Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 97278, 18 pages
doi:10.1155/2007/97278

Research Article

Modification of the Quasilinearization Method for the Inverse Problem

Lenka Čelechovská-Kozáková

Received 25 January 2006; Revised 30 May 2006; Accepted 7 December 2006

Recommended by Virginia Kiryakova

We propose a new modification of Bellman's quasilinearization method such that at any iteration step, it works with an approximate solution of the original nonlinear system and with new approximation of parameters $\alpha^{(k+1)}$ which are close enough to the previous ones. As an output, this approach provides a construction of a convergent sequence of parameters where the limit is the best approximation of parameters of a given system. We apply this method to a mathematical model describing BSP-kinetics in the human liver.

1. Introduction

For solving the inverse problems, in particular, for identification of systems with known structure, the quasilinearization method (QM) is a standard tool. Designed by Bellman et al. [1], this method was later applied to different kinds of identification problems (cf. [2] or [3] for references). We were interested in application of QM to solve the parameter identification problem for the BSP-kinetics in the human liver [4–7]. One of the possible descriptions of this kinetics can be given by the nonlinear system of ordinary differential equations

$$\begin{aligned} \dot{X}(t) &= -c_1 X(K_1 - Y), \\ \dot{Y}(t) &= c_1 X(K_1 - Y) - c_2 Y(K_2 - Z), \\ \dot{Z}(t) &= c_2 Y(K_2 - Z) - c_3 Z, \end{aligned} \tag{1.1}$$

where $X(t), Y(t), Z(t)$ mean the amount of BSP in the blood, in the membranes of hepatic cells, inside the cells at the time t, respectively, and $\alpha = (c_1, c_2, c_3, K_1, K_2)^\top$ is a vector of unknown positive parameters [6]. Suppose a "single injection" in which the amount I

Table 1.1. The amount of BSP in the blood.

Time (min)	t_i	0	3	5	10	20	30	43
BSP (mg)	$r_i = X(t_i)$	250	221	184	141	98	80	64

Table 1.2. The amount of BSP in the bile.

Time (min)	s_j	0	5	10	15	20	25	30
BSP (mg)	$e_j = V(s_j)$	0	0.2	2.5	6	10.5	15.8	21.7
Time (min)	s_j	35	40	45	50	60	70	80
BSP (mg)	$e_j = V(s_j)$	28	34.8	41.8	49	63.8	78.5	92.7
Time (min)	s_j	90	100	110	120	130	140	150
BSP (mg)	$e_j = V(s_j)$	105.7	117	127.1	136.3	144.5	152.1	159.2

(mg) of BSP is injected into the blood at once. This leads to the initial conditions

$$X(0) = I, \qquad Y(0) = Z(0) = 0. \tag{1.2}$$

In order to uniquely determine the unknown positive parameters $\alpha = (K_1, K_2, c_1, c_2, c_3)^\top$, we have to know at least two different data sets. From practical point of view, we can obtain data describing the decreasing level of BSP in the blood (Table 1.1) and in Table 1.2, they are presenting the measurements of BSP in the bile. These data were obtained through medical experiments by Hrnčíř [6].

The first data set corresponds to the function $X(t)$. The second one corresponds to the function $V(t) = I - X(t) - Y(t) - Z(t)$ describing the level of BSP in the bile.

However, the standard approach like in [2, 3], or recent [8, 9] does not provide the reasonable outputs corresponding to the nature of parameters, especially if we solve an identification problem for nonlinear system of ordinary differential equations. (We can obtain negative values of determinated parameters, see Section 5.) Therefore we propose a modification of the quasilinearization method (MQM). The algorithm of the modified QM consists of the steps displayed below. Let us briefly introduce the MQM (see Section 3 for details).

The classical approach used by Bellman (see [2, 3]) is similar to Algorithm 1.1 with the exception of Step 3 (which requires the computation of the solution of the given differential equation in every step of the algorithm) and with the exception of Steps 6 and 7. In the existing sources, like [2, 3, 8, 9], only the linearized differential equation given in Step 4 is used only. This makes things easier from the viewpoint of computation and works properly especially for linear systems of differential equations. The development of computing devices since the eighties of the last century and the software (like the package Mathematica) allow to do the computations fast even if the given differential equation is solved approximately in every step of determining a better approximation of the values of parameters. The problem is that the solution of the differential equation (1.1) for the certain value of the parameter can be far from the solution of this equation linearized around the fixed solution $\mathbf{x}^{(k)}$ from Step 3. This obstacle is removed by Steps 6 and 7 especially in the case of nonlinear differential equations. In this way, the final value of the

Step 1. Consider a nonlinear autonomous initial problem

$$\dot{x} = f(x,\alpha), \qquad x(0) = c,$$

where $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^N$, and $f : \mathbb{R}^{n+N} \to \mathbb{R}^n$ is a continuous function. This problem is equivalent to the Cauchy problem

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{c},$$

where

$$\mathbf{x} = (x,\alpha)^\top = \left(x_1,\dots,x_n,\alpha_1,\dots,\alpha_N\right)^\top \in \mathbb{R}^{n+N},$$
$$\mathbf{g}(\mathbf{x}) = \left(f(x,\alpha),\underbrace{0,\dots,0}_{N}\right)^\top,$$
$$\mathbf{c} = \left(c_1,\dots,c_n,\beta_1,\dots,\beta_N\right)^\top \in \mathbb{R}^{n+N}.$$

Step 2. Choose the initial approximation $\alpha^{(1)}$, the tolerance $\varepsilon > 0$, and put $k = 1$.

Step 3. Compute the solution $\mathbf{x}^{(k)}(t)$ of the system

$$\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}),$$

with the initial condition

$$\mathbf{x}(0) = \left(c_1,\dots,c_n,\alpha_1^{(k)},\dots,\alpha_N^{(k)}\right).$$

Step 4. Evaluate the solution $\mathbf{y}^{(k+1)}(t)$ of the linearized equation in a particular form

$$\mathbf{y}^{(k+1)}(t) = \mathbf{p}^{(k+1)}(t) + \sum_{j=1}^{N} \beta_j \mathbf{h}^{(j,k+1)}(t).$$

Step 5. Determine the minimum β^* of the penalty function $\Psi_{k+1}(\beta) := \Upsilon(\mathbf{y}^{(k+1)})$ and set $\alpha^{(k+1)} := \beta^*$.

Step 6. Choose $\zeta_k > 0$, that is, the maximum allowed distance between the parameters $\alpha^{(k+1)}$ and $\alpha^{(k)}$.

Step 7. If the deviation $S(x^{(k+1)}) < S(x^{(k)})$ and
(a) $\|\alpha^{(k+1)} - \alpha^{(k)}\| \le \zeta_k$, then go to Step 3;
(b) $\|\alpha^{(k+1)} - \alpha^{(k)}\| > \zeta_k$, then suitably change the value $\alpha^{(k+1)}$ (see Lemma 3.5 for details).

Step 8. Set $k := k+1$ and repeat Steps 3, 4, 5, 6, 7(a), respectively, Step 7(b) until the condition

$$0 \le S\left(x^{(k)}\right) - S\left(x^{(k+1)}\right) < \varepsilon$$

is satisfied.

Step 9. If $S(x^{(k+1)}) > S(x^{(k)})$, then go back to Step 2 and start the algorithm with a better choice $\alpha^{(1)}$.

Algorithm 1.1

parameters is reached (according to the criteria for stopping the computatuion given in Steps 8-9).

The organization of this paper is as follows. In Section 2 we give a basic notations and definitions. In Section 3 we describe the modification of quasilinearization method in detail, and in Section 4 we give the convergence theorem. Section 5 includes the numerical results.

2. Notations and definitions

Let $\mathbb{R}^m$ be a vector space with the scalar product

$$(u,v) := u^\top v = \sum_{i=1}^{m} u_i v_i, \tag{2.1}$$

$u = (u_1,\dots,u_m)^\top \in \mathbb{R}^m$, $v = (v_1,\dots,v_m)^\top \in \mathbb{R}^m$. The associated norm is

$$\|u\| := (u,u)^{1/2}. \tag{2.2}$$

Let $A = (a_{ij})$, $i,j = 1,\dots,m$, be an $m \times m$ matrix. Then the matrix norm is given by

$$\|A\| := \left(\sum_{i,j=1}^{m} |a_{ij}|^2 \right)^{1/2}. \tag{2.3}$$

The matrix A is called *positive definite* if there is a constant $K > 0$ such that

$$(u,Au) \geq K\|u\|^2 \tag{2.4}$$

for every $u \in \mathbb{R}^m$.

Lemma 2.1. *Let $\widetilde{\gamma} = (\gamma_1,\dots,\gamma_m)^\top \in \mathbb{R}^m$. Let M be an $m \times m$ symmetric matrix of the form*

$$M = \Gamma + E, \tag{2.5}$$

where $\Gamma = \widetilde{\gamma}\widetilde{\gamma}^\top = (\Gamma_1,\dots,\Gamma_m)$, $\Gamma_i \in \mathbb{R}^m$ for all $i = 1,\dots,m$, and E is the $m \times m$ identity matrix. Then the matrix M is positive definite.

Proof. Denote

$$M_{kk} = (M_1,\dots,M_k) = \begin{pmatrix} m_{11} & \cdots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kk} \end{pmatrix}. \tag{2.6}$$

We can write the matrix M_{kk} in the form

$$M_{kk} = (\Gamma_1 + e_1,\dots,\Gamma_k + e_k), \tag{2.7}$$

where $e_i = (0,\dots,0,1,0,\dots,0)^\top$ is the k-dimensional vector with 1 on the ith position, $i = 1,\dots,k$. The minor $\det M_{kk}$ of the matrix M can be evaluated as follows:

$$\begin{aligned}\det M_{kk} &= \det\left(\Gamma_1 + e_1,\dots,\Gamma_k + e_k\right) = \cdots \\ &= \det E + \sum_{l=1}^{k} \det\left(e_1,\dots,e_{l-1},\Gamma_l,e_{l+1},\dots,e_k\right) + \sum_{j=1}^{2^k-k-1} \det Q_j,\end{aligned} \tag{2.8}$$

where Q_j are the matrices with at least two columns Γ_r,Γ_s. These k-dimensional vectors Γ_r,Γ_s are not linearly independent since

$$\Gamma_i = \gamma_i\tilde{\gamma} = \gamma_i(\gamma_1,\dots,\gamma_k)^\top, \quad \gamma_i \in \mathbb{R}, \tag{2.9}$$

for all $i = 1,\dots,k$. Therefore,

$$\det M_{kk} = \det E + \sum_{l=1}^{k} \det(e_1,\dots,e_{l-1},\Gamma_l,e_{l+1},\dots,e_k) = 1 + \sum_{l=1}^{k} \gamma_l^2, \tag{2.10}$$

and the matrix M is positive definite by Sylvester criterion [10, page 248]. □

Lemma 2.2. *Let M be an $m_1 \times m_1$ symetric positive definite matrix of the form (2.5). Let E be $m_2 \times m_2$ identity matrix. Let $m = m_1 + m_2$. Then the block diagonal $m \times m$ matrix*

$$M_d = \begin{pmatrix} M & 0 \\ 0 & E \end{pmatrix} \tag{2.11}$$

is positive definite too.

The proof is clear.

Lemma 2.3. *Let $L^2_m[0,T]$ be the space of vector functions $h(t) = (h_1(t),\dots,h_m(t))^\top$ with the scalar product*

$$(h,g) = \int_0^T \left(h(t),g(t)\right)_{\mathbb{R}^m} dt. \tag{2.12}$$

Let the matrix M_d have the form (2.11). Then

$$\langle h,g\rangle = \int_0^T (h(t))^\top M_d g(t)dt \tag{2.13}$$

is a scalar product on $L^2_m[0,T]$ too.

The proof follows easily by Lemma 2.2.

Remark 2.4. There are norms of m-dimensional vector function $h(t)$,

$$\|h\|^2 = (h,h), \tag{2.14}$$

$$|||h|||^2 = \langle h,h\rangle, \tag{2.15}$$

associated with the scalar products (2.12), (2.13). Obviously, they are equivalent.

Lemma 2.5. *Let $C^m[0,T]$ be the normed space of continuous m-dimensional vector functions with the norm*

$$\|h\|_C = \max_{t\in[0,T]} \left|\left|h(t)\right|\right|_{\mathbb{R}^m}. \tag{2.16}$$

If the sequence of functions $\{h_n(t)\}_{n=1}^{\infty}$ is uniformly convergent to the function $h(t)$ in the space $C^m[0,T]$, that is, $\lim_{n\to\infty}\|h_n - h\|_C = 0$, then

$$\lim_{n\to\infty} \left|\left|h_n - h\right|\right| = 0, \tag{2.17}$$

where the norm $\|h\|$ is defined by (2.14).

Proof. We can write

$$\begin{aligned}
\left|\left|h_n - h\right|\right|^2 &= (h_n - h, h_n - h) \\
&= \int_0^{\top} \left(h_n(t) - h(t), h_n(t) - h(t)\right)_{\mathbb{R}^m} dt \\
&\le \int_0^{\top} \max_{t\in[0,T]} \left(h_n(t) - h(t), h_n(t) - h(t)\right)_{\mathbb{R}^m} dt \\
&= \int_0^{\top} \max_{t\in[0,T]} \left|\left|h_n - h\right|\right|^2_{\mathbb{R}^m} dt = \int_0^{\top} \left|\left|h_n - h\right|\right|^2_C dt \\
&= T\left|\left|h_n - h\right|\right|^2_C.
\end{aligned} \tag{2.18}$$

Hence

$$\sqrt{T}\left|\left|h_n - h\right|\right|_C \ge \left|\left|h_n - h\right|\right|. \tag{2.19}$$

From this inequality, the assertion of Lemma 2.5 follows. □

Let $D \subset \mathbb{R}^m$ be a convex set. The function $S : D \to \mathbb{R}$ is called a *strictly convex function* if there is a constant $\chi > 0$ such that for every $u, v \in D$ and for every $\alpha \in [0,1]$, the inequality

$$S(\alpha u + (1-\alpha)v) \le \alpha S(u) + (1-\alpha)S(v) - \alpha(1-\alpha)\chi\|u - v\|^2 \tag{2.20}$$

is satisfied. The constant χ is called the *constant of the strict convexity* of the function S on the set D.

Lemma 2.6. *Let $D \subset \mathbb{R}^m$ be a convex closed set. Let $S(u)$ have the form*

$$S(u) = u^{\top} A u + b^{\top} u + c, \tag{2.21}$$

where A is a positive definite $m \times m$ matrix, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}$. Then S is a strictly convex function.

The proof is clear.

3. Modification of the quasilinearization method

Let $Q \subset \mathbb{R}^n$ be a closed convex set of the variables $x = (x_1,\dots,x_n)^\top$ and let $D \subset \mathbb{R}^n$ be a closed convex set of the parameters $\alpha = (\alpha_1,\dots,\alpha_N)^\top$. Let $f : Q \times D \to \mathbb{R}^n$ have continuous bounded partial derivatives up to the second order. Consider a nonlinear autonomous system of ordinary differential equations with the initial condition

$$\begin{aligned} \dot{x}(t) &= f(x,\alpha), \\ x(0) &= c. \end{aligned} \tag{3.1}$$

In order to avoid considering two different types of vectors, we will suppose that the vector α satisfies the differential equation

$$\dot{\alpha}(t) = 0 \tag{3.2}$$

with the initial condition

$$\alpha(0) = \beta, \tag{3.3}$$

where $\beta = (\beta_1,\dots,\beta_N)^\top$. Define a new vector $\mathbf{x}$ by

$$\mathbf{x} = (x,\alpha)^\top = \left(x_1,\dots,x_n,\alpha_1,\dots,\alpha_N\right)^\top \in \mathbb{R}^{n+N}, \tag{3.4}$$

and a vector $\mathbf{c}$ (corresponding to the initial condition) by

$$\mathbf{c} = (c,\beta)^\top = \left(c_1,\dots,c_n,\beta_1,\dots,\beta_N\right)^\top \in \mathbb{R}^{n+N}. \tag{3.5}$$

The vector $\mathbf{x}(t)$ satisfies the nonlinear differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}), \tag{3.6}$$

where $\mathbf{g}(\mathbf{x}) = (f(x,\alpha),\underbrace{0,\dots,0}_{N})^\top$, with the initial condition

$$\mathbf{x}(0) = \mathbf{c}. \tag{3.7}$$

The aim is to find the unknown parameters α such that the solution of the initial problem (3.1) fits in some sense with a given tolerance $\varepsilon > 0$ to the measured data or to the continuous function which approximates these data, respectively.

Assume that the approximating fuction $r(t) = (r_1(t),\dots,r_n(t))^\top$ corresponding to the measured data is given and let $e(t)$ be an approximating function appropriate to a certain linear combination of the components of the solution of (3.1) which is again measured during the experiment (in our case, $r(t) \approx (X(t),Y(t),Z(t))^\top$, $e(t) \approx V(t)$). In this context, let us point out that in practice, the values of $r(t)$ and $e(t)$ are measured in discrete instants of time, $\{t_1,\dots,t_L\}$ and $\{s_1,\dots,s_M\}$, $L,M \in \mathbb{N}$, and the functions $r(t)$, $e(t)$ have to be produced from given measured values. The procedure how to do this is in fact a matter of taste and intuition. It seems to be reasonable to get the functions $r(t)$ and $e(t)$

using spline interpolation. Our motivation is the Cauchy problem given by (1.1), (1.2) described in the intoduction.

The weighted deviation, $\Gamma : C^n[0,T] \to \mathbb{R}$, of a given function $z(t) \in C^n[0,T]$ from the approximating functions $r(t)$ and $e(t)$ can be expressed, in sense of the least-square method, in the form

$$\Gamma(z) = \sum_{l=1}^{n} \left(\int_0^{\top} \left(z_l(t) - r_l(t)\right)^2 dt \right) + \int_0^{\top} \left(\left(\gamma + \sum_{l=1}^{n} \gamma_l z_l(t) \right) - e(t) \right)^2 dt, \tag{3.8}$$

where γ, γ_l are given real weighting constants (in our case, $\gamma = X(0) = I$ and $\gamma_l = -1$ for $l = 1,2,3$).

Lemma 3.1. *Let $C^n[0,T]$ be the space of continuous vector functions $z(t)$ with the norm (2.16), for $m = n$. Let $\Gamma(z)$ have the form (3.8). Then $\Gamma(z)$ is continuous from $C^n[0,T]$ to $\mathbb{R}$.*

The proof follows easily by Lemma 2.5 and Remark 2.4.

Let $\mathbf{x}^{(k)}(t) = (x_1^{(k)}(t),\ldots,x_n^{(k)}(t),\alpha_1^{(k)},\ldots,\alpha_N^{(k)})^{\top}$ (kth iteration) be a solution to (3.6) on the interval $[0,T]$ with the initial condition (3.7) for $\mathbf{c} = (c_1,\ldots,c_n,\alpha_1^{(k)},\ldots,\alpha_N^{(k)})^{\top}$. The solution of the equivalent system (3.1) for $\alpha = \alpha^{(k)} = (\alpha_1^{(k)},\ldots,\alpha_N^{(k)})^{\top}$ is $x^{(k)} = (x_1^{(k)}(t),\ldots, x_n^{(k)}(t))^{\top}$. The deviation between the solution $x^{(k)}(t)$ and measured data has the form (3.8), that is,

$$S\left(x^{(k)}\right) = \sum_{l=1}^{n} \left(\int_0^{\top} \left(x_l^{(k)}(t) - r_l(t)\right)^2 dt \right) + \int_0^{\top} \left(\left(\gamma + \sum_{l=1}^{n} \gamma_l x_l^{(k)}(t) \right) - e(t) \right)^2 dt. \tag{3.9}$$

We would like to find a new vector of parameters $\beta = \alpha^{(k+1)}$ so that

$$S\left(x^{(k+1)}\right) < S\left(x^{(k)}\right). \tag{3.10}$$

The dependence of $\mathbf{x}^{(k)}(t)$, respectively, $x^{(k)}(t)$ on the parameters β ($\beta = \alpha^{(k)}$) is not clear, therefore we approximate $\mathbf{x}^{(k)}(t)$ by the solution $\mathbf{y}^{(k+1)}(t)$ of a linearized system

$$\dot{\mathbf{y}}(t) = \mathbf{g}\left(\mathbf{x}^{(k)}(t)\right) + \mathbf{J}\left(\mathbf{x}^{(k)}(t)\right)\left(\mathbf{y}(t) - \mathbf{x}^{(k)}(t)\right), \tag{3.11}$$

where $\mathbf{J}(\mathbf{x})$ is the Jacobian matrix of $\mathbf{g}(\mathbf{x})$.

Equation (3.11) is a linear system of $n+N$ differential equations and its general solution $\mathbf{y}(t)$ with

$$\mathbf{y}_j(0) = \begin{cases} c_j & \text{for } j = 1,\ldots,n, \\ \beta_{j-n} & \text{for } j = n+1,\ldots,n+N, \end{cases} \tag{3.12}$$

can be represented in the form

$$\mathbf{y}(t) = \mathbf{y}^{(k+1)}(t) = \mathbf{p}^{(k+1)}(t) + \sum_{j=1}^{N} \beta_j \mathbf{h}^{(j,k+1)}(t). \tag{3.13}$$

Here the function $\mathbf{p}^{(k+1)}(t)$ is the (particular) solution of the nonhomogeneous equation

$$\dot{\mathbf{p}}(t) = \mathbf{g}\left(\mathbf{x}^{(k)}(t)\right) + \mathbf{J}\left(\mathbf{x}^{(k)}(t)\right)\left(\mathbf{p}(t) - \mathbf{x}^{(k)}(t)\right) \tag{3.14}$$

which fulfills the initial condition

$$\mathbf{p}(0) = \left(c_1,\ldots,c_n,0,\ldots,0\right)^{\top}, \tag{3.15}$$

the $(n+N)$-column vectors $\mathbf{h}^{(j,k+1)}(t)$, $j = 1,\ldots,N$, are solutions of the homogeneous system

$$\dot{\mathbf{h}}^{(j,k+1)}(t) = \mathbf{J}\left(\mathbf{x}^{(k)}(t)\right)\mathbf{h}^{(j,k+1)}(t) \tag{3.16}$$

with

$$\mathbf{h}_i^{(j,k+1)}(0) = \begin{cases} 0 & \text{for } i \neq n+j, \\ \\ 1 & \text{for } i = n+j,\ i = 1,\ldots,n+N. \end{cases} \tag{3.17}$$

Let

$$\mathbf{H}^{(k+1)}(t) := \left(\mathbf{h}^{(1,k+1)}(t),\ldots,\mathbf{h}^{(N,k+1)}(t)\right) \tag{3.18}$$

be the $(n+N)\times N$ matrix with the columns equal to the solutions of (3.16), (3.17). Then the solution (3.13) can be written in the form

$$\mathbf{y}^{(k+1)}(t) = \mathbf{p}^{(k+1)}(t) + \mathbf{H}^{(k+1)}(t)\beta, \tag{3.19}$$

where $\beta = (\beta_1,\ldots,\beta_N)^{\top}$.

Lemma 3.2. *Let $t \in [0,T]$. Let $\mathbf{x}^{(k)}(t)$ be the solution to (3.6), (3.7) for $\mathbf{x}^{(k)}(0) = (c_1,\ldots,c_n, \alpha_1^{(k)},\ldots,\alpha_N^{(k)})^{\top}$ and let $\mathbf{y}^{(k+1)}(t)$ be the solution to (3.11) with the initial conditions (3.12). If, moreover, $\beta = \alpha^{(k)}$, then*

$$\mathbf{y}^{(k+1)}(t) = \mathbf{x}^{(k)}(t) \tag{3.20}$$

for $t \in [0,T]$. This means that

$$\mathbf{x}^{(k)}(t) = \mathbf{p}^{(k+1)}(t) + \mathbf{H}^{(k+1)}(t)\alpha^{(k)}. \tag{3.21}$$

For the proof, see [4, Lemma 4.1, page 235].

From the equality (3.13), we can see immediately that the dependence of $\mathbf{y}^{(k+1)}(t)$ on the parameters β_j, $j = 1,\ldots,N$, is affine. The parameters β_j, $j = 1,\ldots,N$, are free and they can be used for minimizing the function $\Upsilon : C^{n+N}[0,T] \to \mathbb{R}$,

$$\begin{aligned}\Upsilon\left(\mathbf{y}^{(k+1)}\right) = &\int_0^{\top} \left(\mathbf{y}^{(k+1)}(t) - \mathbf{r}(t)\right)^{\top} \left(\mathbf{y}^{(k+1)}(t) - \mathbf{r}(t)\right) dt \\ &+ \int_0^{\top} \left(\gamma + \sum_{l=1}^{n} \gamma_l \mathbf{y}_l^{(k+1)}(t) - e(t)\right)^2 dt,\end{aligned} \tag{3.22}$$

where $\mathbf{r}(t) = (r_1(t),\ldots,r_n(t),\mathbf{y}_{n+1}^{(k+1)}(t),\ldots,\mathbf{y}_{n+N}^{(k+1)}(t))^{\top}$, $\widehat{\gamma} = (\gamma_1,\ldots,\gamma_n,0,\ldots,0)^{\top} \in \mathbb{R}^{n+N}$.

It is easy to see that $\Upsilon(z_1,\ldots,z_{n+N}) = \Gamma(z_1,\ldots,z_n)$ for all $z_1,\ldots,z_{n+N} \in C[0,T]$.

Since the function $\Upsilon(\mathbf{y}^{(k+1)})$ depends on β, we can look at the function $\Upsilon(\mathbf{y}^{(k+1)})$ as a function of parameters $\beta = (\beta_1,\ldots,\beta_N)^{\top}$. Let

$$\Psi_{k+1}(\beta) := \Upsilon\left(\mathbf{y}^{(k+1)}\right) \tag{3.23}$$

be the function from $\mathbb{R}^N$ to $\mathbb{R}$.

It is easy to show that the function $\Psi_{k+1}(\beta)$ is a quadratic polynomial in the variables $\beta_1,\ldots,\beta_N$, that is,

$$\Psi_{k+1}(\beta) = \beta^{\top} A_{k+1} \beta + b_{k+1}^{\top} \beta + c_{k+1}, \tag{3.24}$$

where the coefficients $A_{k+1}, b_{k+1}^{\top}, c_{k+1}$ are as follows:

$$A_{k+1} = \int_0^{\top} \left(\mathbf{H}^{(k+1)}(t)\right)^{\top} \left(\widehat{\gamma}\widehat{\gamma}^{\top} + E\right) \mathbf{H}^{(k+1)}(t) dt \tag{3.25}$$

is an $N \times N$ matrix, E is $(n+N) \times (n+N)$ unity matrix,

$$\begin{aligned}b_{k+1}^{\top} = 2\int_0^{\top} &\left(\left(\mathbf{p}^{(k+1)}(t) - \mathbf{r}(t)\right)^{\top} + \left(\mathbf{p}^{(k+1)}(t)\right)^{\top} \widehat{\gamma}\widehat{\gamma}^{\top}\right. \\ &\left. + \left(e(t) - \gamma\right)\widehat{\gamma}^{\top}\right) \mathbf{H}^{(k+1)}(t) dt\end{aligned} \tag{3.26}$$

is an N-dimensional row vector, and

$$\begin{aligned}c_{k+1} = &\int_0^{\top} \left(\gamma - e(t)\right)\left(\gamma - e(t) + 2\widehat{\gamma}^{\top} \mathbf{p}^{(k+1)}(t)\right) + \left(\mathbf{p}^{(k+1)}(t)\right)^{\top} \widehat{\gamma}\widehat{\gamma}^{\top} \mathbf{p}^{(k+1)}(t) \\ &+ \left(\mathbf{r}(t) - \mathbf{p}^{(k+1)}(t)\right)^{\top} \left(\mathbf{r}(t) - \mathbf{p}^{(k+1)}(t)\right) dt\end{aligned} \tag{3.27}$$

is a real constant.

The quadratic polynomial (3.24) is continuously differentiable in the variable $\beta = (\beta_1,\ldots,\beta_N)^{\top}$, where for the derivatives, we have

$$\begin{aligned}S'_{k+1}(\beta) &= 2\beta^{\top} A_{k+1} + b_{k+1}^{\top}, \\ S''_{k+1}(\beta) &= 2A_{k+1},\end{aligned} \tag{3.28}$$

and the higher derivatives are zero because S''_{k+1} is an $N \times N$ constant matrix. The matrix A_{k+1} has the form

$$A_{k+1} = \begin{pmatrix} \langle \mathbf{h}^{(1,k+1)}, \mathbf{h}^{(1,k+1)} \rangle & \cdots & \langle \mathbf{h}^{(N,k+1)}, \mathbf{h}^{(1,k+1)} \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{h}^{(1,k+1)}, \mathbf{h}^{(N,k+1)} \rangle & \cdots & \langle \mathbf{h}^{(N,k+1)}, \mathbf{h}^{(N,k+1)} \rangle \end{pmatrix}. \tag{3.29}$$

The elements of the matrix A_{k+1} are scalar products on the space $C^{n+N}[0,T]$ given by (2.13) with the $(n+N) \times (n+N)$ symmetric block diagonal matrix

$$M_d = \Gamma + E = \hat{\gamma}\hat{\gamma}^\top + E. \tag{3.30}$$

In the following lemma, we give the necessary condition for positive definiteness of the matrix A_{k+1}.

Lemma 3.3. *Let $\mathbf{h}^{(j,k+1)}(t)$, $j = 1,\dots,N$, be the solutions of (3.16), (3.17). Then the matrix A_{k+1} is positive definite.*

Proof. Matrix A_{k+1} is the Gramm matrix which is real and symmetric. Since the vectors $\mathbf{h}^{(j,k+1)}(t)$ are linearly independent, we have $\det A_{k+1} \neq 0$. Let λ_j, $j = 1,\dots,N$, be the eigenvalue of the matrix A_{k+1} and let $u^{(j)}$ be the corresponding eigenvector, $\|u^{(j)}\| \neq 0$. Then $\lambda_j \in \mathbb{R}$ and

$$0 < (u^{(j)}, u^{(j)}) = (u^{(j)})^\top A_{k+1} u^{(j)} = (u^{(j)})^\top \lambda_j u^{(j)} = \lambda_j \sum_{i=1}^{N} (u_i^{(j)})^2. \tag{3.31}$$

This inequality implies that all eigenvalues are positive. There are orthogonal matrix O_{k+1} and diagonal matrix $D_{k+1} = \operatorname{diag}(\lambda_1,\dots,\lambda_N)$ so that

$$A_{k+1} = O_{k+1} D_{k+1} O_{k+1}^\top. \tag{3.32}$$

Let $\beta = (\beta_1,\dots,\beta_N)^\top \in \mathbb{R}^N$, $\|\beta\| \neq 0$. Then

$$\begin{aligned} (\beta, A_{k+1}\beta) &= (O_{k+1}^{-1}\beta, D_{k+1} O_{k+1}^{-1}\beta) \geq \min_j \lambda_j (O_{k+1}^{-1}\beta, O_{k+1}^{-1}\beta) \\ &= \min_j \lambda_j (\beta,\beta) = \min_j \lambda_j \|\beta\|^2. \end{aligned} \tag{3.33}$$

□

In the next lemma, we give a set and its property in which we look for the minimum of the function (3.24).

Lemma 3.4. *Let $S_{k+1}(\beta)$ have the form (3.24). Denote $V_k := S(x^{(k)})$, where $x^{(k)}$ is a solution of (3.1) for $\alpha = \alpha_k$. Define*

$$M_{\alpha_k} := \{\beta \mid \beta \in D, \Psi_{k+1}(\beta) \leq V_k\}. \tag{3.34}$$

Then M_{α_k} is a convex set for all $k = 1,2,\dots$.

Proof. Let $\beta_1, \beta_2 \in M_{\alpha_k}$, $a \in (0,1)$. Denote $A = A_{k+1}$, $b = b_{k+1}$ a $c = c_{k+1}$. Then

$$\begin{aligned}\Psi_{k+1}(a\beta_1 + (1-a)\beta_2) &= (a\beta_1 + (1-a)\beta_2)^\top A(a\beta_1 + (1-a)\beta_2) + b(a\beta_1 + (1-a)\beta_2) + c \\ &= a^2\beta_1^\top A\beta_1 + 2a(1-a)\beta_1^\top A\beta_2 + (1-a)^2\beta_2^\top A\beta_2 + ab\beta_1 + (1-a)b\beta_2 + c \\ &= a\beta_1^\top A\beta_1 + ab\beta_1 + ac + (1-a)\beta_2^\top A\beta_2 + (1-a)b\beta_2 + (1-a)c \\ &\quad + 2a(1-a)\beta_1^\top A\beta_2 - a(1-a)\beta_1^\top A\beta_1 - a(1-a)\beta_2^\top A\beta_2 \\ &\le aV_k + (1-a)V_k - a(1-a)(\beta_1 - \beta_2)^\top A(\beta_1 - \beta_2) \le V_k.\end{aligned} \tag{3.35}$$

The last inequality holds since A is positive definite. □

The necessary conditions for determining the local extreme on the set M_{α_k} are given by the equations

$$\frac{\partial \Psi_{k+1}(\beta)}{\partial \beta_j} = 0, \quad j = 1, \ldots, N. \tag{3.36}$$

Let us denote the solution of (3.36) by $\beta^* = (\beta_1^*, \ldots, \beta_N^*)^\top$. Since the matrix A_{k+1} is positive definite by Lemma 3.3 and the function $\Psi_{k+1}(\beta)$ is the strictly convex function by Lemma 2.6, β^* is the unique point of minimum (see [11, page 186]). Put

$$\alpha^{(k+1)} := \beta^* = (\beta_1^*, \ldots, \beta_N^*)^\top. \tag{3.37}$$

In this way, we obtain new initial condition

$$\mathbf{x}^{(k+1)}(0) = (c, \alpha^{(k+1)})^\top \tag{3.38}$$

for the solution $\mathbf{x}^{(k+1)}(t)$ of (3.6). Computing this solution, we get the solution $x^{(k+1)}$ of the equivalent system (3.1) for $\alpha = \alpha^{(k+1)}$. Determine the deviation (3.9). If the inequality (3.10), that is,

$$S(x^{(k+1)}) < S(x^{(k)}), \tag{3.39}$$

holds and the distance between $\alpha^{(k)}$ and $\alpha^{(k+1)}$ is small, that is,

$$||\alpha^{(k+1)} - \alpha^{(k)}|| \le \zeta_k, \tag{3.40}$$

for a given ζ_k small, then we can repeat the whole process of enumeration until the condition

$$0 \le S(x^{(k)}) - S(x^{(k+1)}) < \varepsilon, \tag{3.41}$$

where $\varepsilon > 0$ is a given tolerance, is satisfied.

If the inequality (3.10) is fulfilled, but

$$||\alpha^{(k+1)} - \alpha^{(k)}|| \ge \zeta_k, \tag{3.42}$$

we have to modify the value of the parameter $\alpha^{(k+1)}$. The modification is based on the following lemma.

Lemma 3.5. *Let M_{α_k} have the form (3.34) (cf. Lemma 3.4). Then for arbitrary $\zeta_k > 0$, there is a parameter $\alpha^{(k+1)} \in M_{\alpha_k}$ such that*

$$||\alpha^{(k+1)} - \alpha^{(k)}|| \leq \zeta_k. \tag{3.43}$$

Proof. Let $\beta^* \in M_{\alpha_k}$ be an argument of minima of $\Psi_{k+1}(\beta)$. Since M_{α_k} is a convex set, we can look for the parameter $\alpha^{(k+1)}$ in the form

$$\alpha^{(k+1)} = (1-a)\alpha^{(k)} + a\beta^*, \tag{3.44}$$

where $a \in (0,1)$. The object is to find a proper value a such that the vector $\alpha^{(k+1)}$ has to satisfy the inequality (3.43). We would like to have

$$||\alpha^{(k+1)} - \alpha^{(k)}|| = ||(1-a)\alpha^{(k)} + a\beta^* - \alpha^{(k)}|| = a||\beta^* - \alpha^{(k)}|| \leq \zeta_k. \tag{3.45}$$

Hence, we have to choose a such that $a \leq \zeta_k / \|\beta^* - \alpha^{(k)}\|$. □

We are able to shift the parameter $\alpha^{(k+1)}$ to $\alpha^{(k)}$ such that the distance between $\alpha^{(k+1)}$ and $\alpha^{(k)}$ is arbitrarily small, in particular less than a given tolerance ζ_k.

If $S(x^{(k+1)}) > S(x^{(k)})$ (the value of deviation has increased), we have to stop the whole process of computation and to start with a better choice of the initial approximation $\alpha^{(1)}$.

If $S(x^{(k+1)}) = S(x^{(k)})$ holds, we get the required values of parameters $\alpha = \alpha^{(k)}$ and the algorithm cannot produce better parameter values (for a given $\alpha^{(1)}$) and we are finished.

In the following lemmas, we describe the changes of the distance between the functions $\mathbf{x}^{(k)}(t)$, $\mathbf{x}(t)$ and between $\mathbf{x}^{(k)}(t)$, $\mathbf{y}^{(k+1)}(t)$.

Lemma 3.6. *Let $\mathbf{x}^{(k)}(t)$, $\mathbf{x}(t)$ be the solutions of (3.6), with the initial condition $\mathbf{x}^{(k)}(0) = (c, \alpha^{(k)})^\top$, $\mathbf{x}(0) = (c,\alpha)^\top$. Then for any $\zeta > 0$, there is $\zeta_k > 0$ such that*

$$||\mathbf{x}^{(k)} - \mathbf{x}||_C \leq \zeta, \tag{3.46}$$

whenever

$$||\alpha^{(k)} - \alpha|| \leq \zeta_k. \tag{3.47}$$

Proof. The proposition follows from the continuous dependence of the solution $\mathbf{x}(t)$ of (3.6) on the initial conditions [12, page 94]. □

Corollary 3.7. *Let the function $S(z)$ have the form (3.8). Let $\mathbf{x}^{(k)}(t)$, $\mathbf{x}(t)$ be the solutions of (3.6), with the initial conditions $\mathbf{x}^{(k)}(0) = (c, \alpha^{(k)})^\top$, $\mathbf{x}(0) = (c,\alpha)^\top$. Let $x^{(k)}(t)$, $x(t)$ be the corresponding solutions of (3.1). Then, for every $\varepsilon > 0$, there is $\zeta_k > 0$ such that if*

$$||\alpha^{(k)} - \alpha|| \leq \zeta_k, \tag{3.48}$$

then

$$\left| S(x^{(k)}) - S(x) \right| \leq \varepsilon. \tag{3.49}$$

Proof. The assertion follows from Lemma 3.6 realizing the continuity of $S(z)$ (see Lemma 3.1). □

Lemma 3.8. *Let $t \in [0,T]$ and $k = 1,2,\ldots$. Let $\mathbf{x}^{(k)}(t)$ be the solution of (3.6) with the initial condition $\mathbf{x}^{(k)}(0) = (c,\alpha^{(k)})^{\top}$. Let $\mathbf{y}^{(k+1)}(t)$ be the solution of (3.11) for $\mathbf{y}^{(k+1)}(0) = (c,\alpha^{(k+1)})^{\top}$. Then, for every $\omega_k > 0$, there is $\zeta_k > 0$ such that if*

$$\left|\left|\alpha^{(k+1)} - \alpha^{(k)}\right|\right| \leq \zeta_k, \tag{3.50}$$

then

$$\left|\left|\mathbf{y}^{(k+1)} - \mathbf{x}^{(k)}\right|\right|_C \leq \omega_k. \tag{3.51}$$

Proof. The difference $\mathbf{y}^{(k+1)}(t) - \mathbf{x}^{(k)}(t)$ satisfies the differential equation

$$\frac{d}{dt}\left(\mathbf{y}^{(k+1)}(t) - \mathbf{x}^{(k)}(t)\right) = \mathbf{J}\left(\mathbf{x}^{(k)}(t)\right)\left(\mathbf{y}^{(k+1)}(t) - \mathbf{x}^{(k)}(t)\right). \tag{3.52}$$

Integrating both sides from 0 to $s \in [0,T]$, we get

$$\mathbf{y}^{(k+1)}(s) - \mathbf{x}^{(k)}(s) = \mathbf{y}^{(k+1)}(0) - \mathbf{x}^{(k)}(0) + \int_0^s \mathbf{J}\left(\mathbf{x}^{(k)}(t)\right)\left(\mathbf{y}^{(k+1)}(t) - \mathbf{x}^{(k)}(t)\right)dt. \tag{3.53}$$

Hence

$$\left|\left|\mathbf{y}^{(k+1)} - \mathbf{x}^{(k)}\right|\right| \leq \left|\left|\mathbf{y}^{(k+1)}(0) - \mathbf{x}^{(k)}(0)\right|\right| + \int_0^s \left|\left|\mathbf{J}\left(\mathbf{x}^{(k)}\right)\left(\mathbf{y}^{(k+1)} - \mathbf{x}^{(k)}\right)\right|\right|dt. \tag{3.54}$$

Using the fact that

$$\left|\left|\mathbf{y}^{(k+1)}(0) - \mathbf{x}^{(k)}(0)\right|\right| = \left|\left|\alpha^{(k+1)} - \alpha^{(k)}\right|\right|, \tag{3.55}$$

we have by the Gronwall lemma that

$$\left|\left|\mathbf{y}^{(k+1)} - \mathbf{x}^{(k)}\right|\right| \leq \left|\left|\alpha^{(k+1)} - \alpha^{(k)}\right|\right| \exp\left(\int_0^s \left|\left|\mathbf{J}\left(\mathbf{x}^{(k)}\right)\right|\right|dt\right). \tag{3.56}$$

Since the vector function $\mathbf{x}^{(k)}(t)$ is bounded on the interval $[0,T] \ni s$, we have

$$\int_0^s \left|\left|\mathbf{J}\left(\mathbf{x}^{(k)}\right)\right|\right|dt \leq \mathbf{L}T < \infty, \tag{3.57}$$

where $\mathbf{L}$ is a Lipschitz constant of the function $\mathbf{g}(\mathbf{x})$. Consequently,

$$\left|\left|\mathbf{y}^{(k+1)} - \mathbf{x}^{(k)}\right|\right| \leq \left|\left|\alpha^{(k+1)} - \alpha^{(k)}\right|\right| e^{\mathbf{L}T}. \tag{3.58}$$

Hence, our assertion holds with any $\zeta_k \in (0, \omega_k e^{-LT})$. □

Remark 3.9. Let $\mathbf{x}^{(k)}(t)$, $\mathbf{y}^{(k+1)}(t)$ be the same as in Lemma 3.8. We can express

$$\alpha^{(k+1)} = \alpha^{(k)} + \Delta\alpha^{(k+1)}. \tag{3.59}$$

Then, using (3.13), (3.59), (3.21), we have

$$\begin{aligned}\mathbf{y}^{(k+1)}(t) &= \mathbf{p}^{(k+1)}(t) + \mathbf{H}^{(k+1)}(t)\alpha^{(k+1)} \\ &= \mathbf{p}^{(k)}(t) + \mathbf{H}^{(k+1)}(t)\left(\alpha^{(k)} + \Delta\alpha^{(k+1)}\right) \\ &= \mathbf{x}^{(k)}(t) + \mathbf{H}^{(k+1)}(t)\Delta\alpha^{(k+1)}.\end{aligned} \tag{3.60}$$

In addition, we have

$$\alpha^{(k+1)} = \alpha^{(k)} + \Delta\alpha^{(k+1)} = \alpha^{(1)} + \sum_{i=1}^{k} \Delta\alpha^{(i+1)}. \tag{3.61}$$

4. Convergence of the method

We did not manage to formulate the sufficient conditions for convergence of the sequence $\{\alpha^{(k)}\}_{k=1}^{\infty}$ generated by the modified quasilinearization method (MQM) for arbitrary initial approximation $\alpha^{(1)}$. Nevertheless, the method, if it is successful, constructs a convergent sequence of parameters $\{\alpha^{(k)}\}_{k=1}^{\infty}$.

We can choose a sequence $\{\zeta_k\}_{k=1}^{\infty}$ such that it is decreasing, $\liminf \zeta_k = 0$, and in addition

$$\sum_{k=1}^{\infty} \zeta_k < \infty. \tag{4.1}$$

Due to Lemmas 3.4 and 3.5, the parameter $\alpha^{(k+1)} \in M_{\alpha_k}$ and (3.43) holds. All parameters $\alpha^{(k)}$, $k = 1, 2, \ldots$, are the points of the convex set D defined by

$$D := \operatorname{conv}\left(\bigcup_{k=1}^{\infty} M_{\alpha_k}\right). \tag{4.2}$$

Theorem 4.1. *Let $\{\zeta_k\}_{k=1}^{\infty}$ be the decreasing convergent sequence such that $\zeta_k > 0$ and (4.1) holds. Let $\{\alpha^{(k)}\}_{k=1}^{\infty}$ be a sequence generated by MQM. Then $\{\alpha^{(k)}\}_{k=1}^{\infty}$ is a Cauchy sequence.*

Proof. The sum $\sum_{k=1}^{\infty} \zeta_k$ is a convergent sum which consist of positive real numbers, therefore for every $\varepsilon > 0$, there is $k_0 \in N$ such that $\sum_{l=k_0}^{\infty} \zeta_l \le \varepsilon/2$. Consequently, there is $k_0 \ge k$ so that

$$||\alpha^{(k+p)} - \alpha^{(k)}|| \le ||\alpha^{(k+p)} - \alpha^{(k_0)}|| + ||\alpha^{(k)} - \alpha^{(k_0)}|| \le \sum_{l=k_0}^{k+p} \zeta_l + \sum_{l=k_0}^{k} \zeta_l \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon. \tag{4.3}$$

From the facts above, it follows that for every $\varepsilon > 0$, there is natural number k_0 such that for every natural number p and for every $k \ge k_0$, the inequality

$$||\alpha^{(k+p)} - \alpha^{(k)}|| \le \varepsilon \tag{4.4}$$

is true. This means that the sequence $\{\alpha^{(k)}\}_{k=1}^{\infty}$ is a Cauchy sequence. □

Corollary 4.2. *The sequence $\{\alpha^{(k)}\}_{k=1}^{\infty}$ has a limit $\alpha^{(\infty)}$.*

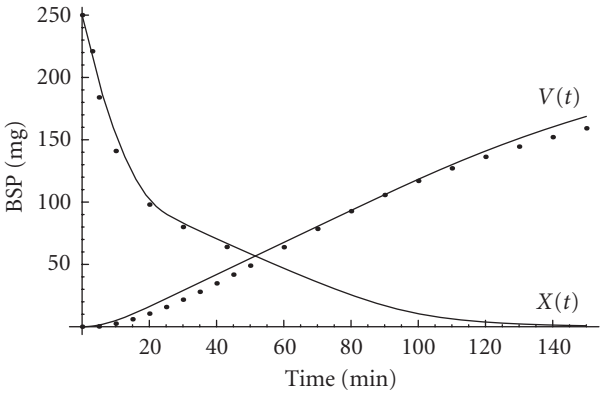

Figure 5.1

The ideal situation is a construction of the sequence $\alpha^{(k)} \to \alpha^{(*)}$ such that $S(x^{(*)}) = 0$, where $x^{(*)}$ is a solution of (3.1) for $\alpha = \alpha^{(*)}$. From practical point of view, this ideal situation is very rare, consequently we take up with a sequence for which the condition (3.41) is satisfied. Using MQM, we receive the best possible approximation $\alpha^{(\infty)}$ depending on an initial choice $\alpha^{(1)}$.

5. Application

In the paper [4], we discussed a simple mathematical model of the human liver. In [5], we presented three other models describing the BSP-kinetics in the human liver. One of them is nonlinear system (1.1) with the initial condition (1.2). In order to determine the positive unknown parameters $\alpha = (K_1, K_2, c_1, c_2, c_3)^\top$, we employ the measured data presented in Tables 1.1 and 1.2. We interpolate these data by cubic splines $SD_3(t), SE_3(t)$ for numerical enumeration. In order to obtain first approximation $x^{(1)}$ of the system (1.1), we have to make an educated guess of the parameters. We start the evaluation with the initial approximation

$$\alpha^{(1)} = \left(K_1^{(1)}, K_2^{(1)}, c_1^{(1)}, c_2^{(1)}, c_3^{(1)}\right)^\top = (13, 130, 0.004, 0.13, 0.0099)^\top. \tag{5.1}$$

The points on Figure 5.1 represent the measurements, see Tables 1.1 and 1.2, in this figure. The function $X(t) = x_1^{(1)}(t)$ and $V(t) = I - X(t) - Y(t) - Z(t) = \gamma + \sum_{i=1}^{n} \gamma_l x_l^{(1)}(t)$, where $n = 3$, $\gamma = X(0) = I$, $\gamma_l = -1$ for $l = 1, 2, 3$. In terms of this graph, we see that the initial approximation is convenient. The value of deviation (3.9) is $S(x^{(1)}) = 5453.89$. Let us put $\varepsilon = 0.0575$.

If we apply the quasilinearization method described by Bellman, we get

$$\alpha^{(2)} = (-33.0488, 172.407, 0.0663514, 0.731521, 0.00749651)^\top. \tag{5.2}$$

This result is not relevant since the parameter K_1 characterizing the capacity of the cell's membranes should be positive. Repeating the classical quasilinearization for the identification problem, we receive a divergent sequence of parameters.

Using our modification described in Section 3, we obtain

$$\alpha^{(700)} = \left(0.482797, 142.108, 0.12435, 1.21995, 0.924285 * 10^{-2}\right)^{\top}, \tag{5.3}$$

for the same initial approximation $\alpha^{(1)}$. We stopped the evaluation after 700 iteration steps since

$$0 \leq S\left(x^{(699)}\right) - S\left(x^{(700)}\right) < 0.0575, \tag{5.4}$$

that is, the condition in Step 8 was satisfied.

Our modification was proved on the simple linear mathematical model of the human liver published in [7]. The advantage of the system describing the simple mathematical model is a knowledge of the exact analytic solution. Modification of the quasilinearization method applied to this simple linear model provides identical results as classical Bellman's quasilinearization method for the inverse problem.

Acknowledgments

This research was supported, in part, by the Grant Agency of Czech Republic, Grant no. 201/06/0318 and by the Czech Ministry of Education, Project MSM 4781305904. Support of these institutions is gratefully acknowledged. The author gratefully acknowledges several useful suggestions of the reviewer.

References

[1] R. Bellman, H. Kagiwada, and R. Kalaba, "Orbit determination as a multi-point boundary-value problem and quasilinearization," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 48, no. 8, pp. 1327–1329, 1962.

[2] R. Bellman and R. Roth, *Quasilinearization and the Identification Problem*, vol. 2 of *Series in Modern Applied Mathematics*, World Scientific, Singapore, 1983.

[3] R. Kalaba and K. Spingarn, *Control, Identification, and Input Optimization*, vol. 25 of *Mathematical Concepts and Methods in Science and Engineering*, Plenum Press, New York, NY, USA, 1982.

[4] L. Čelechovská - Kozáková, "A simple mathematical model of the human liver," *Applications of Mathematics*, vol. 49, no. 3, pp. 227–246, 2004.

[5] L. Čelechovská - Kozáková, "Comparing mathematical models of the human liver based on BSP test," in *Proceedings of 5th International ISAAC Congress*, World Scientific, Catania, Sicily, Italy, July 2005.

[6] E. Hrnčíř, "Personal notes," unpublished.

[7] J. M. Watt and A. Young, "An attempt to simulate the liver on a computer," *The Computer Journal*, vol. 5, pp. 221–227, 1962.

[8] U. G. Abdullaev, "A numerical method for solving inverse problems for nonlinear differential equations," *Computational Mathematics and Mathematical Physics*, vol. 33, no. 8, pp. 1043–1057, 1993.

[9] U. G. Abdullaev, "Quasilinearization and inverse problems of nonlinear dynamics," *Journal of Optimization Theory and Applications*, vol. 85, no. 3, pp. 509–526, 1995.

[10] F. R. Gantmakher, *Matrix Theory*, Gostekhteorizdat, Moscow, Russia, 1953.

[11] F. P. Vasil'yev, *Numerical Methods for the Solution of Extremal Problems*, Nauka, Moscow, Russia, 1980.

[12] P. Hartman, *Ordinary Differential Equations*, S. M. Hartman, Baltimore, Md, USA, 1973.

Lenka Čelechovská-Kozáková: Mathematical Institute, Faculty of Applied Informatics,
Tomas Bata University in Zlín, Mostní 5139, 760 01 Zlín, Czech Republic
Email address: lkozakova@fai.utb.cz