Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 152808, 13 pages
doi:10.1155/2008/152808

Research Article

Multivariate Generalization of the Confluent Hypergeometric Function Kind 1 Distribution

Daya K. Nagar and Fabio Humberto Sepúlveda-Murillo

Departamento de Matemáticas, Universidad de Antioquia, Calle 67, 53–108 Medellín, Colombia

Correspondence should be addressed to Daya K. Nagar, dayaknagar@yahoo.com

Received 27 June 2008; Accepted 4 October 2008

Recommended by Andrei Volodin

The confluent hypergeometric function kind 1 distribution with the probability density function (pdf) proportional to $x^{-1}{}_1F_1(\alpha;\beta;-x)$, $x>0$ occurs as the distribution of the ratio of independent gamma and beta variables. In this article, a multivariate generalization of this distribution is defined and derived. Several pertinent properties of this multivariate distribution are discussed that shed some light on the nature of the distribution.

1. Introduction

The multivariate Liouville family of distributions was proposed by Marshall and Olkin [1]. Sivazlian [2] introduced Liouville distributions as generalizations of gamma and Dirichlet distributions. The Dirichlet and Liouville distributions have applications in such diverse fields as Bayesian analysis, modeling of multivariate data, order statistics, limit laws, multivariate analysis, reliability theory, and stochastic processes. These distributions have also been applied in geology, biology, chemistry, forensic science, and statistical genetics. A comprehensive review on applications and theoretical developments of these distributions are given in [1–5].

In this article, we propose a multivariate generalization of the confluent hypergeometric function kind 1 distribution which is a new member of the multivariate Liouville family of distributions.

The random variable X is said to have a confluent hypergeometric function kind 1 distribution, denoted by $X\sim \mathrm{CH}(\nu,\alpha,\beta,\ \text{kind }1)$, if its probability density function (pdf) is given by Gupta and Nagar [6],

$$\frac{\Gamma(\alpha)\Gamma(\beta-\nu)}{\Gamma(\nu)\Gamma(\beta)\Gamma(\alpha-\nu)}x^{\nu-1}{}_1F_1(\alpha;\beta;-x),\quad x>0,\tag{1.1}$$

where $\beta > \nu > 0,\ \alpha > \nu > 0$, and ${}_1F_1$ is the confluent hypergeometric function kind 1 (Luke [7]). The confluent hypergeometric function kind 1 distribution occurs as the distribution of the ratio of independent gamma and beta variables. Distributions of the product and ratio of independent beta and gamma variates can be found in [8]. For $\alpha = \beta$, the density (1.1) reduces to a gamma density given by

$$\frac{1}{\Gamma(\nu)} x^{\nu-1} \exp(-x), \quad x > 0. \tag{1.2}$$

Recently, Nagar and Sepúlveda-Murillo [9] studied several properties and stochastic representations of the confluent hypergeometric function kind 1 distribution.

In this article we define and study multivariate generalization of the confluent hypergeometric function kind 1 distribution. In Section 3, the multivariate generalization of the confluent hypergeometric function kind 1 distribution is defined and derived as the distribution of the quotient of a set of independent gamma variables variable with an independent beta variable. Several properties of this distribution including marginal and conditional distributions, distribution of partial sums and several factorizations are studied in Section 4.

2. Some useful definitions

Several definitions and results on special functions and integrals used in this article are given in this section. Throughout this work we will use the Pochhammer symbol $(a)_n$ defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots$, and $(a)_0 = 1$.

The generalized hypergeometric function of scalar argument is defined by

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{2.1}$$

where $a_i,\ i = 1, \ldots, p;\ b_j,\ j = 1, \ldots, q$ are complex numbers with suitable restrictions and z is a complex variable. Conditions for the convergence of the series in (2.1) are available in the literature, see Luke [7]. From (2.1) it is easy to see that

$$\begin{gathered} {}_0F_0(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \exp(z), \quad {}_1F_0(a; z) = \sum_{k=0}^{\infty} (a)_k \frac{z^k}{k!} = (1-z)^{-a}, \quad |z| < 1, \\ {}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}, \quad {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1. \end{gathered} \tag{2.2}$$

Also, under suitable conditions,

$$\begin{aligned} &\int_0^{\infty} \exp(-\delta z) z^{\alpha-1} {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; zy) \mathrm{d}z \\ &\quad = \Gamma(\alpha) \delta^{-\alpha} {}_{p+1}F_q(a_1, \ldots, a_p, \alpha; b_1, \ldots, b_q; \delta^{-1} y). \end{aligned} \tag{2.3}$$

The integral representations of the confluent hypergeometric function and the Gauss hypergeometric function are given as

$$\begin{aligned} {}_1F_1(a;c;z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\exp(zt)\mathrm{d}t \\ &\quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\mathrm{d}t, \\ &\quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0, \quad |\arg(1-z)| < \pi, \end{aligned} \tag{2.5}$$

respectively. Note that the series expansions for ${}_1F_1$ and ${}_2F_1$ given in (2.2) can be obtained by expanding $\exp(zt)$ and $(1-zt)^{-b}$, $|zt| < 1$, in (2.4) and (2.5) and integrating t. Substituting $z = 1$ in (2.5) and integrating, we obtain

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0,\ c \neq -1,-2,\ldots. \tag{2.6}$$

The confluent hypergeometric function ${}_1F_1(a;c;z)$ satisfies Kummer's relation

$${}_1F_1(a;c;-z) = \exp(-z){}_1F_1(c-a;c;z). \tag{2.7}$$

The Lauricella hypergeometric function $F_D^{(m)}$ in m variables $z_1,\ldots,z_m$ is defined by

$$F_D^{(m)}[a,b_1,\ldots,b_m;c;z_1,\ldots,z_m] = \sum_{j_1,\ldots,j_m=0}^{\infty} \frac{(a)_{j_1+\cdots+j_m}(b_1)_{j_1}\cdots(b_m)_{j_m} z_1^{j_1}\cdots z_m^{j_m}}{(c)_{j_1+\cdots+j_m} j_1!\cdots j_m!}, \tag{2.8}$$

where $\max\{|z_1|,\ldots,|z_m|\} < 1$. For $m = 1$, the Lauricella hypergeometric function reduces to a Gauss hypergeometric function and for $m = 2$ it slides to an Appell hypergeometric function F_1. Using the result

$$\frac{(a)_j}{(c)_j} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a+j-1}(1-t)^{c-a-1}\,\mathrm{d}t, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0, \tag{2.9}$$

for $j = 0,1,2,\ldots$, and

$$\sum_{j_i=0}^{\infty} \frac{(b_i)_{j_i}(tz_i)^{j_i}}{j_i!} = {}_1F_0(b_i;tz_i) = (1-tz_i)^{-b_i}, \quad |z_i| < 1,\ i = 1,\ldots,m, \tag{2.10}$$

in (2.8), an integral representation of $F_D^{(m)}$ is given by

$$F_D^{(m)}[a,b_1,\dots,b_m;c;z_1,\dots,z_m] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-z_1t)^{-b_1}\cdots(1-z_mt)^{-b_m}\,\mathrm{d}t. \tag{2.11}$$

Further, replacing $(1-z_it)^{-b_i}$ by its equivalent gamma integral, namely,

$$(1-z_it)^{-b_i} = \frac{1}{\Gamma(b_i)}\int_0^\infty \exp[-(1-z_it)v_i]v_i^{b_i-1}\mathrm{d}v_i, \quad \operatorname{Re}(b_i)>0, \tag{2.12}$$

for $i=1,\dots,m$ and integrating out t, one obtains

$$\prod_{i=1}^m\Gamma(b_i)F_D^{(m)}[a,b_1,\dots,b_m;c;z_1,\dots,z_m] = \int_0^\infty\cdots\int_0^\infty \exp\left(-\sum_{i=1}^m v_i\right)\prod_{i=1}^m v_i^{b_i-1}{}_1F_1\left(a;c;-\sum_{i=1}^m z_iv_i\right)\mathrm{d}v_1\cdots\mathrm{d}v_m. \tag{2.13}$$

For further results and properties of this function the reader is referred to [10, 11].

Let $f(\cdot)$ be a continuous function and $\alpha_i>0,\ i=1,\dots,n$. The integral

$$D_n(\alpha_1,\dots,\alpha_n;f) = \int_0^\infty\cdots\int_0^\infty \prod_{i=1}^n x_i^{\alpha_i-1}f\left(\sum_{i=1}^n x_i\right)\prod_{i=1}^n \mathrm{d}x_i \tag{2.14}$$

is known as Liouville's integral. Substituting $y_i=x_i/x,\ i=1,\dots,n-1$ and $x=\sum_{i=1}^n x_i$ with the Jacobian $J(x_1,\dots,x_{n-1},x_n\to y_1,\dots,y_{n-1},x)=x^{n-1}$, it is easy to see that

$$D_n(\alpha_1,\dots,\alpha_n;f) = \frac{\prod_{i=1}^n\Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^n\alpha_i)}\int_0^\infty x^{\sum_{i=1}^n\alpha_i-1}f(x)\mathrm{d}x. \tag{2.15}$$

3. Density function

In this section, we present a multivariate generalization of the hypergeometric function kind 1 distribution and derive it using independent beta and gamma variables.

Definition 3.1. The random variables $X_1,\dots,X_n$ are said to have a multivariate confluent hypergeometric function kind 1 distribution, denoted as $(X_1,\dots,X_n)\sim \mathrm{CH}(\nu_1,\dots,\nu_n;\alpha,\beta,\ \text{kind }1)$, if their joint pdf is given by

$$C(\nu_1,\dots,\nu_n;\alpha,\beta)\prod_{i=1}^n x_i^{\nu_i-1}{}_1F_1\left(\alpha;\beta;-\sum_{i=1}^n x_i\right), \quad x_i>0,\ i=1,\dots,n, \tag{3.1}$$

where $C(\nu_1,\dots,\nu_n;\alpha,\beta)$ is the normalizing constant.

The normalizing constant in (3.1) is given by

$$\begin{aligned}\{C(\nu_1,\dots,\nu_n;\alpha,\beta)\}^{-1} &= \int_0^\infty\cdots\int_0^\infty \prod_{i=1}^n x_i^{\nu_i-1}{}_1F_1\left(\alpha;\beta;-\sum_{i=1}^n x_i\right)\prod_{i=1}^n \mathrm{d}x_i \\ &= \frac{\prod_{i=1}^n\Gamma(\nu_i)}{\Gamma(\sum_{i=1}^n\nu_i)}\int_0^\infty x^{\sum_{i=1}^n\nu_i-1}{}_1F_1(\alpha;\beta;-x)\mathrm{d}x \\ &= \frac{\prod_{i=1}^n\Gamma(\nu_i)}{\Gamma(\sum_{i=1}^n\nu_i)}\int_0^\infty x^{\sum_{i=1}^n\nu_i-1}\exp(-x){}_1F_1(\beta-\alpha;\beta;x)\mathrm{d}x,\end{aligned} \tag{3.2}$$

where the last line has been obtained by using Liouville's integral and (2.7). Now, evaluating the above integral and simplifying the resulting expression using (2.3) and (2.6), we get

$$\{C(\nu_1,\dots,\nu_n;\alpha,\beta)\}^{-1} = \frac{\prod_{i=1}^n\Gamma(\nu_i)\Gamma(\beta)\Gamma(\alpha-\sum_{i=1}^n\nu_i)}{\Gamma(\alpha)\Gamma(\beta-\sum_{i=1}^n\nu_i)}, \tag{3.3}$$

where $\nu_i > 0$, $i = 1,\dots,n$, $\alpha > \sum_{i=1}^n\nu_i$, and $\beta > \sum_{i=1}^n\nu_i$. For $\alpha = \beta$, multivariate confluent hypergeometric function kind 1 density simplifies to the product of n univariate gamma densities. Several special cases of the density (3.1) can be obtained by specializing α and β and using results on confluent hypergeometric function. For example, substitution of $\beta = 2\alpha$ in (3.1) and application of the result

$${}_1F_1(a;2a;z) = \frac{\Gamma(a+1/2)\exp(z/2)}{(z/4)^{a-1/2}}I_{a-1/2}\left(\frac{z}{2}\right), \tag{3.4}$$

where I_δ is the modified Bessel function of the first kind, yield

$$\begin{aligned}&C(\nu_1,\dots,\nu_n;\alpha,2\alpha)4^{\alpha-1/2}\Gamma\left(\alpha+\frac{1}{2}\right)\frac{\exp\left(-\sum_{i=1}^n x_i/2\right)\prod_{i=1}^n x_i^{\nu_i-1}}{\left(\sum_{i=1}^n x_i\right)^{\alpha-1/2}} \\ &\quad\times I_{\alpha-1/2}\left(\frac{\sum_{i=1}^n x_i}{2}\right), \quad x_i > 0,\ i = 1,\dots,n.\end{aligned} \tag{3.5}$$

It may be noted here that the multivariate confluent hypergeometric function kind 1 distribution belongs to the Liouville family of distributions (Sivazlian [2], Gupta and Song [3]). Because of mathematical tractability of the confluent hypergeometric function and its several special cases, the multivariate confluent hypergeometric function kind 1 distribution enriches the class of multivariate Liouville distributions and may serve as an alternative to many existing distributions belonging to this class. The next theorem derives the multivariate confluent hypergeometric function kind 1 distribution using independent gamma and beta variables. First, we define the gamma, beta type 1 and beta type 2 distributions. These definitions can be found in [12].

Definition 3.2. A random variable X is said to have the gamma distribution with shape parameter a, $a > 0$, denoted as $X \sim \mathrm{Ga}(a)$, if its pdf is given by

$$\frac{\exp(-x) x^{a-1}}{\Gamma(a)}, \quad x > 0. \tag{3.6}$$

Definition 3.3. A random variable X is said to have the beta type 1 distribution with parameters (a, b), $a > 0$, $b > 0$, denoted as $X \sim \mathrm{B1}(a, b)$, if its pdf is given by

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1}, \quad 0 < x < 1. \tag{3.7}$$

Definition 3.4. A random variable X is said to have the beta type 2 distribution with parameters (a, b), denoted as $X \sim \mathrm{B2}(a, b)$, $a > 0$, $b > 0$, if its pdf is given by

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1+x)^{-(a+b)}, \quad x > 0. \tag{3.8}$$

If $X \sim \mathrm{B1}(a, b)$, then $1 - X \sim \mathrm{B1}(b, a)$, $X/(1 - X) \sim \mathrm{B2}(a, b)$, and $(1 - X)/X \sim \mathrm{B2}(b, a)$. Further, if $Y \sim \mathrm{B2}(a, b)$, then $1/Y \sim \mathrm{B2}(b, a)$, $Y/(1+Y) \sim \mathrm{B1}(a, b)$, and $1/(1+Y) \sim \mathrm{B1}(b, a)$.

The matrix variate generalizations of gamma, beta type 1, beta type 2 distributions have been defined and studied extensively. For example, see [6].

Theorem 3.5. *Let $Y_1, \dots, Y_n$ and Z be independent, $Y_i \sim \mathrm{Ga}(c_i)$, $i = 1, \dots, n$, and $Z \sim \mathrm{B1}(a, b)$. Then, $(Y_1/Z, \dots, Y_n/Z) \sim \mathrm{CH}(c_1, \dots, c_n; \sum_{i=1}^n c_i + a, \sum_{i=1}^n c_i + a + b,$ kind 1$)$ with the pdf*

$$\begin{aligned} &\frac{\Gamma(a+b)\Gamma\left(\sum_{i=1}^n c_i + a\right)}{\prod_{i=1}^n \Gamma(c_i)\Gamma(a)\Gamma\left(\sum_{i=1}^n c_i + a + b\right)} \prod_{i=1}^n x_i^{c_i - 1} \\ &\quad \times {}_1F_1\left(\sum_{i=1}^n c_i + a; \sum_{i=1}^n c_i + a + b; -\sum_{i=1}^n x_i\right), \quad x_i > 0, \ i = 1, \dots, n. \end{aligned} \tag{3.9}$$

Proof. The joint density of $Y_1, \dots, Y_n$ and Z is given by

$$\frac{\Gamma(a+b)}{\prod_{i=1}^n \Gamma(c_i)\Gamma(a)\Gamma(b)} \prod_{i=1}^n y_i^{c_i - 1} \exp\left(-\sum_{i=1}^n y_i\right) z^{a-1}(1-z)^{b-1}, \tag{3.10}$$

where $0 < z < 1$, $y_i > 0$, $i = 1, \dots, n$. Transforming $X_i = Y_i/Z$, $i = 1, \dots, n$ with the Jacobian $J(y_1, \dots, y_n, z \to x_1, \dots, x_n, z) = z^n$ in (3.10) and integrating out z, we get the marginal density of $(X_1, \dots, X_n)$ as

$$\frac{\Gamma(a+b)}{\prod_{i=1}^n \Gamma(c_i)\Gamma(a)\Gamma(b)} \prod_{i=1}^n x_i^{c_i - 1} \int_0^1 \exp\left(-z\sum_{i=1}^n x_i\right) z^{\sum_{i=1}^n c_i + a - 1}(1-z)^{b-1}\, dz, \tag{3.11}$$

where $x_i > 0$, $i = 1, \dots, n$. Now, evaluation of the above integral using (2.4) yields the desired result. $\square$

Corollary 3.6. *Let $Y_1,\dots,Y_n$ and Z be independent, $Y_i\sim \mathrm{Ga}(c_i)$, $i=1,\dots,n$, and $Z\sim \mathrm{B1}(a,b)$. Then*

$$\left(\frac{Y_1}{1-Z},\dots,\frac{Y_n}{1-Z}\right)\sim \mathrm{CH}\left(c_1,\dots,c_n;\sum_{i=1}^n c_i+b,\sum_{i=1}^n c_i+a+b,\ \text{kind }1\right). \tag{3.12}$$

Corollary 3.7. *Let $Y_1,\dots,Y_n$ and V be independent, $Y_i\sim \mathrm{Ga}(c_i)$, $i=1,\dots,n$, and $V\sim \mathrm{B2}(a,b)$, then*

$$\begin{aligned}\left(\frac{(1+V)Y_1}{V},\dots,\frac{(1+V)Y_n}{V}\right)&\sim \mathrm{CH}\left(c_1,\dots,c_n;\sum_{i=1}^n c_i+a,\sum_{i=1}^n c_i+a+b,\ \text{kind }1\right),\\ ((1+V)Y_1,\dots,(1+V)Y_n)&\sim \mathrm{CH}\left(c_1,\dots,c_n;\sum_{i=1}^n c_i+b,\sum_{i=1}^n c_i+a+b,\ \text{kind }1\right).\end{aligned} \tag{3.13}$$

The Laplace transform of the multivariate confluent hypergeometric function kind 1 density (3.1) is given by

$$C(\nu_1,\dots,\nu_n;\alpha,\beta)\int_0^\infty\cdots\int_0^\infty \exp\left(-\sum_{i=1}^n s_i x_i\right)\prod_{i=1}^n x_i^{\nu_i-1}{}_1F_1\left(\alpha;\beta;-\sum_{i=1}^n x_i\right)\prod_{i=1}^n \mathrm{d}x_i, \tag{3.14}$$

where $\mathrm{Re}(s_i)>0$, $i=1,\dots,n$. Now, rewriting ${}_1F_1$ by applying (2.7) and integrating $x_1,\dots,x_m$ using (2.13), the above expression is evaluated as

$$\frac{\Gamma(\alpha)\Gamma\left(\beta-\sum_{i=1}^n\nu_i\right)}{\Gamma(\beta)\Gamma\left(\alpha-\sum_{i=1}^n\nu_i\right)}F_D^{(n)}\left[\beta-\alpha,\nu_1,\dots,\nu_n;\beta;\frac{1}{1+s_1},\dots,\frac{1}{1+s_n}\right]. \tag{3.15}$$

The joint moments of $X_1,\dots,X_n$ are given by

$$\begin{aligned}E\left(X_1^{r_1}\cdots X_n^{r_n}\right)&=C(\nu_1,\dots,\nu_n;\alpha,\beta)\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^n x_i^{\nu_i+r_i-1}{}_1F_1\left(\alpha;\beta;-\sum_{i=1}^n x_i\right)\prod_{i=1}^n \mathrm{d}x_i\\ &=\frac{C(\nu_1,\dots,\nu_n;\alpha,\beta)}{C\left(\sum_{i=1}^n\nu_i;\alpha,\beta\right)}\frac{\prod_{i=1}^n\Gamma(\nu_i+r_i)}{\Gamma\left[\sum_{i=1}^n(\nu_i+r_i)\right]}E\left[X^{\sum_{i=1}^n r_i}\right],\end{aligned} \tag{3.16}$$

where $X\sim \mathrm{CH}(\sum_{i=1}^n\nu_i,\alpha,\beta,\ \text{kind }1)$. Note that if $X\sim \mathrm{CH}(\nu,\alpha,\beta,\ \text{kind }1)$, then from [9], one gets

$$E(X^h)=\frac{\Gamma(\beta-\nu)\Gamma(\nu+h)\Gamma(\alpha-\nu-h)}{\Gamma(\nu)\Gamma(\alpha-\nu)\Gamma(\beta-\nu-h)}, \tag{3.17}$$

where $\mathrm{Re}(h+\nu)>0$, $\mathrm{Re}(h)<\alpha-\nu$, and $\mathrm{Re}(h)<\beta-\nu$. Now, computing $E[X^{\sum_{i=1}^n r_i}]$ using the above result, substituting for $C(\nu_1,\ldots,\nu_n;\alpha,\beta)$ and $C(\sum_{i=1}^n \nu_i;\alpha,\beta)$ from (3.3) and simplifying the resulting expression, we obtain

$$E\left(X_1^{r_1}\cdots X_n^{r_n}\right)=\frac{\Gamma\left(\beta-\sum_{i=1}^n \nu_i\right)\Gamma\left(\alpha-\sum_{i=1}^n \nu_i-\sum_{i=1}^n r_i\right)\prod_{i=1}^n \Gamma(\nu_i+r_i)}{\prod_{i=1}^n \Gamma(\nu_i)\Gamma\left(\alpha-\sum_{i=1}^n \nu_i\right)\Gamma\left(\beta-\sum_{i=1}^n \nu_i-\sum_{i=1}^n r_i\right)}, \tag{3.18}$$

where $\alpha>\sum_{i=1}^n(\nu_i+r_i)$ and $\beta>\sum_{i=1}^n(\nu_i+r_i)$. Substituting appropriately in the above expression and using definitions of variance, covariance, and correlation coefficient, it is straightforward to show that

$$\begin{aligned}
E(X_j)&=\frac{\nu_j\left(\beta-\sum_{i=1}^n \nu_i-1\right)}{\alpha-\sum_{i=1}^n \nu_i-1},\\
E(X_j^2)&=\frac{\nu_j(\nu_j+1)\left(\beta-\sum_{i=1}^n \nu_i-1\right)\left(\beta-\sum_{i=1}^n \nu_i-2\right)}{\left(\alpha-\sum_{i=1}^n \nu_i-1\right)\left(\alpha-\sum_{i=1}^n \nu_i-2\right)},\\
\mathrm{Var}(X_j)&=\frac{\nu_j\left(\beta-\sum_{i=1}^n \nu_i-1\right)}{\left(\alpha-\sum_{i=1}^n \nu_i-1\right)^2\left(\alpha-\sum_{i=1}^n \nu_i-2\right)}\\
&\quad\times\left[\nu_j(\beta-\alpha)+\left(\beta-\sum_{i=1}^n \nu_i-2\right)\left(\alpha-\sum_{i=1}^n \nu_i-1\right)\right],\\
E(X_jX_k)&=\frac{\nu_j\nu_k\left(\beta-\sum_{i=1}^n \nu_i-1\right)\left(\beta-\sum_{i=1}^n \nu_i-2\right)}{\left(\alpha-\sum_{i=1}^n \nu_i-1\right)\left(\alpha-\sum_{i=1}^n \nu_i-2\right)},\\
\mathrm{Cov}(X_j,X_k)&=\frac{\nu_j\nu_k\left(\beta-\sum_{i=1}^n \nu_i-1\right)(\beta-\alpha)}{\left(\alpha-\sum_{i=1}^n \nu_i-1\right)^2\left(\alpha-\sum_{i=1}^n \nu_i-2\right)},\\
\mathrm{Corr}(X_j,X_k)&=\left[\left(1+\frac{\left(\beta-\sum_{i=1}^n \nu_i-2\right)\left(\alpha-\sum_{i=1}^n \nu_i-1\right)}{\nu_j(\beta-\alpha)}\right)\right.\\
&\quad\left.\times\left(1+\frac{\left(\beta-\sum_{i=1}^n \nu_i-2\right)\left(\alpha-\sum_{i=1}^n \nu_i-1\right)}{\nu_k(\beta-\alpha)}\right)\right]^{-1/2},
\end{aligned} \tag{3.19}$$

where $j\neq k$, $j,k=1,\ldots,n$.

4. Properties

In this section we derive marginal and conditional distributions, distribution of partial sums, and several factorizations of the multivariate confluent hypergeometric function kind 1 distribution.

The following theorem shows that if the joint distribution of $X_1,\ldots,X_n$ is multivariate confluent hypergeometric function kind 1, then the marginal distribution of any subset of $X_1,\ldots,X_n$ is multivariate confluent hypergeometric function kind 1.

Theorem 4.1. *Let* $(X_1,\ldots,X_n)\sim \mathrm{CH}(\nu_1,\ldots,\nu_n;\alpha,\beta,\ \text{kind }1)$. *Then, for* $1\le s\le n$,

$$(X_1,\ldots,X_s)\sim \mathrm{CH}\left(\nu_1,\ldots,\nu_s;\alpha-\sum_{i=s+1}^n \nu_i,\beta-\sum_{i=s+1}^n \nu_i,\ \text{kind }1\right). \tag{4.1}$$

Proof. Replacing ${}_1F_1(\alpha;\beta;-\sum_{i=1}^{n}x_i)$ in (3.1) by its integral representation and integrating out $x_{s+1},\dots,x_n$, the marginal density of $X_1,\dots,X_s$ is derived as

$$C(\nu_1,\dots,\nu_n;\alpha,\beta)\frac{\Gamma(\beta)\prod_{i=s+1}^{n}\Gamma(\nu_i)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\prod_{i=1}^{s}x_i^{\nu_i-1}\int_0^1 t^{\alpha-\sum_{i=s+1}^{n}\nu_i-1}(1-t)^{\beta-\alpha-1}\exp\left(-t\sum_{i=1}^{n}x_i\right)\mathrm{d}t, \quad (4.2)$$

where $x_i>0,\ i=s+1,\dots,n$. Now, evaluating the above integral using (2.4) and simplifying the resulting expression, we get the desired result. □

Corollary 4.2. *If* $(X_1,\dots,X_n)\sim\mathrm{CH}(\nu_1,\dots,\nu_n;\alpha,\beta,\ \text{kind }1)$*, then for any subset of variables* $(X_{j_1},\dots,X_{j_i})$*, it holds that*

$$(X_{j_1},\dots,X_{j_i})\sim\mathrm{CH}\left(\nu_{j_1},\dots,\nu_{j_i};\alpha+\sum_{k=1}^{i}\nu_{j_k}-\sum_{k=1}^{n}\nu_i,\beta+\sum_{k=1}^{i}\nu_{j_k}-\sum_{k=1}^{n}\nu_i,\ \text{kind }1\right), \quad (4.3)$$

where $j_1,\dots,j_i$ *are distinct integers with* $1\le j_1,\dots,j_i\le n,\ 1\le i\le n$.

Corollary 4.3. *Let* $(X_1,\dots,X_n)\sim\mathrm{CH}(\nu_1,\dots,\nu_n;\alpha,\beta,\ \text{kind }1)$*. Then, for* $k=1,\dots,n$, $X_k\sim\mathrm{CH}(\nu_k,\alpha-\sum_{i(\neq k)=1}^{n}\nu_i,\beta-\sum_{i(\neq k)=1}^{n}\nu_i,\ \text{kind }1)$.

Using Theorem 4.1, the conditional density of $(X_{s+1},\dots,X_n)$ given $(X_1,\dots,X_s)$ is obtained as

$$\frac{C(\nu_1,\dots,\nu_n;\alpha,\beta)}{C(\nu_1,\dots,\nu_s;\alpha-\sum_{i=s+1}^{n}\nu_i,\beta-\sum_{i=s+1}^{n}\nu_i)} \times\frac{\prod_{i=s+1}^{n}x_i^{\nu_i-1}{}_1F_1(\alpha;\beta;-\sum_{i=1}^{n}x_i)}{{}_1F_1(\alpha-\sum_{i=s+1}^{n}\nu_i;\beta-\sum_{i=s+1}^{n}\nu_i;-\sum_{i=1}^{n}x_i)}, \quad (4.4)$$

where $0<x_i,\ i=s+1,\dots,n$.

Next, in Theorem 4.5, we give the joint distribution of partial sums of random variables distributed jointly as multivariate confluent hypergeometric function kind 1. Since the theorem requires familiarity with the Dirichlet type 1 distribution, we first give its definition (see [13]).

Definition 4.4. The random variables $U_1,\dots,U_n$ are said to have a Dirichlet type 1 distribution with parameters $\nu_1,\dots,\nu_n;\nu_{n+1}$, denoted by $(U_1,\dots,U_n)\sim\mathrm{D1}(\nu_1,\dots,\nu_n;\nu_{n+1})$, if their joint pdf is given by

$$\frac{\Gamma(\sum_{i=1}^{n+1}\nu_i)}{\prod_{i=1}^{n+1}\Gamma(\nu_i)}\prod_{i=1}^{n}u_i^{\nu_i-1}\left(1-\sum_{i=1}^{n}u_i\right)^{\nu_{n+1}-1}, \quad 0<u_i,\ i=1,\dots,n,\ \sum_{i=1}^{n}u_i<1, \quad (4.5)$$

where $\nu_i>0,\ i=1,\dots,n+1$.

The Dirichlet type 1 distribution, which is a multivariate generalization of the beta type 1 distribution, has been considered by several authors and is well known in the scientific literature. By making the transformation $V_j = U_j/(1-\sum_{i=1}^{n}U_i)$, $j = 1,\ldots,n$, in (4.5), the Dirichlet type 2 density, which is a multivariate generalization of beta type 2 density, is obtained as

$$\frac{\Gamma(\sum_{i=1}^{n+1}\nu_i)}{\prod_{i=1}^{n+1}\Gamma(\nu_i)}\prod_{i=1}^{n}u_i^{\nu_i-1}\left(1+\sum_{i=1}^{n}u_i\right)^{-\sum_{i=1}^{n+1}\nu_i}, \quad \nu_i>0,\ i=1,\ldots,n. \tag{4.6}$$

We will write $(V_1,\ldots,V_n)\sim \mathrm{D2}(\nu_1,\ldots,\nu_n;\nu_{n+1})$ if the joint density of $V_1,\ldots,V_n$ is given by (4.6).

Let $n_1,\ldots,n_\ell$ be nonnegative integers such that $\sum_{i=1}^{\ell}n_i = n$ and define

$$\nu_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*}\nu_j, \quad n_0^*=0, \quad n_i^* = \sum_{j=1}^{i}n_j, \quad i=1,\ldots,\ell. \tag{4.7}$$

Theorem 4.5. *Let* $(X_1,\ldots,X_n)\sim \mathrm{CH}(\nu_1,\ldots,\nu_n;\alpha,\beta,\ \text{kind }1)$. *Define* $Z_j = X_j/X_{(i)}$, $j = n_{i-1}^*+1,\ldots,n_i^*-1$ *and* $X_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*}X_j$, $i=1,\ldots,\ell$. *Then,*

(i) $(X_{(1)},\ldots,X_{(\ell)})$ *and* $(Z_{n_{i-1}^*+1},\ldots,Z_{n_i^*-1})$, $i=1,\ldots,\ell$, *are independently distributed;*

(ii) $(Z_{n_{i-1}^*+1},\ldots,Z_{n_i^*-1})\sim \mathrm{D1}(\nu_{n_{i-1}^*+1},\ldots,\nu_{n_i^*-1};\nu_{n_i^*})$, $i=1,\ldots,\ell$;

(iii) $(X_{(1)},\ldots,X_{(\ell)})\sim \mathrm{CH}(\nu_{(1)},\ldots,\nu_{(\ell)};\alpha,\beta,\ \text{kind }1)$.

Proof. Substituting $x_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*}x_j$ and $z_j = x_j/x_{(i)}$, $j = n_{i-1}^*+1,\ldots,n_i^*-1$, $i=1,\ldots,\ell$ with the Jacobian

$$\begin{aligned} J(x_1,\ldots,x_n &\longrightarrow z_1,\ldots,z_{n_1-1},x_{(1)},\ldots,z_{n_{\ell-1}^*+1},\ldots,z_{n-1},x_{(\ell)}) \\ &= \prod_{i=1}^{\ell}J(x_{n_{i-1}^*+1},\ldots,x_{n_i^*}\longrightarrow z_{n_{i-1}^*+1},\ldots,z_{n_i^*-1},x_{(i)}) \\ &= \prod_{i=1}^{\ell}x_{(i)}^{n_i-1}. \end{aligned} \tag{4.8}$$

in the density of $(X_1,\ldots,X_n)$ given by (3.1), we get the joint density of $Z_{n_{i-1}^*+1},\ldots,Z_{n_i^*-1},X_{(i)}$, $i=1,\ldots,\ell$ as

$$\begin{aligned} &C(\nu_1,\ldots,\nu_n;\alpha,\beta)\prod_{i=1}^{\ell}x_{(i)}^{\nu_{(i)}-1}\,{}_1F_1\left(\alpha;\beta;-\sum_{i=1}^{\ell}x_{(i)}\right) \\ &\quad\times\prod_{i=1}^{\ell}\left[\prod_{j=n_{i-1}^*+1}^{n_i^*-1}z_j^{\nu_j-1}\left(1-\sum_{j=n_{i-1}^*+1}^{n_i^*-1}z_j\right)^{\nu_{n_i^*}-1}\right], \end{aligned} \tag{4.9}$$

where $x_{(i)}>0$, $i=1,\ldots,\ell$, $z_j>0$, $j=n_{i-1}^*+1,\ldots,n_i^*-1$, $\sum_{j=n_{i-1}^*+1}^{n_i^*-1}z_j<1$, $i=1,\ldots,\ell$. From the factorization in (4.9), it is easy to see that $(X_{(1)},\ldots,X_{(\ell)})$ and $(Z_{n_{i-1}^*+1},\ldots,Z_{n_i^*-1})$, $i=1,\ldots,\ell$, are independently distributed. Further $(X_{(1)},\ldots,X_{(\ell)})\sim \mathrm{CH}(\nu_{(1)},\ldots,\nu_{(\ell)};\alpha,\beta,\ \text{kind }1)$ and $(Z_{n_{i-1}^*+1},\ldots,Z_{n_i^*-1})\sim \mathrm{D1}(\nu_{n_{i-1}^*+1},\ldots,\nu_{n_i^*-1};\nu_{n_i^*})$, $i=1,\ldots,\ell$. □

Corollary 4.6. *Let* $(X_1,\ldots,X_n)\sim \mathrm{CH}(\nu_1,\ldots,\nu_n;\alpha,\beta,\ \text{kind } 1)$. *Define* $Z_i = X_i/Z,\ i=1,\ldots,n-1,$ *and* $Z=\sum_{j=1}^n X_j$. *Then* $(Z_1,\ldots,Z_{n-1})$ *and* Z *are independent,* $(Z_1,\ldots,Z_{n-1})\sim \mathrm{D1}(\nu_1,\ldots,\nu_{n-1};\nu_n)$ *and* $Z\sim \mathrm{CH}(\sum_{i=1}^n \nu_i,\alpha,\beta,\ \text{kind } 1)$.

Corollary 4.7. *If* $(X_1,\ldots,X_n)\sim \mathrm{CH}(\nu_1,\ldots,\nu_n;\alpha,\beta,\ \text{kind } 1)$, *then* $\sum_{j=1}^n X_j$ *and* $\sum_{i=1}^s X_i/\sum_{i=1}^n X_i$ *are independent. Further*

$$\frac{\sum_{i=1}^s X_i}{\sum_{i=1}^n X_i}\sim \mathrm{B1}\left(\sum_{i=1}^s \nu_i,\ \sum_{i=s+1}^n \nu_i\right),\quad s<n. \tag{4.10}$$

Theorem 4.8. *Let* $(X_1,\ldots,X_n)\sim \mathrm{CH}(\nu_1,\ldots,\nu_n;\alpha,\beta,\ \text{kind } 1)$. *Define* $W_j = X_j/X_{n_i^*},\ j=n_{i-1}^*+1,\ldots,n_i^*-1$ *and* $X_{(i)}=\sum_{j=n_{i-1}^*+1}^{n_i^*} X_j,\ i=1,\ldots,\ell$. *Then,*

(i) $(X_{(1)},\ldots,X_{(\ell)})$ *and* $(W_{n_{i-1}^*+1},\ldots,W_{n_i^*-1}),\ i=1,\ldots,\ell$, *are independently distributed;*

(ii) $(W_{n_{i-1}^*+1},\ldots,W_{n_i^*-1})\sim \mathrm{D2}(\nu_{n_{i-1}^*+1},\ldots,\nu_{n_i^*-1};\nu_{n_i^*}),\ i=1,\ldots,\ell;$

(iii) $(X_{(1)},\ldots,X_{(\ell)})\sim \mathrm{CH}(\nu_{(1)},\ldots,\nu_{(\ell)};\alpha,\beta,\ \text{kind } 1)$.

Corollary 4.9. *Let* $(X_1,\ldots,X_n)\sim \mathrm{CH}(\nu_1,\ldots,\nu_n;\alpha,\beta,\ \text{kind } 1)$. *Define* $W_i = X_i/X_n,\ i=1,\ldots,n-1,$ *and* $Z=\sum_{j=1}^n X_j$. *Then* $(W_1,\ldots,W_{n-1})$ *and* Z *are independent,* $(W_1,\ldots,W_{n-1})\sim \mathrm{D2}(\nu_1,\ldots,\nu_{n-1};\nu_n)$ *and* $Z\sim \mathrm{CH}(\sum_{i=1}^n \nu_i,\alpha,\beta,\ \text{kind } 1)$.

Corollary 4.10. *If* $(X_1,\ldots,X_n)\sim \mathrm{CH}(\nu_1,\ldots,\nu_n;\alpha,\beta,\ \text{kind } 1)$, *then* $\sum_{i=1}^s X_i/\sum_{i=s+1}^n X_i$ *and* $\sum_{j=1}^n X_j$ *are independent. Further*

$$\frac{\sum_{i=1}^s X_i}{\sum_{i=s+1}^n X_i}\sim \mathrm{B2}\left(\sum_{i=1}^s \nu_i,\ \sum_{i=s+1}^n \nu_i\right),\quad s<n. \tag{4.11}$$

In the following six theorems, we give several factorizations of the multivariate confluent hypergeometric function kind 1 density.

Theorem 4.11. *Let* $(X_1,\ldots,X_n)\sim \mathrm{CH}(\nu_1,\ldots,\nu_n;\alpha,\beta,\ \text{kind } 1)$. *Define* $Y_n=\sum_{j=1}^n X_j$ *and* $Y_i=\sum_{j=1}^i X_j/\sum_{j=1}^{i+1} X_j,\ i=1,\ldots,n-1$. *Then,* $Y_1,\ldots,Y_n$ *are independent,* $Y_i\sim \mathrm{B1}(\sum_{j=1}^i \nu_j,\nu_{i+1}),\ i=1,\ldots,n-1,$ *and* $Y_n\sim \mathrm{CH}(\sum_{i=1}^n \nu_i,\alpha,\beta,\ \text{kind } 1)$.

Proof. Substituting $x_1 = y_n\prod_{i=1}^{n-1} y_i,\ x_2 = y_n(1-y_1)\prod_{i=2}^{n-1} y_i,\ldots,x_{n-1}=y_n(1-y_{n-2})y_{n-1}$ and $x_n = y_n(1-y_{n-1})$ with the Jacobian $J(x_1,\ldots,x_n\to y_1,\ldots,y_n)=\prod_{i=2}^n y_i^{i-1}$ in (3.1), we get the desired result. □

Theorem 4.12. *Let* $(X_1,\ldots,X_n)\sim \mathrm{CH}(\nu_1,\ldots,\nu_n;\alpha,\beta,\ \text{kind } 1)$. *Define* $Z_n=\sum_{j=1}^n X_j$ *and* $Z_i=X_{i+1}/\sum_{j=1}^i X_j,\ i=1,\ldots,n-1$. *Then* $Z_1,\ldots,Z_n$ *are independent,* $Z_i\sim \mathrm{B2}(\nu_{i+1},\sum_{j=1}^i \nu_j),\ i=1,\ldots,n-1,$ *and* $Z_n\sim \mathrm{CH}(\sum_{j=1}^n \nu_j,\alpha,\beta,\ \text{kind } 1)$.

Proof. The desired result follows from Theorem 4.11 by noting that $(1 - Y_i)/Y_i \sim \mathrm{B2}(\nu_{i+1}, \sum_{j=1}^{i} \nu_j)$. □

Theorem 4.13. *Let* $(X_1, \ldots, X_n) \sim \mathrm{CH}(\nu_1, \ldots, \nu_n; \alpha, \beta,\ \text{kind 1})$. *Define* $W_n = \sum_{j=1}^{n} X_j$ *and* $W_i = \sum_{j=1}^{i} X_j / X_{i+1},\ i = 1, \ldots, n-1$. *Then* $W_1, \ldots, W_n$ *are independent,* $W_i \sim \mathrm{B2}(\sum_{j=1}^{i} \nu_j, \nu_{i+1}),\ i = 1, \ldots, n-1$, *and* $W_n \sim \mathrm{CH}(\sum_{i=1}^{n} \nu_i, \alpha, \beta,\ \text{kind 1})$.

Proof. The result follows from Theorem 4.12 by noting that $1/Z_i \sim \mathrm{B2}\ (\sum_{j=1}^{i} \nu_j, \nu_{i+1})$. □

Theorem 4.14. *Let* $(X_1, \ldots, X_n) \sim \mathrm{CH}(\nu_1, \ldots, \nu_n; \alpha, \beta,\ \text{kind 1})$. *Define* $Y_n = \sum_{j=1}^{n} X_j$ *and* $Y_i = X_i / \sum_{j=i}^{n} X_j,\ i = 1, \ldots, n-1$. *Then* $Y_1, \ldots, Y_n$ *are independent,* $Y_i \sim \mathrm{B1}(\nu_i, \sum_{j=i+1}^{n} \nu_j),\ i = 1, \ldots, n-1$, *and* $Y_n \sim \mathrm{CH}(\sum_{i=1}^{n} \nu_i, \alpha, \beta,\ \text{kind 1})$.

Proof. Substituting $x_1 = y_n y_1,\ x_2 = y_n y_2 (1-y_1), \ldots, x_{n-1} = y_n y_{n-1} (1-y_1) \cdots (1-y_{n-2})$, and $x_n = y_n (1-y_1) \cdots (1-y_{n-1})$ with the Jacobian $J(x_1, \ldots, x_n \to y_1, \ldots, y_n) = y_n^{n-1} \prod_{i=1}^{n-2} (1-y_i)^{n-i-1}$ in (3.1), we get the desired result. □

Theorem 4.15. *Let* $(X_1, \ldots, X_n) \sim \mathrm{CH}(\nu_1, \ldots, \nu_n; \alpha, \beta,\ \text{kind 1})$. *Define* $Z_n = \sum_{j=1}^{n} X_j$ *and* $Z_i = X_i / \sum_{j=i+1}^{n} X_j,\ i = 1, \ldots, n-1$. *Then* $Z_1, \ldots, Z_n$ *are independent,* $Z_i \sim \mathrm{B2}(\nu_i, \sum_{j=i+1}^{n} \nu_j),\ i = 1, \ldots, n-1$, *and* $Z_n \sim \mathrm{CH}(\sum_{i=1}^{n} \nu_i, \alpha, \beta,\ \text{kind 1})$.

Proof. The result follows from Theorem 4.14 by noting that $Y_i/(1-Y_i) \sim \mathrm{B2}(\nu_i, \sum_{j=i+1}^{n} \nu_j)$. □

Theorem 4.16. *Let* $(X_1, \ldots, X_n) \sim \mathrm{CH}(\nu_1, \ldots, \nu_n; \alpha, \beta,\ \text{kind 1})$. *Define* $W_n = \sum_{j=1}^{n} X_j$ *and* $W_i = \sum_{j=i+1}^{n} X_j / X_i,\ i = 1, \ldots, n-1$. *Then* $W_1, \ldots, W_n$ *are independent,* $W_i \sim \mathrm{B2}(\sum_{j=i+1}^{n} \nu_j, \nu_i),\ i = 1, \ldots, n-1$, *and* $W_n \sim \mathrm{CH}(\sum_{i=1}^{n} \nu_i, \alpha, \beta,\ \text{kind 1})$.

Proof. The result follows from Theorem 4.15 by noting that $1/W_i \sim \mathrm{B2}(\sum_{j=i+1}^{n} \nu_j, \nu_i)$. □

Now, we consider an approximation of the multivariate confluent hypergeometric function kind 1 distribution when β increases.

Theorem 4.17. *If* $(X_1, \ldots, X_n) \sim \mathrm{CH}(\nu_1, \ldots, \nu_n; \alpha, \beta,\ \text{kind 1})$, *then*

$$\left(\frac{X_1}{\beta}, \ldots, \frac{X_n}{\beta} \right) \xrightarrow{d} (W_1, \ldots, W_n), \tag{4.12}$$

where $(W_1, \ldots, W_n) \sim \mathrm{D2}(\nu_1, \ldots, \nu_n; \alpha - \sum_{i=1}^{n} \nu_i)$, *and "* $\xrightarrow{d}$ *" denotes the convergence in distribution.*

Proof. Substituting $x_i = \beta u_i,\ i = 1, \ldots, n$ with the Jacobian $J(x_1, \ldots, x_n \to u_1, \ldots, u_n) = \beta^n$ in (3.1), the joint p.d.f. of $U_1, \ldots, U_n$ is given by

$$g(u_1, \ldots, u_n) = \frac{\Gamma(\alpha)}{\Gamma(\alpha - \sum_{i=1}^{n} \nu_i)} \left(\prod_{i=1}^{n} \frac{u_i^{\nu_i - 1}}{\Gamma(\nu_i)} \right) \frac{\Gamma(\beta - \sum_{i=1}^{n} \nu_i)}{\Gamma(\beta) \beta^{-\sum_{i=1}^{n} \nu_i}} {}_1F_1\left(\alpha; \beta; -\beta \sum_{i=1}^{n} u_i \right). \tag{4.13}$$

Now, using the results

$$\lim_{\beta\to\infty} {}_1F_1\left(\alpha;\beta;-\beta\sum_{i=1}^{n}u_i\right) = {}_1F_0\left(\alpha;-\sum_{i=1}^{n}u_i\right) = \left(1+\sum_{i=1}^{n}u_i\right)^{-\alpha},$$
$$\lim_{\beta\to\infty}\frac{\Gamma\left(\beta-\sum_{i=1}^{n}\nu_i\right)}{\Gamma(\beta)\beta^{-\sum_{i=1}^{n}\nu_i}} = 1 \tag{4.14}$$

it is easy to see that

$$\lim_{\beta\to\infty} F(u_1,\ldots,u_n) = G(u_1,\ldots,u_n), \tag{4.15}$$

where $F(u_1,\ldots,u_n)$ is the joint cumulative distribution function (cdf) of $(X_1,\ldots,X_n)/\beta$ and $G(u_1,\ldots,u_n)$ is the joint cdf of Dirichlet type 2 variables with parameters $(\nu_1,\ldots,\nu_n;\alpha-\sum_{i=1}^{n}\nu_i)$. $\square$

Acknowledgment

The research work of DKN was supported by the Comité para el Desarrollo de la Investigación, Universidad de Antioquia research Grant no. E 01252.

References

[1] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, vol. 143 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1979.

[2] B. D. Sivazlian, "On a multivariate extension of the gamma and beta distributions," *SIAM Journal on Applied Mathematics*, vol. 41, no. 2, pp. 205–209, 1981.

[3] A. K. Gupta and D. Song, "Generalized Liouville distribution," *Computers & Mathematics with Applications*, vol. 32, no. 2, pp. 103–109, 1996.

[4] R. D. Gupta and D. St. P. Richards, "The history of the Dirichlet and Liouville distributions," *International Statistical Review*, vol. 69, no. 3, pp. 433–446, 2001.

[5] D. Song and A. K. Gupta, "Properties of generalized Liouville distributions," *Random Operators and Stochastic Equations*, vol. 5, no. 4, pp. 337–348, 1997.

[6] A. K. Gupta and D. K. Nagar, *Matrix Variate Distributions*, vol. 104 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2000.

[7] Y. L. Luke, *The Special Functions and Their Approximations, Vol. 1*, Mathematics in Science and Engineering, Vol. 53, Academic Press, New York, NY, USA, 1969.

[8] D. K. Nagar and E. Zarrazola, "Distributions of the product and the quotient of independent Kummer-beta variables," *Scientiae Mathematicae Japonicae*, vol. 61, no. 1, pp. 109–117, 2005.

[9] D. K. Nagar and F. H. Sepúlveda-Murillo, "Properties of confluent hypergeometric function kind 1 distribution," *Journal of Interdisciplinary Mathematics*, vol. 11, no. 4, pp. 495–504, 2008.

[10] H. Exton, *Multiple Hypergeometric Functions and Applications*, Mathematics & Its Application, Ellis Horwood, Chichester, UK, 1976.

[11] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Series: Mathematics and Its Applications, John Wiley & Sons, New York, NY, USA, 1985.

[12] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions. Vol. 2*, Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics, John Wiley & Sons, New York, NY, USA, 2nd edition, 1995.

[13] S. Kotz, N. Balakrishnan, and N. L. Johnson, *Continuous Multivariate Distributions. Vol. 1*, John Wiley & Sons, New York, NY, USA, 2nd edition, 2000.