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Research Article

On Rate of Approximation by Modified Beta Operators

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We establish the rate of convergence for the modified Beta operators $B_n(f,x)$, for functions having derivatives of bounded variation.

1. Introduction

In the year 1995 Gupta and Ahmad [1] proposed modified Beta operators so as to approximate Lebesgue integrable functions on R^+ as

$$B_n(f,x) = \frac{n-1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)\,dt, \quad x \in R^+, \tag{1.1}$$

where

$$b_{n,k}(x) = \frac{1}{B(k+1,n)}\frac{x^k}{(1+x)^{n+k+1}}, \qquad p_{n,k}(t) = \binom{n+k-1}{k} t^k (1+t)^{-n-k}. \tag{1.2}$$

These operators are linear positive operators and reproduce only the constantfunctions.

We set

$$\lambda_n(x,t) = \frac{n-1}{n}\int_0^t \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(s)\,ds. \tag{1.3}$$

In particular, it can be observed by the definition of Beta function that

$$\lambda_n(x,\infty) = 1. \tag{1.4}$$

Gupta and Ahmad [1] estimated asymptotic formula and error estimate for these operators in simultaneous approximation. Later Gupta et al. [2] studied the approximation properties of these operators in L_p norm and they obtained some direct results for the linear combinations. Here we continue our studies on these operators. First we consider the following class of functions.

By $DB_r(0,\infty),\ r \geq 0$ we mean the class of absolutely continuous functions f defined on $(0,\infty)$, which satisfy the following:

(i) $f(t) = O(t^r),\ t \to \infty$,

(ii) this class has a derivative f' on the interval $(0,\infty)$ coinciding a.e. with a function which is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that all functions $f \in BD_r(0,\infty)$ possess for each $c > 0$ the representation

$$f(x) = f(c) + \int_c^x \psi(t)\,dt, \quad x \geq c. \tag{1.5}$$

Very recently Gupta and Agrawal [3] and Gupta et al. [4] have obtained an interesting result on the rate of convergence of certain Durrmeyer type operators in ordinary and simultaneous approximation. İspir et al. [5] estimated similar results for Kantorovich operators for functions with derivatives of bounded variation. We now extend the study for the modified Beta operators and estimate the rate of convergence of the operators (1) for functions having derivatives of bounded variation.

2. Auxiliary Results

We shall use the following lemmas to prove our main theorem.

Lemma 2.1 ([1]). *Let the function $T_{n,m}(x)$, $m \in \aleph^0$, be defined as*

$$T_{n,m}(x) = \frac{n-1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} p_{n,k}(t)(t-x)^m\,dt. \tag{2.1}$$

Then

$$T_{n,0}(x) = 1, \qquad T_{n,1}(x) = \frac{3x+1}{n-2},$$
$$T_{n,2}(x) = \frac{2(n+7)x^2 + 2(n+5)x + 2}{(n-2)(n-3)}. \tag{2.2}$$

Also for each $x \in [0,\infty)$, *one has*

$$T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right). \tag{2.3}$$

Remark 2.2. From Lemma 2.1 and using Cauchy-Schwarz inequality, for n sufficiently large, it follows that

$$B_n(|t-x|, x) \le [T_{n,2}(x)]^{1/2} \le \sqrt{\frac{Cx(x+1)}{n}}, \tag{2.4}$$

where $C > 2$.

Lemma 2.3. *Let* $x \in (0,\infty)$, *then for* $C > 2$ *and* n *sufficiently large, one has*

$$\lambda_n(x,y) \le \frac{Cx(x+1)}{n(x-y)^2}, \quad 0 \le y < x,$$
$$1 - \lambda_n(x,z) \le \frac{Cx(x+1)}{n(z-x)^2}, \quad x < z < \infty. \tag{2.5}$$

Proof. By using Lemma 2.1, for n sufficiently large, we have

$$\lambda_n(x,y) = \frac{n-1}{n}\int_0^y \sum_{k=0}^{\infty} b_{n,k}(x)p_{n,k}(t)dt \le \frac{n-1}{n}\int_0^y \frac{(x-t)^2}{(x-y)^2}\sum_{k=0}^{\infty} b_{n,k}(x)p_{n,k}(t)dt$$
$$\le (x-y)^{-2}T_{n,2}(x) \le \frac{Cx(x+1)}{n(x-y)^2}. \tag{2.6}$$

The proof of the second inequality is similar, and we skip the details. □

3. Rate of Approximation

Our main result is stated as follows.

Theorem 3.1. *Let $f \in DB_r(0,\infty)$, $r \in N$, and $x \in (0,\infty)$. Then for n sufficiently large, one has*

$$\begin{aligned}
|B_n(f,x) - f(x)| \leq{}& \frac{C(x+1)}{n}\left(\sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-x/k}^{x+x/k}((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)\right) \\
&+ \frac{C(x+1)}{nx}\left(|f(2x) - f(x) - xf'(x^+)| + |f(x)|\right) \\
&+ \sqrt{\frac{Cx(x+1)}{n}}\left(O\left(n^{-r/2}\right) + |f'(x^+)|\right) \\
&+ \sqrt{\frac{Cx(x+1)}{4n}}|f'(x^+) - f'(x^-)| + \frac{1}{2}|f'(x^+) + f'(x^-)|\frac{3x+1}{n-2},
\end{aligned} \tag{3.1}$$

where the auxiliary function f_x is given by

$$f_x(t) = \begin{cases} f(t) - f(x^-), & 0 \leq t < x, \\ 0, & t = x, \\ f(t) - f(x^+), & x < t < \infty. \end{cases} \tag{3.2}$$

$\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$.

Proof. By mean value theorem, we have

$$\begin{aligned}
B_n(f,x) - f(x) &= \frac{n-1}{n}\int_0^\infty \sum_{k=0}^{\infty} b_{n,k}(x)p_{n,k}(t)(f(t) - f(x))dt \\
&= \int_0^\infty \int_x^t \frac{n-1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)p_{n,k}(t)(f'(u)du)dt.
\end{aligned} \tag{3.3}$$

Applying the identity

$$\begin{aligned}
f'(u) ={}& \frac{1}{2}[f'(x^+) + f'(x^-)] + (f')_x(u) + \frac{1}{2}[f'(x^+) - f'(x^-)]\operatorname{sgn}(u-x) \\
&+ \left[f'(x) - \frac{1}{2}[f'(x^+) + f'(x^-)]\right]\chi_x(u),
\end{aligned} \tag{3.4}$$

Equation (3.3) becomes (keeping in mind that the last term of the above identity vanishes)

$$\begin{aligned}
\left|B_n(f,x)-f(x)\right| \le{}& \left|\int_x^{\infty}\left(\int_x^t (f')_x(u)du\right)\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)dt\right.\\
&\left.-\int_0^{x}\left(\int_x^t (f')_x(u)du\right)\frac{n-1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)dt\right|\\
&+\frac{1}{2}\left|f'(x^+)-f'(x^-)\right|B_n(|t-x|,x)\\
&+\frac{1}{2}\left|f'(x^+)+f'(x^-)\right|B_n((t-x),x).
\end{aligned} \tag{3.5}$$

As

$$\begin{aligned}
&\frac{n-1}{n}\int_0^{\infty}\left(\int_x^t \frac{1}{2}\left[f'(x^+)-f'(x^-)\right]\operatorname{sgn}(u-x)du\right)\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)dt\\
&\quad=\frac{1}{2}\left[f'(x^+)-f'(x^-)\right]B_n(|t-x|,x),\\
&\frac{n-1}{n}\int_0^{\infty}\left(\int_x^t \frac{1}{2}\left[f'(x^+)+f'(x^-)\right]du\right)\sum_{k=0}^{\infty}b_{n,k}(x)p_{n,k}(t)dt\\
&\quad=\frac{1}{2}\left[f'(x^+)+f'(x^-)\right]B_n((t-x),x).
\end{aligned} \tag{3.6}$$

Thus in view of the above values, Lemma 2.1, and Remark 2.2, (3.5) reduces to

$$\begin{aligned}
\left|B_n(f,x)-f(x)\right| \le{}& \left|E_n(f,x)+F_n(f,x)\right| +\frac{1}{2}\left|f'(x^+)-f'(x^-)\right|\sqrt{\frac{Cx(x+1)}{n}}\\
&+\frac{1}{2}\left|f'(x^+)+f'(x^-)\right|\frac{3x+1}{n-2}.
\end{aligned} \tag{3.7}$$

In order to complete the proof of the theorem, it is sufficient to estimate the terms $E_n(f,x)$ and $F_n(f,x)$. Applying integration by parts, using Lemma 2.3 and taking $y = x - x/\sqrt{n}$, we have

$$|F_n(f,x)| = \left|\int_0^x \left(\int_x^t (f')_x(u)du\right) d_t\lambda_n(x,t)\right|$$

$$\begin{aligned}\int_0^x \lambda_n(x,t)(f')_x(t)dt &\leq \left(\int_0^y + \int_y^x\right)|(f')_x(t)||\lambda_n(x,t)|dt \\ &\leq \frac{Cx(x+1)}{n}\int_0^y \bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt + \int_y^x \bigvee_t^x((f')_x)dt \\ &\leq \frac{Cx(x+1)}{n}\int_0^y \bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}((f')_x).\end{aligned} \tag{3.8}$$

Let $u = x/(x-t)$. Then we have

$$\begin{aligned}\frac{Cx(x+1)}{n}\int_0^y \bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt &= \frac{Cx(x+1)}{n}\int_1^{\sqrt{n}} \bigvee_{x-x/u}^{x}((f')_x)du \\ &\leq \frac{Cx(x+1)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x}((f')_x).\end{aligned} \tag{3.9}$$

Thus,

$$|F_n(f,x)| \leq \frac{Cx(x+1)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/u}^{x}((f')_x) + \frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x}((f')_x). \tag{3.10}$$

On the other hand, we have

$$\begin{aligned}|E_n(f,x)| &= \left|\frac{n-1}{n}\int_x^\infty \left(\int_x^t (f')_x(u)du\right)\sum_{k=0}^{\infty} b_{n,k}(x)p_{n,k}(t)dt\right| \\ &= \left|\frac{n-1}{n}\int_{2x}^\infty \left(\int_x^t (f')_x(u)du\right)\sum_{k=0}^{\infty} b_{n,k}(x)p_{n,k}(t)dt\right. \\ &\quad \left.+\int_x^{2x}\left(\int_x^t (f')_x(u)du\right)d_t(1-\lambda_n(x,t))\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \left| \frac{n-1}{n} \int_{2x}^{\infty} (f(t) - f(x)) \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) dt \right| \\
&\quad + \left| f'(x^+) \right| \left| \frac{n-1}{n} \int_{2x}^{\infty} (t-x) \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) dt \right| \\
&\quad + \left| \int_{x}^{2x} (f')_x(u) du \right| |1 - \lambda_n(x, 2x)| + \int_{x}^{2x} \left| (f')_x(t) \right| |1 - \lambda_n(x,t)| dt \\
&\leq \frac{C_1}{x} \frac{n-1}{n} \int_{2x}^{\infty} \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) t^r |t-x| dt \\
&\quad + \frac{|f(x)|}{x^2} \frac{n-1}{n} \int_{2x}^{\infty} \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) (t-x)^2 dt \\
&\quad + \left| f'(x^+) \right| \frac{n-1}{n} \int_{2x}^{\infty} \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) |t-x| dt + \frac{C(x+1)}{nx} \left| f(2x) - f(x) - x f'(x^+) \right| \\
&\quad + \frac{C(x+1)}{n} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x}^{x+x/k} ((f')_x) + \frac{x}{\sqrt{n}} \bigvee_{x}^{x+x/\sqrt{n}} ((f')_x).
\end{aligned}
\tag{3.11}
$$

For the estimation of the first two terms in the right-hand side of (3.11). we proceed as follows.

Applying Holder's inequality, Remark 2.2, and Lemma 2.1, we have

$$
\begin{aligned}
&\frac{C_1}{x} \frac{n-1}{n} \int_{2x}^{\infty} \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) t^r |t-x| dt + \frac{|f(x)|}{x^2} \frac{n-1}{n} \int_{2x}^{\infty} \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) (t-x)^2 dt \\
&\quad \leq \frac{C_1}{x} \left(\frac{n-1}{n} \int_{2x}^{\infty} \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) t^{2r} dt \right)^{1/2} \left(\int_{0}^{\infty} \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) (t-x)^2 dt \right)^{1/2} \\
&\qquad + \frac{|f(x)|}{x^2} \left(\frac{n-1}{n} \int_{2x}^{\infty} \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) (t-x)^2 dt \right) \\
&\quad \leq C_1 O\left(n^{-r/2}\right) \frac{\sqrt{Cx(x+1)}}{\sqrt{n}} + |f(x)| \frac{C(x+1)}{nx}.
\end{aligned}
\tag{3.12}
$$

Also, by Remark 2.2, the third term of the right side of (3.11) is given by

$$
\begin{aligned}
\left| f'(x^+) \right| \frac{n-1}{n} \int_{2x}^{\infty} \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) |t-x| dt &\leq \left| f'(x^+) \right| \frac{n-1}{n} \int_{0}^{\infty} \sum_{k=0}^{\infty} b_{n,k}(x) p_{n,k}(t) |t-x| dt \\
&\leq \left| f'(x^+) \right| \frac{\sqrt{Cx(x+1)}}{\sqrt{n}}.
\end{aligned}
\tag{3.13}
$$

Thus

$$
\begin{aligned}
E_n(f,x) \le{}& C_1 O\left(n^{-r/2}\right)\frac{\sqrt{Cx(x+1)}}{\sqrt{n}} + \left|f(x)\right|\frac{C(x+1)}{nx} + \left|f'(x^+)\right|\frac{\sqrt{Cx(x+1)}}{\sqrt{n}} \\
&+ \frac{C(x+1)}{nx}\left|f(2x)-f(x)-xf'(x^+)\right| \\
&+ \frac{C(x+1)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+x/k}\left((f')_x\right) + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+x/\sqrt{n}}\left((f')_x\right).
\end{aligned}
\tag{3.14}
$$

Combining the estimates (3.7), (3.10), and (3.14) we get the desired result. This completes the proof of the theorem. □

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