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Research Article

A Connection between $C^\infty(\mathbb{T}^n)$ and $\mathcal{S}(\mathbb{R}^n)$

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We interpret $C^\infty(\mathbb{T}^n)$ as a quotient space of $\mathcal{S}(\mathbb{R}^n)$.

In measure-theoretic sense, the n-torus $\mathbb{T}^n$ is the cube $[0,1]^n$ with Lebesgue measure. A function f in $C^\infty(\mathbb{R}^n)$ is said to be in $C^\infty(\mathbb{T}^n)$ if $f(x+m)=f(x)$ for all $x\in\mathbb{R}^n$ and $m\in\mathbb{Z}^n$. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of rapidly decreasing functions.

Given $f\in L^1(\mathbb{R}^n)$, we denote its Fourier transform by

$$\widehat{f}(\xi)=\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx,\quad \xi\in\mathbb{R}^n. \tag{1}$$

Given $f\in L^1(\mathbb{T}^n)$, we denote its Fourier coefficients by

$$\widetilde{f}(m)=\int_{\mathbb{R}^n} f(x)e^{-2\pi i m\cdot x}dx,\quad m\in\mathbb{Z}^n. \tag{2}$$

We have $\sup_{m\in\mathbb{Z}^n}|\widetilde{f}(m)|\le \|f\|_{L^1(\mathbb{T}^n)}$.

Lemma 1. *Suppose that f, $\widehat{f}$ are in $L^1(\mathbb{R}^n)$, then it can be assumed that f and $\widehat{f}$ are both continuous since they can be expressed in terms of each other via Fourier inversion. If they satisfy*

$$|f(x)|+|\widehat{f}(x)|\le C(1+|x|)^{-n-\delta} \tag{3}$$

for some $C,\delta > 0$, *then*

$$\sum_{m\in\mathbb{Z}^n} \widehat{f}(m)e^{2\pi i m\cdot x} = \sum_{m\in\mathbb{Z}^n} f(x+m), \tag{4}$$

for all $x \in \mathbb{R}^n$, *and in particular,*

$$\sum_{m\in\mathbb{Z}^n} \widehat{f}(m) = \sum_{m\in\mathbb{Z}^n} f(m). \tag{5}$$

(See [1, Theorem 3.1.17].)

Lemma 2. *Let* $s \in Z$ *with* $s \geq 0$, *suppose that* f *is in* $C^s(\mathbb{T}^n)$, *then*

$$|\widetilde{f}(m)| \leq c_{n,s}\frac{\max\left(\|f\|_{L^1(\mathbb{T}^n)}, \sup_{|\alpha|=s}|\widetilde{\partial^\alpha f}(m)|\right)}{(1+|m|)^s}, \tag{6}$$

for some constant $c_{n,s}$.

(See [1, Corollary 3.2.10].)

We are in the position to get the following theorem.

Theorem 3. *If* ϕ *is in* $\mathcal{S}(\mathbb{R}^n)$ *and*

$$g(x) = \sum_{m\in\mathbb{Z}^n} \phi(x+m), \tag{7}$$

then $g \in C^\infty(\mathbb{T}^n)$. *Conversely, for every* $g \in C^\infty(\mathbb{T}^n)$, *there exists* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *such that*

$$g(x) = \sum_{m\in\mathbb{Z}^n} \phi(x+m). \tag{8}$$

Proof. The proof of the first part is trivial.

Now assume that $g \in C^\infty(\mathbb{T}^n)$ and set

$$G(x) = \sum_{m\in\mathbb{Z}^n} \widetilde{g}(m)\mathcal{X}_{B(m,\lambda)}(x), \tag{9}$$

where $B(m,\lambda) = \{x \in \mathbb{R}^n : |x-m| < \lambda\}$, $0 < \lambda < 2/5$, and $\mathcal{X}_{B(m,\lambda)}$ denotes the characteristic function of $B(m,\lambda)$.

According to Lemma 2, for all positive integers N, we have

$$|\widetilde{g}(m)| \leq c_{n,N}\frac{\max\left(\|g\|_{L^1(\mathbb{T}^n)}, \sup_{|\alpha|=N}|\widetilde{\partial^\alpha g}(m)|\right)}{(1+|m|)^N} \tag{10}$$

$$\leq c_{n,N}\frac{\max\left(\|g\|_{L^1(\mathbb{T}^n)}, \sup_{|\alpha|=N}\|\partial^\alpha g\|_{L^1(\mathbb{T}^n)}\right)}{(1+|m|)^N}. \tag{11}$$

So, it is easily seen that $G(x) \in L^1(\mathbb{R}^n)$.

Set

$$k(x) = \begin{cases} ce^{1/(|x|^2-1)}, & |x| \le 1, \\ 0, & |x| > 1, \end{cases} \tag{12}$$

where c is a constant such that $\int_{\mathbb{R}^n} k(x)dx = 1$.

For $\varepsilon > 0$, set $k_\varepsilon(x) = \varepsilon^{-n}k(\varepsilon^{-1}x)$, and denote

$$G_1(x) = (G*k_{\lambda/4})(x). \tag{13}$$

Then by the property of convolution, $G_1 \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha G_1 = G*\partial^\alpha k_{\lambda/4}$.

Also, since $\partial^\gamma k_{\lambda/4}(y)$ is continuous and supported in $B(0,\lambda/4)$. So for any multi-index γ and nonnegative integer N,

$$\begin{aligned}
&(1+|x|)^N |\partial^\gamma G_1(x)| \\
&\quad = (1+|x|)^N \left| \int_{\mathbb{R}^n} G(x-y)\partial^\gamma k_{\lambda/4}(y)dy \right| \\
&\quad \le C(1+|x|)^N \sup_{y\in B(0,\lambda/4)} |G(x-y)| \\
&\quad \le C(1+|m|)^N |\tilde{g}(m)|,
\end{aligned} \tag{14}$$

here m is the only point with integer coordinates that is in $B(x,5\lambda/4)$ (if there is one such m, otherwise $(1+|x|)^N|\partial^\gamma G_1(x)|$ is 0). C depends only on γ and N. So by (11), G_1 is in $\mathcal{S}(\mathbb{R}^n)$.

And

$$G_1(m) = \int_{B(0,\lambda/4)} G(m-y)k_{\lambda/4}(y)dy = G(m)\int_{B(0,\lambda/4)} k_{\lambda/4}(y)dy = G(m) = \tilde{g}(m). \tag{15}$$

Suppose that ϕ is the function in $\mathcal{S}(\mathbb{R}^n)$ such that $\hat{\phi} = G_1$. Clearly, ϕ and G_1 satisfy the conditions of Lemma 1, and so we have

$$g(x) = \sum_{m\in\mathbb{Z}^n} \tilde{g}(m)e^{2\pi i m\cdot x} = \sum_{m\in\mathbb{Z}^n} G_1(m)e^{2\pi i m\cdot x} = \sum_{m\in\mathbb{Z}^n} \phi(x+m). \tag{16}$$

□

$C^\infty(\mathbb{T}^n)$ is generally topologized by the family of seminorms

$$\rho_\alpha(f) = \sup_x |\partial^\alpha f(x)|, \tag{17}$$

where α ranges over all multi-indices. In this topology, $\phi_j \to \phi$ means

$$\sup_x |\partial^\alpha \phi_j(x) - \partial^\alpha \phi(x)| \longrightarrow 0 \tag{18}$$

for all multi-indices α. $C^\infty(\mathbb{T}^n)$ is a Fréchet space and it can be regarded as a quotient space of $\mathcal{S}(\mathbb{R}^n)$ up to isomorphism of topological vector spaces.

Theorem 4. *Set*

$$H = \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sum_{m \in \mathbb{Z}^n} \phi(x+m) \equiv 0 \right\}, \tag{19}$$

then H is a closed subspace of $\mathcal{S}(\mathbb{R}^n)$, and there is a linear one-to-one correspondence between the quotient space $\mathcal{S}(\mathbb{R}^n)/H$ and $C^\infty(\mathbb{T}^n)$ which is a homomorphism.

Proof. It is easy to see that H is closed in $\mathcal{S}(\mathbb{R}^n)$.

Define $\Lambda : \mathcal{S}(\mathbb{R}^n)/H \to C^\infty(\mathbb{T}^n)$ by

$$\Lambda(\phi + H) = \sum_{m \in \mathbb{Z}^n} \phi(x+m). \tag{20}$$

It is obvious that Λ is well defined, linear, one-to-one, and onto. It remains to prove that Λ is continuous and open.

If d is an invariant metric on $\mathcal{S}(\mathbb{R}^n)$ compatible with its topology, then

$$\rho(\phi + H, \varphi + H) = \inf\{d(\phi - \varphi, \psi) : \psi \in H\} \tag{21}$$

defines an invariant metric on $\mathcal{S}(\mathbb{R}^n)/H$ which is compatible with the quotient topology.

Suppose $\phi_j + H \to \phi + H$ $(j \to \infty)$ in the quotient topology of $\mathcal{S}(\mathbb{R}^n)/H$, we have

$$\rho(\phi_j + H, \phi + H) = \inf\{d(\phi_j - \phi, \psi) : \psi \in H\} \longrightarrow 0, \quad (j \longrightarrow \infty). \tag{22}$$

For each j, there is $\psi_j \in H$ such that

$$d(\phi_j - \phi, \psi_j) \le 2\inf\{d(\phi_j - \phi, \psi) : \psi \in H\}. \tag{23}$$

So,

$$\lim_{j\to\infty} d(\phi_j - \psi_j, \phi) = \lim_{j\to\infty} d(\phi_j - \phi, \psi_j) = 0. \tag{24}$$

That is, $\phi_j - \psi_j \to \phi$ $(j \to \infty)$ in $\mathcal{S}(\mathbb{R}^n)$. Hence, it is easy to see that

$$\lim_{j\to\infty} \sum_{m \in \mathbb{Z}^n} (\phi_j(x+m) + \psi_j(x+m)) = \lim_{j\to\infty} \sum_{m \in \mathbb{Z}^n} \phi_j(x+m) = \sum_{m \in \mathbb{Z}^n} \phi(x+m) \tag{25}$$

in the topology of $C^\infty(\mathbb{T}^n)$.

That is,

$$\lim_{j\to\infty} \Lambda(\phi_j + H) = \Lambda(\phi + H), \tag{26}$$

so Λ is continuous.

Since both $\mathcal{S}(\mathbb{R}^n)/H$ and $C^\infty(\mathbb{T}^n)$ are F-spaces, Λ is also open, by the open mapping theorem. This completes the proof. □

The elements of the dual space $\mathcal{D}'(\mathbb{T}^n)$ of $C^\infty(\mathbb{T}^n)$ are called distributions on $\mathbb{T}^n$. The above result may shed some light on the relation between $\mathcal{D}'(\mathbb{T}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions on $\mathbb{R}^n$. For example, for every $u \in \mathscr{D}'(\mathbb{T}^n)$, $u \circ \Lambda \circ \pi$ is in $\mathscr{S}'(\mathbb{R}^n)$, where $\pi(\phi) = \phi + H$ is the quotient mapping from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)/H$. Hence, $\mathscr{D}'(\mathbb{T}^n)$ can be imbedded into $\mathscr{S}'(\mathbb{R}^n)$ in a natural way.

References

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