mathcal{U}}_D)$, where $\phi(x)$ is the collection of all nets in $(X,\mathcal{U})$ which converge to x.

Theorem 2.20 (see [1, page 211]). *The space* $(\overline{X}_D, \overline{\mathcal{U}}_D)$ *is quiet and D-complete.*

Proposition 2.21. *Let* $(X, \mathcal{U})$ *be a quasi-uniform space and let* $(x_a)_{a\in A}$ *be a* $\mathcal{U}^\star$*-Cauchy net of* X.

(1) *The filter* $\mathcal{F}((x_a)_{a\in A})$ *of tails of the net* $(x_a)_{a\in A}$ *is a* $\mathcal{U}^\star$*-Cauchy filter.*

(2) *The family* $\mathcal{T}((x_a)_{a\in A}) = \{U(K) \mid K \in T_a$ *and* U *is a symmetric member of* $\mathcal{U}^\star\}$ *is a base for the minimal* $\mathcal{U}^\star$*-Cauchy filter coarser than* $\mathcal{F}((x_a)_{a\in A})$.

Proof. (1) Let $\{T_a \mid a \in A\}$ be the family of tails of the net $(x_a)_{a\in A}$. Since $(x_a)_{a\in A}$ is a $\mathcal{U}^\star$-Cauchy net, for each $U \in \mathcal{U}$, there exists $a_U \in A$ such that $(x_a, x_{a'}) \in U$ for all $a, a' \in A$ with $a, a' \geq a_U$. Hence $T_{a_U} \times T_{a_U} \subseteq U$ which implies that $\mathcal{F}((x_a)_{a\in A})$ is a $\mathcal{U}^\star$-Cauchy filter.

(2) is an immediate consequence of [5, Proposition 3.30]. □

Proposition 2.22. *If* $(X, \mathcal{U})$ *is a quiet* **-weakly pair Cauchy bounded quasi-uniform space, then for the D-completion* $(\overline{X}_D, \overline{\mathcal{U}}_D)$ *and the bicompletion* $(\widetilde{X}, \widetilde{\mathcal{U}})$ *of* $(X, \mathcal{U})$*, respectively, the relation*

$$f : (\overline{X}_D, \overline{\mathcal{U}}_D) \longrightarrow (\widetilde{X}, \widetilde{\mathcal{U}}) \quad \text{defined by } f(\xi) = \mathcal{T}((z_\gamma)_{\gamma\in\Gamma}), \tag{2.2}$$

where $(z_\gamma)_{\gamma\in\Gamma}$ *is a* $\mathcal{U}^\star$*-Cauchy net of* $\mathcal{B}_\xi$*, is a quasi-uniformly continuous mapping.*

Proof. The relation f *is a mapping.* Let $\xi = (\mathcal{A}_\xi, \mathcal{B}_\xi)$ be a point of the domain of f and let $(x_a)_{a\in A} \in \mathcal{A}_\xi$. Then, for $(y_\beta)_{\beta\in B} \in \mathcal{B}_\xi$, we have that $(y_\beta, x_a) \to 0$. Since $(X, \mathcal{U})$ is *-weakly pair Cauchy bounded, there exists a $\mathcal{U}^\star$-Cauchy net $(z_\gamma)_{\gamma\in\Gamma}$ on X with $(cl_{\tau(\mathcal{U})}\mathcal{F}((y_\beta)_{\beta\in B}) \subseteq \mathcal{F}((x_\gamma)_{\gamma\in\Gamma}))$ such that $(z_\gamma, x_a) \to 0$. Hence $(z_\gamma)_{\gamma\in\Gamma} \in \mathcal{B}_\xi$, and thus f is well defined. We show that $f(\xi)$ does not depend on the special choice of the $\mathcal{U}^\star$-Cauchy nets of $\mathcal{B}_\xi$ and hence $f(\xi)$ is uniquely defined. Indeed, let $(x_\delta)_{\delta\in\Delta}$ be an arbitrary $\mathcal{U}^\star$-Cauchy net of $\mathcal{B}_\xi$ and let $U, V \in \mathcal{U}$ such that V is quiet to U. For some a_0, γ_0, δ_0, we have $(z_\gamma, x_a) \in V$ whenever $\gamma \geq \gamma_0$, $a \geq a_0$, and $(z_{\delta'}, z_\delta) \in V$ whenever $\delta, \delta' \geq \delta_0$. On the other hand, $(z_{\delta'}, x_a) \to 0$. It follows that $(z_\gamma, z_\delta) \in U$ for $\gamma \geq \gamma_0$, $\delta \geq \delta_0$. Similarly, there are $\delta_0' \in \Delta$, $\gamma_0' \in \Gamma$ such that $(z_\delta, z_\gamma) \in U$ for $\delta \geq \delta_0'$, $\gamma \geq \gamma_0'$. Hence $(z_\gamma, z_\delta) \to 0$ in $\tau(\mathcal{U}^\star)$-topology. It remains to prove that $\mathcal{T}((z_\gamma)_{\gamma\in\Gamma}) = \mathcal{T}((z_\delta)_{\delta\in\Delta})$. Indeed, let U, V be members of $\mathcal{U}^\star$ such that $V \circ V \subseteq U$. There are $\gamma_V \in \Gamma$ and $\delta_V \in \Delta$ such that $(x_\gamma, x_\delta) \in V$ and $(x_\delta, x_\gamma) \in V$ for each $\gamma \geq \gamma_V$ and $\beta \geq \beta_V$. Hence $T_{\gamma_V} \subseteq V(T_{\delta_V})$ and $T_{\delta_V} \subseteq V(T_{\gamma_V})$, and finally $V(T_{\gamma_V}) \subseteq U(T_{\delta_V})$ and $V(T_{\delta_V}) \subseteq U(T_{\beta_V})$. Thus $\mathcal{T}((z_\gamma)_{\gamma\in\Gamma}) = \mathcal{T}((z_\delta)_{\delta\in\Delta})$.

f is quasi-uniformly continuous. Let $U, K, V, W \in \mathcal{U}$ be such that $W \circ W \subseteq V$, V quiet to K and $K \circ K \circ K \subseteq U$. Suppose that $\xi, \xi' \in \overline{X}_D$ with $(\xi, \xi') \in \overline{W}_D$. Then if $(y_\beta)_{\beta\in B}$ is a conet of ξ and $(x_a)_{a\in A}$ is a net of ξ', there are $\beta_W \in B$, $a_W \in A$ such that $(y_\beta, x_a) \in W$ whenever $\beta \geq \beta_W$ and $a \geq a_W$. Hence $cl_{\tau_\mathcal{U}} T_{\beta_W} \times T_{a_W} \subseteq W \circ W \subseteq V$. Since $(X, \mathcal{U})$ is *-weakly pair Cauchy bounded, there is a $\mathcal{U}^\star$-Cauchy net $(z_\gamma)_{\gamma\in\Gamma}$ on X with $T_\gamma \subseteq cl_{\tau_\mathcal{U}} T_{\beta_W}$, such that $(z_\gamma, x_a) \in V$ whenever $\gamma \geq \gamma_V$ and $a \geq a_W = a_V$. Let $(z_\delta)_{\delta\in\Delta}$ be a $\mathcal{U}^\star$-Cauchy conet of ξ'. Then there is $\delta_V \in \Delta$ such that $(z_{\delta'}, z_\delta) \in V$ for $\delta', \delta \geq \delta_V$. Since V is quiet to K and $(z_{\delta'}, x_a) \to 0$, we conclude that $(z_\gamma, z_\delta) \in K$ whenever $\gamma \geq \gamma_V$ and $\delta \geq \delta_V$. Hence $K^\star(T_{\gamma_V}) \times K^\star(T_{\delta_V}) \subseteq K \circ K \circ K \subseteq U$ which implies that $(\mathcal{T}((z_\gamma)_{\gamma\in\Gamma}), \mathcal{T}((z_\delta)_{\delta\in\Delta})) \in \widetilde{U}$. □

We recall some definitions from [12].

Definition 2.23 (see [12, page 65]). (i) If $(x_a)_{a\in A}$ is a net in a set X, and $S\subseteq X$, say that $x_a\in S$ is *eventually* in S if there exists $a_0\in A$ such that $x_a\in S$ for all $a\geq a_0$. (ii) Let $\mathcal{F}$ be a filter in X and let $\Lambda=\{(x,F)\mid F\in\mathcal{F},\ x\in X\}$ be a set directed by the order

$$(x,F)\leq(x',F')\quad \text{iff } F'\subseteq F. \tag{2.3}$$

The net $g:\Lambda\to X$, with $g_{(x,F)}=x$, is called the *associated* net of the filter $\mathcal{F}$.

In this case the filter $\mathcal{F}$ is precisely the family of all sets F such that the net $(g_{(x,F)})_{(x,F)\in\Lambda}$ is eventually in F.

Definition 2.24. Let $\mathcal{F}$ be a filter on a quasi-uniform space $(X,\mathcal{U})$ and let $(g_{(x,F)})_{(x,F)\in\Lambda}$ be an associated net of $\mathcal{F}$. Consider all the nets $\mathcal{B}_{\mathcal{F}}=\{(y^j_\beta)_{\beta\in B_j}\mid j\in J\}$ such that $(cl_{\tau(\mathcal{U})}T_{g_{(x,F)}}, T_{y^j_\beta})$ constitutes a base for a pair filter on X and next we pick up all the nets $\mathcal{A}_{\mathcal{F}}=\{(x^i_a)_{a\in A_i}\mid i\in I\}$ such that $(T_{y^j_\beta},T_{x^i_a})$ is a base for a pair filter on X.

The ordered couple $(\mathcal{A}_{\mathcal{F}},\mathcal{B}_{\mathcal{F}})$ has the following properties.

(a) For every $U\in\mathcal{U}$ and every $(x^i_a)_{a\in A_i}\in\mathcal{A}_{\mathcal{F}}$, $(y^j_\beta)_{\beta\in B_j}\in\mathcal{B}_{\mathcal{F}}$, there are indexes a^i_U, β^j_U such that $(y^j_\beta,x^i_a)\in U$ whenever $a\geq a^i_U$ and $\beta\geq\beta^j_U$.

(b) $\mathcal{B}_{\mathcal{F}}$ contains all the conets of all the nets which are elements of $\mathcal{A}_{\mathcal{F}}$ and conversely $\mathcal{A}_{\mathcal{F}}$ contains all the nets whose conets are all the elements of $\mathcal{A}_{\mathcal{F}}$.

The ordered pair $(\mathcal{A}_{\mathcal{F}},\mathcal{B}_{\mathcal{F}})=\xi_{\mathcal{F}}$ is called the *associated cut of* $\mathcal{F}$ (see [13, Definition 1.1(3)]).

It is clear that in quiet spaces, $\mathcal{A}_{\mathcal{F}}$ constitutes an equivalence class of D-Cauchy nets on X.

Lemma 2.25 (see [13, Proposition 4.8]). *Let $(X,\mathcal{U})$ be a quasi-uniform space, let $\mathcal{F}$ be a minimal $\mathcal{U}^\star$-Cauchy filter on it and let $(g_{(x,F)})_{(x,F)\in\Lambda}$, $(g_{(y,F)})_{(y,F)\in K}$ be two associated nets of $\mathcal{F}$.*

(i) The nets $(g_{(x,F)})_{(x,F)\in\Lambda}$ and $(g_{(y,F)})_{(y,F)\in K}$ are $\mathcal{U}^\star$-Cauchy nets.

(ii) $(g_{(x,F)},g_{(y,F)})\to 0$ in $\tau(\mathcal{U}^\star)$.

Lemma 2.26 (see [13, Proposition 4.7]). *If two minimal $\mathcal{U}^\star$-Cauchy filters in a quasi-uniform space $(X,\mathcal{U})$ have a common associated net, then they coincide.*

Proposition 2.27. *Let $(X,\mathcal{U})$ be a quiet quasi-uniform space and let $\mathcal{F}$ be a minimal $\mathcal{U}^\star$-Cauchy filter on X. Then for the bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ and the standard D-completion $(\overline{X}_D,\overline{\mathcal{U}}_D)$, the relation*

$$g:(\widetilde{X},\widetilde{\mathcal{U}})\longrightarrow(\overline{X}_D,\overline{\mathcal{U}}_D)\quad \textit{defined by } g(\mathcal{F})=\xi_{\mathcal{F}} \tag{2.4}$$

is a quasi-uniformly continuous mapping.

Proof. The relation g is a mapping. Let $\mathcal{F}\in\widetilde{X}$ be a point of the domain of g. Suppose that g carries $\mathcal{F}$ into the points ξ' and ξ'' of $\overline{X}_D$. Then there exist two associated nets of $\mathcal{F}$, let $(g_{(x,F)})_{(x,F)\in\Lambda}$ and $(g_{(y,F)})_{(y,F)\in\Lambda}$ which generate ξ' and ξ'', respectively. It follows from Lemma 2.25 that $(g_{(x,F)})_{(x,F)\in\Lambda}$ and $(g_{(y,F)})_{(y,F)\in\Lambda}$ are *-Cauchy nets such that $(g_{(x,F)}, g_{(y,F)}) \to 0$ in $\tau(\mathcal{U}^\star)$. It immediately follows from Definition 2.24 that $\mathcal{A}_{\xi'} = \mathcal{A}_{\xi''}$ and thus $\xi' = \xi''$. Hence the relation g is a mapping.

The mapping g is quasi-uniformly continuous. Let $U,V,W,M \in \mathcal{U}$ such that M be quiet to W, $W \circ W \subseteq V$ and $V \circ V \subseteq U$. Let $\mathcal{F}$, $\mathcal{G}$ be minimal $\mathcal{U}^\star$-filters of X such that $(\mathcal{F},\mathcal{G}) \in \widetilde{M}$. Then there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq M$. Thus if $(g_{(x,F)})_{(x,F)\in\Lambda}$, $(g_{(y,G)})_{(y,G)\in K}$ are associated nets of $\mathcal{F}$ and $\mathcal{G}$, respectively, there exist $(x_M,F_M) \in \Lambda$ and $(y_M,G_M) \in K$ such that $(g_{(x,F)},g_{(y,G)}) \in M$ whenever $(x,F) \geq (x_M,F_M)$ and $(y,G) \geq (y_M,G_M)$. Let now $(y_\beta)_{\beta\in B} \in \mathcal{B}_{\xi_{\mathcal{F}}}$ and $(x_a)_{a\in A} \in \mathcal{A}_{\xi_{\mathcal{F}}}$. Then there are $a_M \in A$, $\beta_M \in B$, and $(x_M,F_M) \in \Lambda$ such that $(y_\beta,x_a) \in M$ whenever $\beta \geq \beta_M$, $a \geq a_M$ and $(g_{(x',F')}, g_{(x'',F'')}) \in M$ whenever $(x',F'),(x'',F'') \geq (x_M,F_M)$. Since $(X,\mathcal{U})$ is quiet, from $(g_{(x',F')}, x_a) \to 0$, we conclude that $(y_\beta,g_{(x'',F'')}) \in W$ whenever $\beta \geq \beta_M$ and $(x'',F'') \geq (x_M,F_M)$. Hence from $(y_\beta,g_{(x_M,F_M)}) \in W$ and $(g_{(x_M,F_M)},g_{(y,G)}) \in M$, we conclude that $(y_\beta,g_{(y,G)}) \in M \circ W \subseteq V$ whenever $\beta \geq \beta_M$ and $(y,G) \geq (y_M,G_M)$. Since $(x_a)_{a\in A} \in \mathcal{A}_{\xi_{\mathcal{G}}}$, there are $(y_V,G_V) \in K$ and $a_V \in A$ such that $(g_{(y,G)},x_a) \in V$ for $(y,G) \geq (y_V,G_V)$ and $a \geq a_V$. If $(y^\star,G^\star) = \sup\{(y_M,G_M),(y_V,G_V)\}$, then from $(y_\beta,g_{(y^\star,G^\star)}) \in V$ and $(g_{(y^\star,G^\star)},x_a) \in V$, we conclude that $(y_\beta,x_a) \in U$ for $\beta \geq \beta_M$ and $a \geq a_V$. Hence $(\xi_{\mathcal{F}},\xi_{\mathcal{G}}) \in \overline{U}$. □

Theorem 2.28. *Let $(X,\mathcal{U})$ be a quiet *-pair Cauchy bounded quasi-uniform space. Then the standard D-completion $(\overline{X}_D,\overline{\mathcal{U}}_D)$ and the bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ of $(X,\mathcal{U})$ coincide (up to quasi-uniformism).*

Proof. Let f and g be the mappings defined in Propositions 2.22 and 2.27, respectively. Suppose that $\xi \in \overline{X}_D$. Then $(g \circ f)(\xi) = g(f(\xi)) = g(\mathcal{T}((x_\gamma)_{\gamma\in\Gamma}))$ for some $\mathcal{U}^\star$-Cauchy net $(x_\gamma)_{\gamma\in\Gamma} \in \mathcal{B}_\xi$. On the other hand, $g(\mathcal{T}((x_\gamma)_{\gamma\in\Gamma})) = \xi_{\mathcal{T}((x_\gamma)_{\gamma\in\Gamma})}$, where $\xi_{\mathcal{T}((x_\gamma)_{\gamma\in\Gamma})} \in \overline{X}_D$ is generated by an associated net, let $(x_k)_{k\in K}$, of the minimal $\mathcal{U}^\star$-Cauchy filter $\mathcal{T}((x_\gamma)_{\gamma\in\Gamma})$. Since $\mathcal{T}((x_\gamma)_{\gamma\in\Gamma}))$ has as a base the family of all sets T_γ such that the net $(x_\gamma)_{\gamma\in\Gamma}$ is eventually in T_γ, we conclude that $(x_\gamma)_{\gamma\in\Gamma}$ is an associated net of $\mathcal{T}((x_\gamma)_{\gamma\in\Gamma})$. Hence, Lemma 2.25 implies that $(x_\gamma,x_k) \to 0$ in $\tau(\mathcal{U}^\star)$. Consequently by Definition 2.24, we deduce that $\mathcal{A}_\xi = \mathcal{A}_{\xi_{\mathcal{T}((x_\gamma)_{\gamma\in\Gamma})}}$ and thus $\xi = \xi_{\mathcal{T}((x_\gamma)_{\gamma\in\Gamma})}$. Hence $(g \circ f)(\xi) = \xi$, that is, $g \circ f = I_{\overline{X}_D}$, where $I_{\overline{X}_D}$ is the identity map of $\overline{X}_D$.

Let now $\mathcal{F} \in (\widetilde{X},\widetilde{\mathcal{U}})$. Then $(f \circ g)(\mathcal{F}) = f(g(\mathcal{F})) = f(\xi_{\mathcal{F}})$, where $\xi_{\mathcal{F}} \in \overline{X}_D$ is generated by an associated net of $\mathcal{F}$, let $(x_a)_{a\in A}$. By definition of f, we have that $f(\xi_{\mathcal{F}}) = \mathcal{T}((x_\beta)_{\beta\in B})$ for some $\mathcal{U}^\star$-Cauchy net $(x_\beta)_{\beta\in B} \in \mathcal{B}_{\xi_{\mathcal{F}}}$. Since $(x_a)_{a\in A} \in \mathcal{B}_{\xi_{\mathcal{F}}}$, as in the proof of Proposition 2.21, we conclude that $(x_a,x_\beta) \to 0$ in $\tau(\mathcal{U}^\star)$. Hence $(x_a)_{a\in A}$ is eventually in every member of $\mathcal{T}((x_\beta)_{\beta\in B})$ and consequently it is one of its associated nets. Thus $(x_a)_{a\in A}$ is a common associated net both of $\mathcal{F}$ and $\mathcal{T}((x_\beta)_{\beta\in B})$. By Lemma 2.26, we conclude that $\mathcal{F} = \mathcal{T}((x_\beta)_{\beta\in B}) = (f \circ g)(\mathcal{F})$. Hence $g \circ f = I_{\widetilde{X}}$ where $I_{\widetilde{X}}$ is the identity map of $\widetilde{X}$. Therefore, f is a quasi-uniformism. □

It is well known that the bicompletion and the standard D-completion of a uniform space coincide (up to uniformism) with the usual uniform completion. We give an example of a quiet *-pair Cauchy bounded quasi-uniform space, but nonuniform space, in which the standard D-completion and the bicompletion of $(X,\mathcal{U})$ coincide (up to quasi-uniformism).

Example 2.29. We consider the subset $X = \mathbb{R} \setminus \{0\}$ of reals, and we define a quasi-uniformity $\mathcal{U}$ by the quasimetric

$$d(x,y) = \begin{cases} |x-y| & \text{if } x,y<0, \\ \min\{y-x,1\} & \text{if } x<0<y, \\ 0 & \text{if } x,y>0,\ x=y, \\ 1 & \text{otherwise.} \end{cases} \tag{2.5}$$

It is clear that $(X,\mathcal{U})$ is a quiet *-pair Cauchy bounded quasi-uniform space. Let $\mathcal{F}$ be the filter generated by $\{(-\epsilon,0) \mid \epsilon > 0\}$. Then $\mathcal{F} = \widetilde{\mathcal{F}}$. Hence $\widetilde{X} = \{i(x) \mid x \in X\} \cup \{\widetilde{\mathcal{F}}\}$ (if $x > 0$, then $\{x\}$ is open in the topology induced by the quasi-uniformity $\mathcal{U}$ on X). On the other hand, if $\xi = (\mathcal{A}_\xi,\mathcal{B}_\xi)$, where $\mathcal{A}_\xi$ contains all the nets which has, as a conet, the net $\{(-\epsilon,0) \mid \epsilon > 0\}$ and $\mathcal{B}_\xi$ contains all the conets of all the nets which are elements of $\mathcal{A}_\xi$, then $\overline{X}_D = \{\phi(x) \mid x \in X\} \cup \{\xi\}$. The rest is obvious.

2.2. Bicompletion and *-half-completion. Romaguera and Sánchez in [3] have introduced the notion of T_1 *-half-completion as follows.

Definition 2.30. Let $(X,\mathcal{U})$ be a quasi-uniform space. A T_1 quasi-uniform space $(Y,\mathcal{V})$ is called a T_1 *-*half-completion* of $(X,\mathcal{U})$ if $(Y,\mathcal{V})$ is half-complete and $(X,\mathcal{U})$ is quasi-isomorphic to a $\tau(\mathcal{V}^\star)$-dense subspace of $(Y,\mathcal{V})$. Say that $(X,\mathcal{U})$ is T_1 *-*half-completable* if it has a T_1 *-half-completion.

Let $(X,\mathcal{U})$ be a T_0 quasi-uniform space and $(\widetilde{X},\widetilde{\mathcal{U}})$ its bicompletion. We will denote by $G(X)$ the set of closed points of $(X,\mathcal{U})$.

The following result is proved in [3].

Lemma 2.31. *Let $(X,\mathcal{U})$ be a T_1 *-half-completable quasi-uniform space. Then any T_1 *-half-completion of $(X,\mathcal{U})$ is quasi-isomorphic to $(G(X),\widetilde{\mathcal{U}}/G(X)\times G(X))$.*

Proposition 2.32. *If $(X,\mathcal{U})$ is a T_0 quasi-uniform space and Y is a $\tau(\mathcal{U}^\star)$-dense subset of X such that $(Y,\mathcal{U}/Y\times Y)$ is weakly quiet, then $(X,\mathcal{U})$ is weakly quiet.*

Proof. Let $U,V,W,S \in \mathcal{U}$ be such that $V\circ V\circ V \subseteq U$, $W/Y\times Y$ weakly quiet to $V/Y\times Y$ and $S\circ S\circ S \subseteq W$. We prove that S is weakly quiet to U. Suppose that $x,y \in X$, $(x_a)_{a\in A}$ and $(y_\beta)_{\beta\in B}$ are nets on X such that $(x,x_a) \in S^\star$ for $a \in A$, $(y_\beta,y) \in S$ for $\beta \in B$, and $(y_\beta,x_a) \to 0$.

For each $K \in \mathcal{U}$ and each $a \in A$, since Y is $\tau(\mathcal{U}^\star)$-dense in X, there exists a $t_{(a,K)} \in K^*(x_a)\cap Y$. Consider the set $A_\mathcal{U} = \{(a,K) \mid a \in A,\ K \in \mathcal{U}\}$ ordered by $(a',\Lambda) \geq (a,K)$ if and only if $a' \geq a$ and $\Lambda \subseteq K$. Then $(t_{(a,K)})_{(a,K)\in A_\mathcal{U}}$ is a net in $(Y,\mathcal{U}/Y\times Y)$. Similarly, we construct a net $(s_{(\beta,K)})_{(\beta,K)\in B_\mathcal{U}}$ in $(Y,\mathcal{U}/Y\times Y)$, where $s_{(\beta,K)} \in K^*(y_\beta)\cap Y$ and $B_\mathcal{U} = \{(\beta,K) \mid \beta \in B,\ K \in \mathcal{U}\}$. It is clear that for any $\Lambda \in \mathcal{U}$ we have $(s_{(\beta,M)},y_\beta) \in \Lambda^\star$ and

$(x_a, t_{(a,M)}) \in \Lambda^\star$ whenever $M \subset \Lambda$. On the other hand, also for any $\Lambda \in \mathcal{U}$, there are $a_\Lambda \in A$ and $\beta_\Lambda \in B$ such that $(y_\beta, x_a) \in \Lambda$ whenever $a \geq a_\Lambda$, $\beta \geq \beta_\Lambda$. Let now $K \in \mathcal{U}$ and let $\Lambda \in \mathcal{U}$ satisfy $\Lambda \circ \Lambda \circ \Lambda \subseteq K$. Then $(s_{(\beta,M)}, t_{(a,M)}) \in \Lambda \circ \Lambda \circ \Lambda \subseteq K$ for $(a,M) \geq (a_\Lambda, \Lambda)$ and $(\beta, M) \geq (\beta_\Lambda, \Lambda)$. It follows that $(s_{(\beta,M)}, t_{(a,M)}) \to 0$.

Suppose that $\mu \in S^\star(x) \cap Y$ and $\nu \in S^\star(y) \cap Y$. Then, from $(\mu, x) \in S^\star$, $(x, x_a) \in S^\star$ for $a \geq a_{S^\star}$ and $(x_a, t_{(a,M)}) \in S^\star$ for $a \geq a_{S^\star}$, $M \subset S$, we conclude that $(\mu, t_{(a,M)}) \in S^\star \circ S^\star \circ S^\star \subseteq W^\star/Y \times Y$ whenever $a \geq a_{S^\star}$ and $M \subset S$. Similarly, we have $(s_{(\beta,M)}, \nu) \in W^\star/Y \times Y$ whenever $\beta \geq \beta_{S^\star}$ and $M \subset S$. Since the space $(Y, \mathcal{U}/Y \times Y)$ is weakly quiet, from $(\mu, t_{(a,M)}) \in W^\star/Y \times Y$, $(s_{(\beta,M)}, \nu) \in W^\star/Y \times Y$, and $(s_{(\beta,M)}, t_{(a,M)}) \to 0$, we conclude that $(\mu, \nu) \in V/Y \times Y$. From this fact and the assumption that $\mu \in S^\star(x)$ and $\nu \in S^\star(y)$, it follows that $(x, y) \in S^\star \circ V \circ S^\star \subseteq U$. The proof is over. □

Theorem 2.33. *Let $(X, \mathcal{U})$ be a T_0 weakly quiet quasi-uniform space. Then its bicompletion is the T_1 $\star$-half-completion of $(X, \mathcal{U})$.*

Proof. Let $(\widetilde{X}, \widetilde{\mathcal{U}})$ be the T_0 bicompletion of $(X, \mathcal{U})$. Then, $(X, \mathcal{U})$ is quasi-uniformly embedded as a $\tau(\widetilde{\mathcal{U}}^\star)$-dense subspace of $(\widetilde{X}, \widetilde{\mathcal{U}})$ by the mapping $i : X \to \widetilde{X}$ such that, for each $x \in X$, $i(x)$ is the $\tau(\mathcal{U}^\star)$-neighborhood filter at x. Hence the space $(i(X), \widetilde{\mathcal{U}}/i(X) \times i(X))$ is a $\tau(\widetilde{\mathcal{U}}^\star)$-dense subspace of $(\widetilde{X}, \widetilde{\mathcal{U}})$ and weakly quiet. Thus Propositions 2.10 and 2.32 imply that $(\widetilde{X}, \widetilde{\mathcal{U}})$ is T_1. Since $(\widetilde{X}, \widetilde{\mathcal{U}})$, is bicomplete we conclude that it is half-complete too. Then Lemma 2.31 implies that $(\widetilde{X}, \widetilde{\mathcal{U}})$ is the T_1 $\star$-half-completion of $(X, \mathcal{U})$ which is unique up to quasi-uniformism (clearly $G(X) = \widetilde{X}$ whenever $(\widetilde{X}, \widetilde{\mathcal{U}})$ is a T_1 quasi-uniform space). □

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