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Research Article

On Constructing Finite, Finitely Subadditive Outer Measures, and Submodularity

Charles Traina

Department of Mathematics & Computer Science, St. John's University, 8000 Utopia Parkway Queens, NY 11439, USA

Correspondence should be addressed to Charles Traina, trainac@stjohns.edu

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Given a nonempty abstract set X, and a covering class $\mathcal{C}$, and a finite, finitely subadditive outer measure ν, we construct an outer measure $\overline{\nu}$ and investigate conditions for $\overline{\nu}$ to be submodular. We then consider several other set functions associated with ν and obtain conditions for equality of these functions on the lattice generated by $\mathcal{C}$. Lastly, we describe a construction of a finite, finitely subadditive outer measure given an arbitrary family of subsets, $\mathcal{B}$, of X and a nonnegative, finite set function τ defined on $\mathcal{B}$.

1. Introduction

Let X be a nonempty set, and let $\mathcal{C}$ be a covering class of subsets of X such that $\emptyset,\ X \in \mathcal{C}$, with $\mathcal{C}$ closed under finite intersections, and let ν be a finite, finitely subadditive outer measure defined for all subsets of X. In [1, 2], we considered a finite, finitely subadditive outer measure $\overline{\nu}$ defined by means of ν. The outer measure $\overline{\nu}$ generalized results obtained by [3] for the case of a 0-1 valued, finitely additive measure μ defined on the algebra of subsets generated by a lattice of subsets $\mathcal{L}$ of X. In investigating properties of $\overline{\nu}$, the hypothesis that $\overline{\nu}$ be submodular is often present. We determine conditions for $\overline{\nu}$ to be submodular.

We also consider for a finite, finitely subadditive outer measure ν and covering class $\mathcal{C}$ under the hypothesis that $C \subset \mathcal{S}_\nu$, the finite, finitely additive measure $\mu = \nu/_{\mathcal{A}(\mathcal{L})} \in M(\mathcal{L})$, where $\mathcal{L} = \mathcal{L}(\mathcal{C})$ is the lattice generated by $\mathcal{C}$. There are several set functions associated with μ and ν, namely, $\nu',\ \mu',\ \mu_i,\ \rho,\ \rho'$ and it is shown that the inequality $\rho' \le \mu_i \le \rho \le \nu \le \mu' \le \nu'$ always holds for all subsets of X. We determine when an equality of all these set functions holds on $\mathcal{L}$. This is useful since for instance it allows one to conclude that μ is an $\mathcal{L}$-*regular* measure on $\mathcal{A}(\mathcal{L})$. Once again, the condition of submodularity plays an essential role.

Lastly, we consider the general problem of constructing a finite, finitely subadditive outer measure from an arbitrary family, $\mathcal{B}$, of subsets of X, with $\emptyset,\ X \in \mathcal{B}$ and $\mathcal{B}$ closed under finite unions, and a nonnegative, finite set function τ defined on $\mathcal{B}$, with $\tau(\emptyset) = 0$. Assuming that τ is finitely subadditive on $\mathcal{B}$, the function $\lambda(E) = \inf\{\tau(B)/E \subset B,\ B \in \mathcal{B}\}$ for $E \subset X$ is a finite, finitely subadditive outer measure defined for all subsets of X. We obtain a characterization of those sets $E \subset X$ that are *λ-measurable.* This provides us with a condition for the sets of $\mathcal{B}$ to be *λ-measurable* without requiring that λ be submodular.

2. Background and notation

We introduce the necessary outer measure, and covering class definitions, and note some known properties that we shall need.

The definitions and notations are standard and are consistent with those found in, for example, [1, 4–6]. We collect the ones we need and some of their properties for the reader's convenience.

Throughout this section and the rest of the paper, X will denote a nonempty abstract set. For $E \subset X$, E' will denote the complement of E. The power set of X will be denoted by $\mathcal{P}(X)$. We will be concerned with covering classes of subsets of X. By a *covering class*, we will always mean a nonempty class $\mathcal{C}$ of subsets of X such that $\emptyset,\ X \in \mathcal{C}$ and such that for any $E \subset X$, there is a finite family of sets $C_1, C_2, \dots, C_n \in \mathcal{C}$ with $E \subset \bigcup_{i=1}^{n} C_i$. We always assume that C is closed under finite intersections. We define $\mathcal{L} = \mathcal{L}(\mathcal{C})$ to be the lattice generated by $\mathcal{C}$. $\mathcal{L}(\mathcal{C})$ is the set of all finite unions of sets from $\mathcal{C}$. The algebra generated by $\mathcal{L}$ is $\mathcal{A}(\mathcal{L})$. It is known that $\mathcal{A}(\mathcal{L}) = \mathcal{A}(\mathcal{C})$, the algebra generated by $\mathcal{C}$. The set of complements of the members of $\mathcal{C}$ is denoted by $\mathcal{C}'$.

Definition 2.1. A covering class $\mathcal{C}$ is said to *coallocate itself* if whenever $C, C_1, C_2 \in \mathcal{C}$, and $C \subset C_1' \cup C_2'$, there are sets $A, B \in \mathcal{C}$ with $C = A \cup B$ and $A \subset C_1',\ B \subset C_2'$.

Definition 2.2. A covering class $\mathcal{C}$ is said to be *normal* if for any $A, B \in \mathcal{C}$ with $A \cap B = \emptyset$, there exist $C, D \in \mathcal{C}$ with $A \subset C',\ B \subset D'$, and $C' \cap D' = \emptyset$.

Definition 2.3. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be covering classes with $\mathcal{C}_1 \subset \mathcal{C}_2$. $\mathcal{C}_1$ is said to *coseparate* $\mathcal{C}_2$ if whenever $B_1, B_2 \in \mathcal{C}_2,\ B_1 \cap B_2 = \emptyset$, there exist $A_1, A_2 \in \mathcal{C}_1$ with $B_1 \subset A_1',\ B_2 \subset A_2'$, and $A_1' \cap A_2' = \emptyset$.

It is known that if a covering class coallocates itself, then it is a normal covering class.

Let ν be a finite, finitely subadditive outer measure defined for all subsets of X and let $\mathcal{C}$ be a covering class of the subsets of X. In general, there are several set functions that we can associate with this outer measure ν. For $E \subset X$, we define $\rho(E) = \nu(X) - \nu(E')$. ρ is finite valued and in general is not an inner measure. Conditions when ρ is an inner measure have been thoroughly investigated in [4, 5]. We also define a set function ν' by

$$\nu'(E) = \inf\left\{\nu(C')/E \subset C',\ C \in \mathcal{C}\right\} \quad \text{for } E \subset X, \tag{2.1}$$

where ν' is always a finite, finitely subadditive outer measure, and so it has associated with it a ρ', which may be expressed as

$$\rho'(E) = \sup\left\{\rho(C)/C \subset E,\ C \in \mathcal{C}\right\}, \quad E \subset X. \tag{2.2}$$

It is always true that $\rho' \leq \rho \leq \nu \leq \nu'$ on $\mathcal{P}(X)$. When a set function ν is an outer measure, the family of *ν-measurable sets* is denoted by $\mathcal{S}_\nu$, that is,

$$\mathcal{S}_\nu = \{M \subset X/\nu(A) = \nu(A \cap M) + \nu(A \cap M'),\ \forall A \subset X\}. \tag{2.3}$$

It is well known that $\mathcal{S}_\nu$ is an algebra and the restriction of ν to $\mathcal{S}_\nu$, that is, $\nu/_{\mathcal{S}_\nu}$ is a finite, finitely additive measure. We also define $\mathcal{S}^\nu = \{E \subset X/\nu(E) = \rho(E)\}$. $\mathcal{S}^\nu$ is not in general an algebra of sets, and we always have $\mathcal{S}_\nu \subset \mathcal{S}^\nu$.

An outer measure ν satisfies Condition A: if $\nu = \rho'$ on $\mathcal{C}'$. ν satisfies Condition B: if $\nu(C') = \sup\{\nu(A)/A \subset C',\ A \in \mathcal{C}\}$, for $C \in \mathcal{C}$.

Definition 2.4. Let $\mathcal{E}$ be a collection of subsets of X and let f be a real-valued set function defined on $\mathcal{E}$.

If for all $A, B \in \mathcal{E}$ with $A \cup B,\ A \cap B \in \mathcal{E}$, we have

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B), \tag{2.4}$$

then f is said to be *submodular on* $\mathcal{E}$. If the reverse inequality holds, then f is said to be *supermodular on* $\mathcal{E}$. f is said to be *modular on* $\mathcal{E}$ if equality holds. If

$$f(A \cup B) \geq f(A) + f(B) \quad \text{whenever } A, B \in \mathcal{E} \text{ with } A \cap B = \varnothing, \tag{2.5}$$

then f is said to be *superadditive on* $\mathcal{E}$.

If $\mathcal{E} = \mathcal{P}(X)$, we usually say f is *submodular.*

Let $\mathcal{L}$ be a lattice of subsets of X such that $\varnothing, X \in \mathcal{L}$. $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$, and $M(\mathcal{L})$ denotes the set of all nontrivial, nonnegative, finitely additive measures on $\mathcal{A}(\mathcal{L})$. $M_R(\mathcal{L})$ denotes the set of all those $\mu \in M(\mathcal{L})$ which are *$\mathcal{L}$-regular*, that is, those $\mu \in M(\mathcal{L})$ with the property that for any $A \in \mathcal{A}(\mathcal{L})$ $\mu(A) = \sup\{\mu(L)/L \subset A,\ L \in \mathcal{L}\}$.

$M(\mathcal{L})$, $M_R(\mathcal{L})$, and several other subsets of $M(\mathcal{L})$ have been extensively studied in the literature; we cite just a few recent papers [4–8]. For $\mu \in M(\mathcal{L})$, the set function μ' defined for $E \subset X$ by

$$\mu'(E) = \inf\{\mu(L')/E \subset L',\ L \in \mathcal{L}\} \tag{2.6}$$

is a finite, finitely subadditive outer measure on $\mathcal{P}(X)$. We also define the set function μ_i by

$$\mu_i(E) = \mu'(X) - \mu'(E') = \sup\{\mu(L)/L \subset E,\ L \in \mathcal{L}\}. \tag{2.7}$$

3. Submodularity of the outer measure $\overline{\nu}$

In [3], a 0-1 measure μ was used to give necessary and sufficient conditions for a lattice to be normal, by constructing a 0-1, finitely subadditive outer measure $\overline{\mu}$. In [1, 2], the measure μ was replaced by a finite, finitely subadditive outer measure ν, and the outer measure $\overline{\mu}$ was generalized by a finite, finitely subadditive outer measure $\overline{\nu}$. However, we now need to impose a separation property on the covering class, and require submodularity of the outer measures. In this section, we focus on the latter set functions, and determine conditions for $\overline{\nu}$ to be submodular.

We begin by recalling the basic construction and properties of $\overline{\nu}$. For emphasis, let X be a nonempty set and let $\mathcal{C}$ be a covering class of subsets of X, with $\emptyset, X \in \mathcal{C}$ and $\mathcal{C}$ closed under finite intersections. Let ν be a finite, finitely subadditive outer measure defined for all subsets of X. Assume that $\mathcal{C}$ coallocates itself and the associated set function ρ is finitely subadditive on $\mathcal{P}(X)$. Since ρ is finitely subadditive on $\mathcal{P}(X)$, the set function ρ' is also finitely subadditive on $\mathcal{P}(X)$ (see, e.g., [2]). For $E \subset X$ define

$$\overline{\nu}(E) = \inf\left\{\rho'(C')/E \subset C',\ C \in \mathcal{C}\right\}, \tag{3.1}$$

where $\overline{\nu}$ is a finite, finitely subadditive outer measure and

(1) $\overline{\nu} = \rho'$ on $\mathcal{C}'$,

(2) $\overline{\rho} = \nu'$ on $\mathcal{C}$,

(3) $\overline{\rho}' \leq \overline{\rho} \leq \overline{\nu} \leq \overline{\nu}'$ on $\mathcal{P}(X)$,

(4) $\overline{\nu}$ satisfies condition A, that is, $\overline{\nu} = \overline{\rho}'$ on $\mathcal{C}'$ which is equivalent to $\overline{\rho} = \overline{\nu}'$ on $\mathcal{C}$.

(The functions $\overline{\nu}'$, $\overline{\rho}$, $\overline{\rho}'$ are the usual ones associated with the outer measure $\overline{\nu}$ as defined for a general finite, finitely subadditive outer measure in Section 2).

We now determine conditions for $\overline{\nu}$ to be submodular. It will be shown that by requiring that ν or its associated set functions satisfy certain conditions on either $\mathcal{C}$ or $\mathcal{C}'$, and $\mathcal{C}$ is a lattice, the submodularity of $\overline{\nu}$ will follow. The importance of $\overline{\nu}$ being submodular can be seen from the following known facts [2].

It is known that if $\overline{\nu}$ is submodular then

(1) $\mathcal{C} \subset \mathcal{S}_{\overline{\nu}} = \mathcal{S}^{\overline{\nu}}$,

(2) $\overline{\nu}$ is $\mathcal{C}$ regular on $\mathcal{L}'$, where $\mathcal{L} = \mathcal{L}(\mathcal{C})$,

(3) $\overline{\nu}/_{\mathcal{A}(\mathcal{L})} \in M_R(\mathcal{L})$,

(4) $\nu \leq \overline{\nu}$ on $\mathcal{C}$.

Theorem 3.1. *If $\mathcal{C}$ is a lattice, and if ν' is supermodular on $\mathcal{C}$, then ρ' is submodular on $\mathcal{C}'$ and so $\overline{\nu}$ is submodular on $\mathcal{C}'$.*

Proof. Let $C_1, C_2 \in \mathcal{C}$. Since ν' is supermodular on the lattice $\mathcal{C}$,

$$\nu'(C_1 \cup C_2) + \nu'(C_1 \cap C_2) \geq \nu'(C_1) + \nu'(C_2). \tag{3.2}$$

Therefore,

$$\nu'(X) - \nu'(C_1 \cup C_2) + \nu'(X) - \nu'(C_1 \cap C_2) \leq \nu'(X) - \nu'(C_1) + \nu'(X) - \nu'(C_2), \tag{3.3}$$

and so

$$\rho'(C_1' \cap C_2') + \rho'(C_1' \cup C_2') \leq \rho'(C_1') + \rho'(C_2'). \tag{3.4}$$

Since $\overline{\nu} = \rho'$ on $\mathcal{C}'$, the above inequality shows

$$\overline{\nu}(C_1' \cap C_2') + \overline{\nu}(C_1' \cup C_2') \leq \overline{\nu}(C_1') + \overline{\nu}(C_2'). \tag{3.5}$$

Since $C_1, C_2 \in \mathcal{C}$ are arbitrary, this shows that ρ' is submodular on $\mathcal{C}'$ and $\overline{\nu}$ is submodular on $\mathcal{C}'$. □

Theorem 3.2. *Suppose $\mathcal{C}$ is a lattice. If $\overline{\nu}$ is submodular on $\mathcal{C}'$, then $\overline{\nu}$ is submodular.*

Proof. Let $A, B \subset X$ and let $\epsilon > 0$ be given and arbitrary. By definition of $\overline{\nu}$ there exist $C_1, C_2 \in \mathcal{C}$ such that $A \subset C_1'$, $B \subset C_2'$ and

$$\overline{\nu}(A) \leq \rho'(C_1') < \overline{\nu}(A) + \frac{\epsilon}{2}, \qquad \overline{\nu}(B) \leq \rho'(C_2') < \overline{\nu}(B) + \frac{\epsilon}{2}. \tag{3.6}$$

Now $A \cup B \subset C_1' \cup C_2'$, $A \cap B \subset C_1' \cap C_2'$,

$$\overline{\nu}(A \cup B) + \overline{\nu}(A \cap B) \leq \overline{\nu}(C_1' \cup C_2') + \overline{\nu}(C_1' \cap C_2'). \tag{3.7}$$

Since $\overline{\nu}$ is submodular on $\mathcal{C}'$,

$$\overline{\nu}(C_1' \cup C_2') + \overline{\nu}(C_1' \cap C_2') \leq \overline{\nu}(C_1') + \overline{\nu}(C_2') < \rho'(C_1') + \rho'(C_2'). \tag{3.8}$$

By (3.7), (3.8), (3.6), we have

$$\overline{\nu}(A \cup B) + \overline{\nu}(A \cap B) < \overline{\nu}(A) + \overline{\nu}(B) + \epsilon. \tag{3.9}$$

Since $\epsilon > 0$ is arbitrary, (3.9) shows

$$\overline{\nu}(A \cup B) + \overline{\nu}(A \cap B) \leq \overline{\nu}(A) + \overline{\nu}(B), \tag{3.10}$$

and so $\overline{\nu}$ is submodular. □

Combining Theorems 3.1 and 3.2, we have the following theorem.

Theorem 3.3. *If $\mathcal{C}$ is a lattice, and if ν' is supermodular on $\mathcal{C}$ then $\overline{\nu}$ is submodular.*

We recall that given an outer measure ν, a *measurable cover* for a set $E \subset X$ is a set $M \in \mathcal{S}_\nu$ such that $\nu(E) = \nu(M)$.

Definition 3.4. Let ν be a finite, finitely subadditive outer measure, and let $\mathcal{S}_\nu$ be the ν-measurable sets. We define for $E \subset X$

$$\nu^0(E) = \inf\{\nu(M)/E \subset M \in \mathcal{S}_\nu\}, \tag{3.11}$$

and say that ν is *approximately regular* if $\nu = \nu^0$.

ν^0 is a finite, finitely subadditive, submodular outer measure, and $\nu^0(M) = \nu(M)$ for $M \in \mathcal{S}_\nu$.

It is known that when a finite, finitely subadditive outer measure ν is approximately regular then ν is submodular [5].

Theorem 3.5. *Let $\mathcal{C}$ be a covering class and let ν be a finite, finitely subadditive outer measure. If every $C' \in \mathcal{C}'$ has a measurable cover with respect to $\overline{\nu}$, then $\overline{\nu}$ is approximately regular and therefore submodular.*

Proof. We first show that the hypothesis that each $C' \in \mathcal{C}'$ has a measurable cover with respect to $\overline{\nu}$ implies that $\overline{\nu} = \overline{\nu}^0$ on $\mathcal{C}'$.

Let $C' \in \mathcal{C}'$. We always have for any $M \in \mathcal{S}_{\overline{\nu}}$ with $C' \subset M$, $\overline{\nu}(C') \le \overline{\nu}(M)$. Therefore, by the definition of $\overline{\nu}^0(C')$,

$$\overline{\nu}(C') \le \overline{\nu}^0(C'). \tag{3.12}$$

Now by hypothesis, there exists an $M_0 \in \mathcal{S}_{\overline{\nu}}$ such that $C' \subset M_0$ and $\overline{\nu}(C') = \overline{\nu}(M_0)$. By monotonicity of $\overline{\nu}^0$,

$$\overline{\nu}^0(C') \le \overline{\nu}(M_0) = \overline{\nu}(C'). \tag{3.13}$$

By (3.12) and (3.13), $\overline{\nu}^0(C') = \overline{\nu}(C')$. Since $C' \in \mathcal{C}'$ is arbitrary, $\overline{\nu} = \overline{\nu}^0$ on $\mathcal{C}'$, so that $\overline{\nu}$ is approximately regular on $\mathcal{C}'$.

We next show that when $\overline{\nu}$ is approximately regular on $\mathcal{C}'$ implies that $\overline{\nu}$ is approximately regular.

Let $E \subset X$. Let $M \in \mathcal{S}_{\overline{\nu}}$ with $E \subset M$. By monotonicity of $\overline{\nu}, \overline{\nu}(E) \le \overline{\nu}(M)$. It follows from the definition that

$$\overline{\nu}(E) \le \overline{\nu}^0(E). \tag{3.14}$$

Suppose now that $C' \in \mathcal{C}'$ and $E \subset C'$. By monotonicity of $\overline{\nu}^0$,

$$\overline{\nu}^0(E) \le \overline{\nu}^0(C') = \overline{\nu}(C'). \tag{3.15}$$

We always have $\overline{\nu} = \rho'$ on $\mathcal{C}'$, so (3.15) shows that

$$\overline{\nu}^0(E) \le \rho'(C'). \tag{3.16}$$

Since $\mathcal{C}'$ is any set from $\mathcal{C}'$ with $E \subset C'$, (3.16) shows that

$$\overline{\nu}^0(E) \le \overline{\nu}(E). \tag{3.17}$$

By (3.14) and (3.17) , we have $\overline{\nu}(E) = \overline{\nu}^0(E)$ for any $E \subset X$. Therefore, $\overline{\nu} = \overline{\nu}^0$, so that $\overline{\nu}$ is approximately regular. It follows from a result in [5] that $\overline{\nu}$ is submodular. $\square$

An important question to consider is whether or not $\mathcal{C} \subset \mathcal{S}_{\overline{\nu}}$ without requiring that $\overline{\nu}$ is submodular. If this is true, then since $\mathcal{S}_{\overline{\nu}}$ is an algebra, $\mathcal{A}(\mathcal{L}(\mathcal{C})) = \mathcal{A}(\mathcal{L}) \subset \mathcal{S}_{\overline{\nu}}$ and thus $\overline{\nu}/_{\mathcal{A}(\mathcal{L})} \in M(\mathcal{L})$. Our next theorem shows that this is indeed the case when $\mathcal{C}$ is a lattice and ρ is superadditive on $\mathcal{C}$.

Theorem 3.6. *If $\mathcal{C}$ is a lattice, and if ρ is superadditive on $\mathcal{C}$, then $\mathcal{C} \subset \mathcal{S}_{\overline{\nu}}$.*

Proof. It suffices to show by a theorem in [2] that each $C \in \mathcal{C}$ splits all sets of $\mathcal{C}'$ additively with respect to $\overline{\nu}$.

Let $C \in \mathcal{C}$ and $A' \in \mathcal{C}'$. We always have $\overline{\nu}(A') \leq \overline{\nu}(A' \cap C) + \overline{\nu}(A' \cap C')$. For the reverse inequality, we proceed as follows. Let $\epsilon > 0$ be given and arbitrary. There exists $B, D \in \mathcal{C}$ such that

$$\begin{aligned} &B \subset A' \cap C', \qquad \rho'(A' \cap C') - \frac{\epsilon}{2} < \rho(B), \\ &D \subset A' \cap B', \qquad \rho'(A' \cap B') - \frac{\epsilon}{2} < \rho(D). \end{aligned} \tag{3.18}$$

Now $B \cap D \subset B \cap (A' \cap B') = \varnothing$, and $B \cup D \subset (A' \cap C') \cup (A' \cap B') \subset A'$. Since $\mathcal{C}$ is a lattice, $B \cup D \in \mathcal{C}$. We have $\overline{\nu}(A') = \rho'(A')$, so by the definition of $\rho'(A')$, we have

$$\overline{\nu}(A') > \rho'(A') \geq \rho(B \cup D). \tag{3.19}$$

By the superadditivity of ρ on $\mathcal{C}$,

$$\rho(B \cup D) \geq \rho(B) + \rho(D). \tag{3.20}$$

Using (3.19), (3.20), and (3.18), we have

$$\overline{\nu}(A') > \rho'(A' \cap C') + \rho'(A' \cap B') - \epsilon. \tag{3.21}$$

Again, since $\mathcal{C}$ is a lattice and $A, B, C \in \mathcal{C}$ and $\overline{\nu} = \rho'$ on $\mathcal{C}'$, we have by (3.21)

$$\overline{\nu}(A') > \overline{\nu}(A' \cap C') + \overline{\nu}(A' \cap B') - \epsilon. \tag{3.22}$$

Since $A' \cap B' \supset A' \cap (A \cup C) = A' \cap C$, by monotonicity of $\overline{\nu}$,

$$\overline{\nu}(A' \cap B') > \overline{\nu}(A' \cap C). \tag{3.23}$$

Therefore, by (3.22) and (3.23), we get $\overline{\nu}(A') > \overline{\nu}(A' \cap C') + \overline{\nu}(A' \cap C) - \epsilon$. Since $\epsilon > 0$ is arbitrary, we get $\overline{\nu}(A') \geq \overline{\nu}(A' \cap C') + \overline{\nu}(A' \cap C)$. Therefore, $\overline{\nu}(A') = \overline{\nu}(A' \cap C) + \overline{\nu}(A' \cap C')$, and C splits A' additively with respect to $\overline{\nu}$. Since $A' \in \mathcal{C}'$ is arbitrary, it follows that C splits all sets of $\mathcal{C}'$ additively with respect to $\overline{\nu}$ and so $C \in \mathcal{S}_{\overline{\nu}}$. The arbitrariness of $C \in \mathcal{C}$ shows that $\mathcal{C} \subset \mathcal{S}_{\overline{\nu}}$.

Consequently, $\mathcal{A}(\mathcal{L}) \subset \mathcal{S}_{\overline{\nu}}$ and so $\overline{\nu}/_{\mathcal{A}(\mathcal{L})} \in M(\mathcal{L})$. □

4. Equality of ν and its associated set functions on $\mathcal{L}$

In this section, we again consider a covering class $\mathcal{C}$ of subsets of a nonempty set X, $\mathcal{C}$ being closed under finite intersections, and such that $\emptyset, X \in \mathcal{C}$, and ν a finite, finitely subadditive outer measure defined for all subsets of X. We assume that $\mathcal{C} \subset \mathcal{S}_\nu$, the set of ν-measurable sets. Since $\mathcal{S}_\nu$ is an algebra of sets, $\mathcal{L} = \mathcal{L}(\mathcal{C}) \subset \mathcal{S}_\nu$, hence $\mathcal{A}(\mathcal{L}) \subset \mathcal{S}_\nu$. Therefore, $\mu = \nu/_{\mathcal{A}(\mathcal{L})} \in M(\mathcal{L})$. We consider the following set functions obtained from ν, which are defined in Section 2: $\mu', \mu_i, \rho, \nu', \rho'$. We obtain an inequality between all these set functions that holds for all subsets of X, and determine conditions for equality of these functions on $\mathcal{L}$. Equality on $\mathcal{L}$ becomes useful, since for instance we will then have $\mu \in M_R(\mathcal{L})$.

Theorem 4.1. *$\rho' \le \mu_i \le \rho \le \nu \le \mu' \le \nu'$ on $\mathcal{P}(X)$.*

Proof. Let $E \subset X$ be fixed but arbitrary. We always have

$$\rho(E) \le \nu(E), \tag{4.1}$$

so it is the other inequalities that we must establish.

Let $L \in \mathcal{L}$ with $E \subset L'$. The definition of $\mu'(E)$ shows that $\mu'(E) \le \mu'(L') = \nu(L')$. By monotonicity of ν, we have $\nu(E) \le \nu(L') = \mu(L')$ whenever $E \subset L'$, $L \in \mathcal{L}$. Thus, $\nu(E)$ is a lower bound for the set $\{\mu(L')/E \subset L', L \in \mathcal{L}\}$. Therefore,

$$\nu(E) \le \mu'(E). \tag{4.2}$$

Now suppose $C \in \mathcal{C}$ and $E \subset C'$. Since $C \in \mathcal{L}$, $\mu'(E) \le \mu(C') = \nu(C')$. Therefore, $\mu'(E)$ is a lower bound for the set $\{\nu(C')/E \subset C', C \in \mathcal{C}\}$. The definition of $\nu'(E)$ gives

$$\mu'(E) \le \nu'(E). \tag{4.3}$$

Next, consider any $C \in \mathcal{C}$ with $C \subset E$. By the definition of $\mu_i(E)$, we have $\mu(C) \le \mu_i(E)$. Since $\mu(C) = \nu(C)$, we have $\nu(C) \le \mu_i(E)$ for all $C \subset E$ with $C \in \mathcal{C}$. Since $\rho(C) \le \nu(C)$, we have $\rho(C) \le \mu_i(E)$ for all $C \in \mathcal{C}$, with $C \subset E$. Therefore, $\mu_i(E)$ is an upper bound for the set $\{\rho(C)/C \subset E, C \in \mathcal{C}\}$. The definition of $\rho'(E)$ thus gives

$$\rho'(E) \le \mu_i(E). \tag{4.4}$$

To show that $\mu_i(E) \le \rho(E)$, we argue by contradiction. Thus suppose that $\rho(E) < \mu_i(E)$. Then there exists an $L_0 \in \mathcal{L}$ such that $L_0 \subset E$ and $\rho(E) < \mu(L_0) \le \mu_i(E)$. By monotonicity of ρ, $\rho(L_0) \le \rho(E)$, so we have $\rho(L_0) < \mu(L_0)$. Now $\rho(L_0) = \nu(X) - \nu(L_0') = \mu(X) - \mu(L_0')$, hence $\mu(X) < \mu(L_0) + \mu(L_0')$. This contradicts the finite additivity of μ. Therefore, it must be

$$\mu_i(E) \le \rho(E). \tag{4.5}$$

Combining inequalities (4.1) through (4.5), we have

$$\rho'(E) \le \mu_i(E) \le \rho(E) \le \nu(E) \le \mu'(E) \le \nu'(E). \tag{4.6}$$

Since $E \subset X$ was arbitrary, we have shown that $\rho' \le \mu_i \le \rho \le \nu \le \mu' \le \nu'$ on $\mathcal{P}(X)$. □

We now determine conditions when the inequality of Theorem 4.1 will be an equality on $\mathcal{L}$. Before doing so, we indicate what happens when this is true. We always have that μ' is submodular. Assuming $\rho' = \mu_i = \rho = \nu = \mu' = \nu'$ on $\mathcal{L}$, suppose $L \in \mathcal{L}$. Then $\mu_i(L') = \mu'(X) - \mu'(L) = \mu(X) - \mu(L) = \mu(L')$ since $\mu \in M(\mathcal{L})$. Now $\mu_i(L') = \sup\{\mu(K)/K \subset L',\ K \in \mathcal{L}\}$ so we have $\mu(L') = \sup\{\mu(K)/K \subset L',\ K \in \mathcal{L}\}$. By [5], this shows since $L \in \mathcal{L}$ is arbitrary that $\mu \in M_R(\mathcal{L})$. It follows from this that $\mathcal{L} \subset \mathcal{S}_{\mu'}$, and so $(\mu')^0 = \mu,'$ so that μ' is approximately regular, hence submodular.

Theorem 4.2. *If $\mathcal{C}$ is a lattice, ν satisfies condition A, and ν is submodular on $\mathcal{C}'$, then $\rho' = \mu_i = \rho = \nu = \mu' = \nu'$ on $\mathcal{L}$.*

Proof. We show that ν' is submodular. Since ν satisfies condition A, $\mathcal{C} \subset \mathcal{S}^{\nu'}$. Let $E, F \subset X$ and let $\epsilon > 0$ be arbitrary. By the definition of ν', there exist $C_1, C_2 \in \mathcal{C}$ such that $E \subset C_1'$ and $\nu'(E) \leq \nu(C_1') < \nu'(E) + \epsilon/2$, and $F \subset C_2'$ and

$$\nu'(F) \leq \nu(C_2') < \nu'(F) + \frac{\epsilon}{2}. \tag{4.7}$$

Now $E \cup F \subset C_1' \cup C_2'$, $E \cap F \subset C_1' \cap C_2'$, where $C_1 \cap C_2$ and $C_1 \cup C_2$ belong to the lattice $\mathcal{C}$. Therefore,

$$\nu'(E \cup F) + \nu'(E \cap F) \leq \nu(C_1' \cup C_2') + \nu(C_1' \cap C_2'). \tag{4.8}$$

Since ν is submodular on $\mathcal{C}'$, $\nu(C_1' \cup C_2') + \nu(C_1' \cap C_2') \leq \nu(C_1') + \nu(C_2')$. Combining this last inequality with (4.8) and (4.7), we have

$$\nu'(E \cup F) + \nu'(E \cap F) < \nu'(E) + \nu'(F) + \epsilon. \tag{4.9}$$

Since $\epsilon > 0$ is arbitrary, this shows that $\nu'(E \cup F) + \nu'(E \cap F) \leq \nu'(E) + \nu'(F)$, where $E, F \subset X$ are arbitrary. Therefore, ν' is submodular. Hence, $\mathcal{S}^{\nu'} = \mathcal{S}_{\nu'}$ where $\mathcal{S}_{\nu'}$ is an algebra. By our initial observation, $\mathcal{C} \subset \mathcal{S}^{\nu'} = \mathcal{S}_{\nu'}$, and so $\mathcal{L} \subset \mathcal{S}^{\nu'}$. Thus for all $L \in \mathcal{L}$, $\nu'(L) = \rho'(L)$, and so by Theorem 4.1, we have $\rho' = \mu_i = \rho = \nu = \mu' = \nu'$ on $\mathcal{L}$. □

Theorem 4.3. *If $\nu = \nu^0$ on $\mathcal{C}'$, $\mathcal{C}$ coseparates $\mathcal{L}(\mathcal{C})$, and ν satisfies condition B, then $\rho' = \mu_i = \rho = \nu = \mu' = \nu'$ on $\mathcal{L}$.*

Proof. The hypothesis that $\nu = \nu^0$ on $\mathcal{C}'$ shown by [6] that ν' is submodular and so $\mathcal{S}_{\nu'} = \mathcal{S}^{\nu'}$. We show that the hypotheses that $\mathcal{C}$ coseparates $\mathcal{L}(\mathcal{C})$ and ν satisfies condition B give $\rho' = \mu_i = \rho = \nu = \mu' = \nu'$ on $\mathcal{C}$.

Since $\mathcal{C} \subset \mathcal{S}_\nu \subset \mathcal{S}^\nu$, $\nu(C) = \rho(C)$ for all $C \in \mathcal{C}$. Since for any $C \in \mathcal{C}$, $\rho'(C) = \rho(C) = \mu_i(C)$, we have

$$\rho' = \mu_i = \rho = \nu \quad \text{on } \mathcal{C}. \tag{4.10}$$

We next show $\mu' = \nu'$ on $\mathcal{C}$. To do so, we argue by contradiction. Suppose there is a $C_0 \in \mathcal{C}$ such that $\mu'(C_0) < \nu'(C_0)$. Then there is an $L_0 \in \mathcal{L}$ such that $C_0 \subset L_0'$ and

$\mu'(C_0) \le \mu(L_0') < \nu'(C_0)$. Since $C_0 \cap L_0 = \varnothing$ with $C_0, L_0 \in \mathcal{L}$ and $\mathcal{C}$ coseparates $\mathcal{L}$, there exist $A, B \in \mathcal{C}$ with $C_0 \subset A'$, $L_0 \subset B'$ and $A' \cap B' = \varnothing$. Therefore, $C_0 \subset A' \subset B \subset L_0'$ and

$$\mu'(C_0) \le \mu'(A') = \mu(A') \le \mu(B) \le \mu(L_0') < \nu'(C_0). \tag{4.11}$$

We also have $\mu(A') = \nu(A')$. Thus, we have an $A \in \mathcal{C} \subset \mathcal{L}$, with $C_0 \subset A'$ and $\nu(A') < \nu'(C_0)$. This contradicts the definition of $\nu'(C_0)$. Therefore, we must have $\mu' = \nu'$ on $\mathcal{C}$.

We next show $\nu = \mu'$ on $\mathcal{C}$. We reason by contradiction. Suppose there is a $C \in \mathcal{C}$ with $\nu(C) < \mu'(C)$. Therefore, $\nu(X) - \nu(C) > \nu(X) - \mu'(C)$. But we also have $\nu(X) = \nu(C) + \nu(C')$ so $\nu(C') = \nu(X) - \nu(C)$ and $\nu(X) = \mu(X) = \mu'(X)$. Therefore, $\nu(C') > \mu'(X) - \mu'(C) = \mu_i(C')$. Since ν satisfies condition B, $\nu(C') = \sup\{\nu(A)/A \subset C',\ A \in \mathcal{C}\} > \mu_i(C')$. Thus, there is an $A_0 \in \mathcal{C}$ with $A_0 \subset C'$ and $\mu_i(C') < \nu(A_0) \le \nu(C')$, and $A_0 \cap C = \varnothing$. Since $\mathcal{C}$ coseparates $\mathcal{L}$, there exist $B, D \in \mathcal{C}$ such that $A_0 \subset B'$, $C \subset D'$ and $B' \cap D' = \varnothing$. Therefore, $A_0 \subset B' \subset D \subset C'$. This gives

$$\mu_i(C') < \nu(A) \le \nu(B') \le \nu(D) \le \nu(C'). \tag{4.12}$$

But $\nu(D) = \mu(D)$, so we have a $D \in \mathcal{C} \subset \mathcal{L}$ with $D \subset C'$ and $\mu_i(C') < \mu(D)$. This contradicts the definition of $\mu_i(C')$. Therefore, we must have $\nu = \mu'$ on $\mathcal{C}$. Thus we have $\rho' = \mu_i = \rho = \nu = \mu' = \nu'$ on $\mathcal{C}$. Since $\rho' = \nu'$ on $\mathcal{C}$, $\mathcal{C} \subset \mathcal{S}^{\nu'}$. By the submodularity of ν', $\mathcal{S}^{\nu'} = \mathcal{S}_{\nu'}$. Therefore, $\mathcal{C} \subset \mathcal{S}_{\nu'}$ and so $\mathcal{L} \subset \mathcal{S}_{\nu'}$. Therefore, for all $L \in \mathcal{L}$, $\rho'(L) = \nu'(L)$ and so $\rho' = \mu_i = \rho = \nu = \mu' = \nu'$ on $\mathcal{L}$. □

Thus far, in order to have the equality

$$\rho' = \mu_i = \rho = \nu = \mu' = \nu' \quad \text{on } \mathcal{L}, \tag{$*$}$$

the property of submodularity of ν or ν' on $\mathcal{C}$ or $\mathcal{C}'$ played a major role. A natural question to ask is whether this equality can be achieved without submodularity. To obtain some insight into how one may proceed, we make the following observation. For the outer measure ν', it is always true that $\mathcal{S}_{\nu'} \subset \mathcal{S}^{\nu'}$, where $\mathcal{S}_{\nu'}$ is an algebra of sets. If $\mathcal{C} \subset \mathcal{S}_{\nu'}$, then $\mathcal{L} = \mathcal{L}(\mathcal{C}) \subset \mathcal{S}_{\nu'}$, and consequently $\mathcal{C} \subset \mathcal{S}^{\nu'}$, whence $\nu'(L) = \rho'(L)$ for all $L \in \mathcal{L}$. It follows from this that the equality $(*)$ holds on $\mathcal{L}$. It seems reasonable then to seek conditions for $\mathcal{C} \subset \mathcal{S}_{\nu'}$.

We have the following theorem.

Theorem 4.4. *Suppose each $C \in \mathcal{C}$ splits all sets of $\mathcal{C}'$ additively with respect to ν'. Then $\mathcal{C} \subset \mathcal{S}_{\nu'}$ and $\rho' = \mu_i = \rho = \nu = \mu' = \nu'$ on $\mathcal{L}$.*

The result follows from [2].

The results of this section prove useful in studying the restriction of a finite, finitely subadditive outer measure to the algebra generated by the covering class, when the sets of the covering class are measurable, which will be investigated in a subsequent paper . Although Theorem 4.4 may be difficult to implement in practice, it leads naturally to the content of the next section.

5. An outer measure construction

In this section, we consider a nonempty set X and the construction of a finite, finitely subadditive outer measure given an arbitrary family of subsets $\mathcal{B}$ of X, and a finite, nonnegative set function τ on $\mathcal{B}$. By the standard method, we construct a finite, finitely subadditive outer measure λ from τ and $\mathcal{B}$. We seek a characterization of those sets that split all sets of $\mathcal{B}$ additively with respect to λ.

For emphasis, let X be a nonempty set and let $\mathcal{B}$ be an arbitrary family of subsets of X with $\varnothing, X \in \mathcal{B}$ and $\mathcal{B}$ is closed under finite unions. Let τ be a finite, nonnegative set function defined on $\mathcal{B}$ with $\tau(\varnothing) = 0$, and τ subadditive on $\mathcal{B}$. For $E \subset X$ let

$$\lambda(E) = \inf\left\{\tau(B)/E \subset B,\ B \in \mathcal{B}\right\}. \tag{5.1}$$

It follows by a standard argument that λ is a finite, finitely subadditive outer measure defined for all subsets of X. Let $\mathcal{S}_\lambda$ be the algebra of λ-*measurable* subsets of X. We determine conditions when a set or a family of sets splits all sets of $\mathcal{B}$ additively with respect to λ, this set or family will split all subsets of X additively with respect to λ. The importance of this was hinted at in the last section.

We observe that if $B \in \mathcal{B}$, then $\lambda(B) \leq \tau(B)$. Also, if τ is monotone on $\mathcal{B}$, then $\lambda(B) = \tau(B)$ for all $B \in \mathcal{B}$.

The following theorem is known, and its proof can be found in [2], for instance.

Theorem 5.1. *If $E \subset X$ splits all sets of $\mathcal{B}$ additively with respect to λ, then $E \in \mathcal{S}_\lambda$.*

Our motivation for what follows comes from the characterization of the measurable sets for a σ-finite measure which can be found in [8, 9]. We begin with the observation that there are always at least two sets in $\mathcal{B}$ that split all sets of $\mathcal{B}$ additively with respect to λ, namely, $\varnothing$ and X (they may be the only ones). We seek a characterization of such sets.

Theorem 5.2. *Let $B_0 \in \mathcal{B}$ split all sets of $\mathcal{B}$ additively with respect to λ. Let $N \in \mathcal{B}$ be such that $\tau(N) = 0$. Then $E = B_0 - N$ splits all sets of $\mathcal{B}$ additively with respect to λ, and so $E \in \mathcal{S}_\lambda$.*

Proof. Let $B \in \mathcal{B}$ be arbitrary. We can write $B = (B \cap E) \cup (B \cap E')$. Now, $B \cap E = B \cap (B_0 \cap N') \subset B \cap B_0$ and $B \cap E' = B \cap (B_0' \cup N) \subset (B \cap B_0') \cup N$. By the monotonicity and finite subadditivity of λ,

$$\lambda(B \cap E) + \lambda(B \cap E') \leq \lambda\big(B \cap B_0\big) + \lambda\big(B \cap B_0'\big) + \lambda(N). \tag{5.2}$$

Since $N \in \mathcal{B}$, $\lambda(N) \leq \tau(N) = 0$, so $\lambda(N) = 0$ and (5.2) becomes

$$\lambda(B \cap E) + \lambda\big(B \cap E'\big) \leq \lambda\big(B \cap B_0\big) + \lambda\big(B \cap B_0'\big). \tag{5.3}$$

The hypothesis on B_0 gives $\lambda(B \cap B_0) + \lambda(B \cap B_0') = \lambda(B)$ so by (5.3), $\lambda(B \cap E) + \lambda(B \cap E') \leq \lambda(B)$. The reverse inequality is always true by finite subadditivity of λ. Thus by Theorem 5.1,

$$E \in \mathcal{S}_\lambda. \tag{5.4}$$

□

Thus we see that a sufficient condition for a set E to be λ-*measurable* is: E must be expressible as the difference of a set from $\mathcal{B}$ that splits all sets of $\mathcal{B}$ additively with respect to λ and a subset of X for which τ is 0.

An example may help to illustrate the theorem.

Example 5.3. Let $X = \{a,b,c\}$ and take $\mathcal{B} = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}$. Define τ on $\mathcal{B}$ by $\tau(\varnothing) = \tau(\{a\}) = 0$, $\tau(\{b\}) = \tau(\{a,b\}) = 1$, $\tau(X) = 2$. The outer measure λ determined by $\mathcal{B}$ and τ is $\lambda(\varnothing) = \lambda(\{a\}) = 0$, $\lambda(\{b\}) = \lambda(\{a,b\}) = 1$, $\lambda(\{c\}) = \lambda(\{a,c\}) = \lambda(\{b,c\}) = \lambda(X) = 2$. Now $\mathcal{S}_\lambda = \{\varnothing, X, \{a\}, \{b,c\}\}$. We see that $\varnothing = X - \varnothing$, $X = X - \varnothing$, $\{a\} = \{a\} - \varnothing$, $\{b,c\} = X - \{a\}$.

We return to characterizing those sets E that split all sets of $\mathcal{B}$ additively with respect to λ. In the style of Munroe [10], we have the following theorem.

Theorem 5.4. *Let $E \subset X$ be any set. There is a sequence of sets $\{B_n\}$ from $\mathcal{B}$ such that $E \subset B_n$ all n and $\lambda(E) = \lambda(\bigcap_{n=1}^{\infty} B_n)$.*

Proof. Let $E \subset X$ be arbitrary. By the definition of $\lambda(E)$, for each $n \in \mathbb{N}$ there is a $B_n \in \mathcal{B}$ such that $E \subset B_n$ and

$$\lambda(E) \le \tau(B_n) < \lambda(E) + \frac{1}{n}. \tag{5.5}$$

Let $B = \bigcap_{n=1}^{\infty} B_n$. Then $E \subset B \subset B_n$ all n. By monotonicity of λ,

$$\lambda(E) \le \lambda(B) \le \lambda(B_n) \quad \text{all } n. \tag{5.6}$$

Now $\lambda(B_n) \le \tau(B_n)$ all n, so by (5.6) and (5.5) we get $\lambda(E) \le \lambda(B) < \lambda(E) + 1/n$ for any $n \in \mathbb{N}$. Thus for any given $\epsilon > 0$, we can find an $n \in N$ large enough so that $0 < 1/n < \epsilon$. For any such n, $\lambda(E) \le \lambda(B) < \lambda(E) + \epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $\lambda(E) = \lambda(B)$. □

Observation 1. Suppose $E \subset X$ splits all sets $A \subset X$ additively with respect to λ. By Theorem 5.4, there exist $B_n \in \mathcal{B}$, $n \in \mathbb{N}$ with $E \subset \bigcap_{n=1}^{\infty} B_n$ and $\lambda(E) = \lambda(\bigcap_{n=1}^{\infty} B_n)$. Let $B = \bigcap_{n=1}^{\infty} B_n$. We can write $B = E \cup (B \cap E')$. Since $E \in \mathcal{S}_\lambda$, $\lambda(B) = \lambda(E) + \lambda(B \cap E')$, and since all quantities are finite, $\lambda(B \cap E') = \lambda(B) - \lambda(E) = 0$. Since $E = B - (B - E')$, let $N = B - E$ so that $\lambda(N) = 0$.

This observation establishes the following theorem.

Theorem 5.5. *If $E \in \mathcal{S}_\lambda$, then there is a sequence of sets $B_n \in \mathcal{B}$, and a set $N \subset X$ such that $E \subset B_n$ all n, and $E = (\bigcap_{n=1}^{\infty} B_n) - N$, where $\lambda(N) = 0$.*

We see that our result is analogous to what happens in the case of constructing an outer measure from a measure in the general case, see for instance Munroe [10].

In summary, to obtain a set E that splits all sets of $\mathcal{B}$ additively with respect to λ: let $B_0 \in \mathcal{B}$ be a set that splits all sets of $\mathcal{B}$ additively with respect to λ (for instance X) and let $N \subset X$ with $\tau(N) = 0$. Then $E = B_0 - N$ will split all sets of $\mathcal{B}$ additively with respect to λ, so that $E \in \mathcal{S}_\lambda$.

If $E \in \mathcal{S}_\lambda$, then there is a sequence of sets $\{B_n\}$ from $\mathcal{B}$ such that $E \subset B_n$ all n , and a set $N \subset X$ such that $\lambda(N) = 0$ and $E = (\bigcap_{n=1}^{\infty} B_n) - N$.

We note that if $\mathcal{B}$ is a δ-lattice that is, a lattice closed under countable intersections, then the set $B = \bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$.

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