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Research Article

On the Existence, Uniqueness, and Basis Properties of Radial Eigenfunctions of a Semilinear Second-Order Elliptic Equation in a Ball

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We consider the following eigenvalue problem: $-\Delta u + f(u) = \lambda u$, $u = u(x)$, $x \in B = \{x \in \mathbb{R}^3 : |x| < 1\}$, $u(0) = p > 0$, $u|_{|x|=1} = 0$, where p is an arbitrary fixed parameter and f is an odd smooth function. First, we prove that for each integer $n \geq 0$ there exists a radially symmetric eigenfunction u_n which possesses precisely n zeros being regarded as a function of $r = |x| \in [0,1)$. For $p > 0$ sufficiently small, such an eigenfunction is unique for each n. Then, we prove that if $p > 0$ is sufficiently small, then an arbitrary sequence of radial eigenfunctions $\{u_n\}_{n=0,1,2,\ldots}$, where for each n the nth eigenfunction u_n possesses precisely n zeros in $[0,1)$, is a basis in $L_2^r(B)$ ($L_2^r(B)$ is the subspace of $L_2(B)$ that consists of radial functions from $L_2(B)$. In addition, in the latter case, the sequence $\{u_n/\|u_n\|_{L_2(B)}\}_{n=0,1,2,\ldots}$ is a Bari basis in the same space.

1. Introduction, Notation and Definitions, and Results

In the present paper, we consider the problem

$$-\Delta u + f(u) = \lambda u, \quad u = u(|x|),\ x \in B = \left\{x \in \mathbb{R}^3 : |x| < 1\right\}, \tag{1.1}$$

$$u(0) = p > 0, \tag{1.2}$$

$$u|_{|x|=1} = 0, \tag{1.3}$$

where f is an odd continuously differentiable function, $\lambda \in \mathbb{R}$ is a spectral parameter, and p is an arbitrary positive fixed parameter. Problems of this type may arise in particular in the solid state physics, heat and diffusion theory, in the theory of nonlinear waves, and so forth. Hereafter in the paper, all the quantities we deal with are real.

We restrict our attention to the radial eigenfunctions of problem (1.1)–(1.3), that is, to the eigenfunctions u that depend only on $r = |x|$. Under our assumptions, the problem has an infinite sequence of radial eigenfunctions $\{u_n\}_{n=0,1,2,\ldots}$ such that for each integer n the nth eigenfunction u_n regarded as a function of r has precisely n zeros in the interval $[0,1)$. The main question we are interested in the present paper is whether such a sequence of eigenfunctions is a basis in a commonly used space, such as the subspace $L_2^r(B)$ of $L_2(B)$ that consists of all radial functions from $L_2(B)$. According to our result below, this is true if $p>0$ is sufficiently small.

For a discussion of the pertinence of our formulation of the problem (note that problem (1.1)–(1.3) includes an unusual normalization condition (1.2)) and for a longer list of references, we refer the reader to our quite recent review paper [1]. Here we note only that our formulation of the problem is "good" in the sense that, as in the linear case, our problem has an infinite sequence of radial eigenfunctions u_n, where for each integer $n \geq 0$ the nth eigenfunction u_n, regarded as a function of the argument $r \in [0,1)$, possesses precisely n zeros and, if $p > 0$ is sufficiently small, such a sequence of eigenfunctions is a basis in the space $L_2^r(B)$. But if one excludes the normalization condition (1.2) from the statement of the problem, then the set of all eigenfunctions becomes too wide; it would contain "a lot of" bases. It is a separate question what normalization condition should be imposed. The author believes that this question may be answered only in the future if/when the field becomes developed sufficiently; in particular, an applied problem may give such an answer. In this context, the reader may consider our system (1.1)–(1.3) as a model problem.

We mention especially our paper [2] (see also [1]) in which a problem analogous to (1.1)–(1.3) was studied in the spatial dimension 1. It is proved in these two articles that if assumption (f) is valid (see below) and if in addition $f(u)/u$ is a nondecreasing function of $u > 0$, then this one-dimensional problem has a unique sequence of eigenfunctions $\{\varphi_n\}_{n=0,1,2,\ldots}$ such that for each n the nth eigenfunction has precisely n zeros in $[0,1)$ and this sequence of eigenfunctions is a basis (in addition, a Riesz basis) in $L_2(0,1)$ while the sequence of normalized eigenfunctions $\{\varphi_n/\|\varphi_n\|_{L_2(0,1)}\}_{n=0,1,2,\ldots}$ is a Bari basis in the same space (we will establish precise definitions in what follows).

Now, we will introduce some *notation* and *definitions*. Let $L_2 = L_2(B)$ be the standard Lebesgue space of functions $g, h, \ldots$ square integrable over B, equipped with the scalar product $(g,h)_{L_2(B)} = \int_B g(x)h(x)dx$ and the corresponding norm $\|\cdot\|_{L_2(B)} = (\cdot,\cdot)_{L_2(B)}^{1/2}$. By $L_2^r = L_2^r(B)$ we denote the subspace of the space $L_2(B)$ that consists of all radial functions from $L_2(B)$ and is equipped with the same scalar product and the norm. Let $L_{2,r^2}(0,1)$ denote the usual weighted Lebesgue space of functions g measurable in $(0,1)$ for which

$$\|g\|_{L_{2,r^2}(0,1)} := \left\{\int_0^1 r^2 g^2(r)dr\right\}^{1/2} < \infty. \tag{1.4}$$

The space $L_{2,r^2}(0,1)$ is equipped with the corresponding scalar product. In fact, $L_2(B)$, L_2^r, and $L_{2,r^2}(0,1)$ are Hilbert spaces.

Let H be a separable Hilbert space over the field of real numbers in which the scalar product and the norm are denoted $(\cdot,\cdot)_H$ and $\|\cdot\|_H$, respectively. We recall that a sequence $\{h_n\}_{n=0,1,2,\ldots} \subset H$ is called a *(Schauder) basis* in H if for any $h \in H$ there exists a unique sequence of real numbers $\{a_n\}_{n=0,1,2,\ldots}$ such that

$$h = \sum_{n=0}^{\infty} a_n h_n \quad \text{in } H. \tag{1.5}$$

Two sequences $\{h_n\}_{n=0,1,2,\ldots}$ and $\{e_n\}_{n=0,1,2,\ldots}$ from H are called *quadratically close to each other* (or the sequence $\{h_n\}_{n=0,1,2,\ldots}$ is called *quadratically close* to the sequence $\{e_n\}_{n=0,1,2,\ldots}$) if

$$\sum_{n=0}^{\infty} \|h_n - e_n\|_H^2 < \infty. \tag{1.6}$$

A basis $\{h_n\}_{n=0,1,2,\ldots}$ in H quadratically close to an orthonormal basis $\{e_n\}_{n=0,1,2,\ldots}$ in H is called a *Bari basis* in H. According to Corollary 2.5 in [1], if $\{h_n\}_{n=0,1,2,\ldots}$ is an arbitrary sequence of elements of H and if

$$\sum_{n=0}^{\infty} \|h_n - e_n\|_H^2 < 1, \tag{1.7}$$

where $\{e_n\}_{n=0,1,2,\ldots}$ is an orthonormal basis in H, then $\{h_n\}_{n=0,1,2,\ldots}$ is a Bari basis in H. Bari bases being compared with bases have additional nice properties that we do not discuss in the present paper (on this subject, see, e.g., [3]). Some general aspects of the theory of nonorthogonal expansions in a Hilbert space are considered in [1, 3].

We call a sequence of radial eigenfunctions $\{u_n\}_{n=0,1,2,\ldots}$ of problem (1.1)–(1.3) *standard* if for each integer $n \geq 0$ the nth eigenfunction u_n regarded as a function of r possesses precisely n zeros in the interval $[0, 1)$. Everywhere we assume the following.

(f) *Let f be a continuously differentiable odd function in $\mathbb{R}$ and let $f'(0) = 0$.*

Note that the assumption that $f'(0) = 0$ is not restrictive. One can achieve this for an arbitrary odd continuously differentiable function by a shift of the spectrum.

Consider the following linear eigenvalue problem:

$$\begin{gathered} -\Delta v = \mu v, \quad v = v(|x|), \ x \in B, \\ v(0) = p, \\ v|_{|x|=1} = 0, \end{gathered} \tag{1.8}$$

where $\mu \in \mathbb{R}$ is a spectral parameter. Denote by $\{v_n\}_{n=0,1,2,\ldots}$ the sequence of the radial eigenfunctions of problem (1.8) where, for each integer $n \geq 0$, the nth eigenfunction v_n regarded as a function of $r \in [0, 1)$ possesses precisely n zeros. By

$$\mu_0 < \mu_1 < \cdots < \mu_n < \cdots \tag{1.9}$$

we denote the corresponding sequence of eigenvalues. Note that $\{v_n\}_{n=0,1,2,\ldots}$ is an orthogonal basis in L_2^r. Our main results here are as follows.

Theorem 1.1. *Under assumption (f)*

(a) *for any integer $n \geq 0$ problem* (1.1)–(1.3) *has a radial eigenfunction u_n which, being regarded as a function of r, possesses precisely n zeros in the interval* $[0, 1)$*;*

(b) $|u(r)| \leq p$ *for any $r \in [0, 1]$ and for an arbitrary radial eigenfunction u of problem* (1.1)–(1.3)*;*

(c) *let in addition to assumption (f) $f(u)/u$ be a nondecreasing function of $u > 0$. Then, the positive radial eigenfunction u_n is unique;*

(d) *under assumption (f) there exists $p_0 > 0$ such that for any $p \in (0, p_0]$ and any integer $n \geq 0$ the radial eigenfunction u_n of problem* (1.1)–(1.3) *that, being regarded as a function of r, has precisely n zeros in $[0,1)$ is unique.*

Theorem 1.2. *Let assumption (f) be valid. Then, there exists $\kappa = \kappa(p) > 0$ defined for all $p > 0$ and going to 0 as $p \to +0$ such that for any $p > 0$*

$$\sum_{n=0}^{\infty} \left\| \frac{u_n}{\|u_n\|_{L_2}} - \frac{v_n}{\|v_n\|_{L_2}} \right\|_{L_2}^2 < \kappa(p) \tag{1.10}$$

for an arbitrary standard sequence $\{u_n\}_{n=0,1,2,\dots}$ of eigenfunctions of problem (1.1)–(1.3). *Consequently, if $p > 0$ is sufficiently small, an arbitrary standard sequence of eigenfunctions, which is unique for $p > 0$ sufficiently small by Theorem 1.1, is a basis in L_2^r and, in addition, the sequence $\{u_n/\|u_n\|_{L_2}\}_{n=0,1,2,\dots}$ is a Bari basis in the same space.*

Remark 1.3. In view of Theorem 1.2 and the Bari's theorem (see [1, 3]), if one proves the linear independence, in the sense of the space L_2^r, of a standard sequence of eigenfunctions of problem (1.1)–(1.3) when $p > 0$ is not necessarily sufficiently small, then this sequence of eigenfunctions will be proved to be a basis in L_2^r and the sequence of the normalizations of these eigenfunctions to 1 is a Bari basis in the same space, too. However, in the present paper, we leave open the question about the linear independence of such a system when $p > 0$ is not small.

In the next section, we will prove Theorem 1.1, and in Section 3, Theorem 1.2.

2. Proof of Theorem 1.1

Proofs of results of the type of Theorem 1.1(a) are known now (on this subject, see, e.g., [4]), so we will establish only a sketch of the proof of this claim. In the class of radial solutions, problem (1.1)–(1.3) reduces to the following one:

$$-\left(r^2 u'\right)' + r^2 f(u) = \lambda r^2 u, \quad u = u(r), \ r \in (0,1), \tag{2.1}$$

$$u(0) = p, \tag{2.2}$$

$$u'(0) = u(1) = 0, \tag{2.3}$$

where the prime means the derivative in r. Equation (2.1) can also be rewritten in the following equivalent form:

$$-u'' - \frac{2}{r}u' + f(u) = \lambda u, \quad u = u(r), \ r \in (0,1). \tag{2.4}$$

We supply (2.4) with the following initial data:

$$u(0) = p, \qquad u'(0) = 0. \tag{2.5}$$

A solution of (2.4) and (2.5) that satisfies the condition $u(1) = 0$ is a solution of problem (2.1)–(2.3). In (2.1) and (2.4), $r = 0$ is a singular point. However, for problem (2.4)-(2.5), local existence, uniqueness, and continuous dependence theorems in their usual form are valid (for proofs of these claims, see, e.g., [4]). Let $u(r)$ be a solution of (2.4)-(2.5). Then, $u''(0) = (1/3)[f(p) - \lambda p]$, the derivative $u_\lambda = du/d\lambda$ exists, is continuous and it satisfies the equations

$$-\left(r^2 u_\lambda'\right)' + r^2 f'(u(r))u_\lambda = \lambda r^2 u_\lambda + r^2 u, \quad u_\lambda = u_\lambda(r), \ r \in (0,1), \tag{2.6}$$

$$u_\lambda(0) = u_\lambda'(0) = 0. \tag{2.7}$$

As above, for problem (2.6)-(2.7) local existence and uniqueness theorems in their usual form, as far as the theorem of the continuous dependence on λ, take place. In addition, since (2.6) is linear with respect to u_λ, the solution u_λ of problem (2.6)-(2.7) exists for all those values of $r > 0$, for which the solution $u(r)$ of problem (2.4)-(2.5) exists.

Let us prove statement (a) of Theorem 1.1. Multiply (2.4) by $u'(r)$ and integrate the result from 0 to r. Then, we obtain the identity

$$\left\{\frac{1}{2}u'^2(r) + \frac{\lambda}{2}\left[u^2(r) - p^2\right] + F(p) - F(u(r))\right\}' = -\frac{2}{r}u'^2(r), \tag{2.8}$$

where $F(u) = \int_0^u f(s)ds$. Denote $E(u(r)) = (1/2)u'^2(r) + (\lambda/2)[u^2(r) - p^2] + F(p) - F(u(r))$.

Lemma 2.1. *If $\lambda \in \mathbb{R}$ is such that $u''(0) > 0$, where $u(r)$ is the corresponding solution of problem* (2.4)-(2.5)*, then there is no point $r > 0$ such that $u(r) = p$. In particular, $u(r) > 0$ for any $r > 0$ so that u is not an eigenfunction of problem* (2.1)–(2.3).

Proof. On the contrary, suppose that $u''(0) > 0$ and there exists $r > 0$ such that $u(r) = p$. But then, $u'^2(r) \geq 0$, therefore $E(u(r)) \geq 0$ which contradicts (2.8). □

Lemma 2.2. *Let $u''(0) \leq 0$. Then, $|u(r)| \leq p$ for all $r \in (0,1]$.*

Proof. It can be made by analogy (if $u''(0) = 0$, then $u(r) \equiv p$ by the uniqueness theorem). □

Note that Lemmas 2.1 and 2.2 yield statement (b) of Theorem 1.1. Note in addition that if $u''(0) \leq 0$, then the solution of problem (2.4)-(2.5) is global (i.e., it can be continued on the entire half-line $r > 0$).

Observe now that $u''(0) > 0$ for all sufficiently large $|\lambda|$, $\lambda < 0$, and $u''(0) < 0$ for all sufficiently large $\lambda > 0$. By Lemma 2.2, $|u(r)| \leq p$ for all $\lambda > 0$ sufficiently large. Therefore, comparing (2.4)-(2.5) and (1.8) (one should rewrite system (1.8) in the form analogous to (2.4)-(2.5)), we see that, according to the standard comparison theorem, the number of zeros in $(0,1)$ of $u(r)$ increases unboundedly when $\lambda > 0$ unboundedly increases.

Take an arbitrary integer $n \geq 0$ and denote by Λ_n the set of all values of λ for each of which the solution $u(r)$ of (2.4)-(2.5) has at least $(n+1)$ zeros in $(0,1)$. Let $\lambda_n = \inf \Lambda_n$. Denote by $u_n(r)$ the solution of problem (2.4)-(2.5) taken with $\lambda = \lambda_n$. Then, $|u_n(r)| \le p$ for all r so that this solution is global. Observe that, if $u(r) \not\equiv 0$ is a solution of (2.4) and if $u(r_0) = 0$, then $u'(r_0) \neq 0$ by the uniqueness theorem. Therefore, zeros of $u_n(r)$ are isolated and hence, u_n has a finite number m of zeros in $(0,1)$. If $m > n$, then there exists $\lambda < \lambda_n$ sufficiently close to λ_n such that the corresponding solution $u(r)$ of (2.4)-(2.5) has no less than $m > n$ zeros in this interval, which contradicts our definition of the set Λ_n. By analogy, if $m < n$ or if $m = n$ and $u_n(1) \neq 0$, then any solution of (2.4) and (2.5) taken for $\lambda > \lambda_n$ sufficiently close to λ_n has no more than n zeros in $(0,1)$ which contradicts our definition of the set Λ_n. Thus, $u_n(r)$ has precisely n zeros in the interval $(0,1)$ and $u_n(1) = 0$. So, claim (a) of Theorem 1.1 is proved.

Let us prove claim (c). On the contrary, suppose that there exist two positive eigenfunctions u^1 and u^2 of problem (2.1)–(2.3) corresponding to the eigenvalues λ^1 and λ^2, respectively, where $\lambda^1 < \lambda^2$. By (2.8), $u^{i'}(r) < 0$ for any $r \in (0,1]$. Indeed, if it would be $u^i(r) = u^{i'}(r) = 0$, then $u^i \equiv 0$, while if $u^i(r) > 0$ and $u^{i'}(r) < 0$ for $r \in (0, r_0)$ and if $u^{i'}(r_0) = 0$, then $u^i(r) \ge u^i(r_0)$ for all $r > r_0$ in view of (2.4) and by the same arguments as in the proof of Lemma 2.1.

Now, we apply a variant of the result from [5].

Lemma 2.3. *One has $u^{2'}(r)/u^2(r) < u^{1'}(r)/u^1(r)$ for any $r \in (0,1)$.*

Proof. We have $u^{2''}(0) < u^{1''}(0) < 0$, therefore

$$\frac{u^{1'}(r)}{u^1(r)} > \frac{u^{2'}(r)}{u^2(r)}, \quad u^2(r) < u^1(r) < p \tag{2.9}$$

in a right half-neighborhood of the point $r = 0$. Let us prove that (2.9) holds everywhere in $(0,1)$. Suppose that the first inequality (2.9) holds in some interval $(0,a)$, where $a \in (0,1)$. Integrate it from 0 to $r \in (0,a]$. Then

$$\ln u^1(r) - \ln p > \ln u^2(r) - \ln p, \tag{2.10}$$

therefore $u^1(r) > u^2(r)$ until the first inequality (2.9) holds.

Suppose that the first inequality (2.9) is valid in an interval $(0,a)$, $a \in (0,1)$, and that it is violated at the point $r = a$. Note that, as is proved above, $u^1(a) > u^2(a)$. But then, by (2.4),

$$\left(\frac{u^{1'}(r)}{u^1(r)}\right)'\bigg|_{r=a} > \left(\frac{u^{2'}(r)}{u^2(r)}\right)'\bigg|_{r=a}, \tag{2.11}$$

hence, by continuity (2.11) is still valid in a left half-neighborhood of the point $r = a$ so that it must be $u^{1'}(a)/u^1(a) > u^{2'}(a)/u^2(a)$, which is a contradiction. □

Now, suppose that (2.9) holds everywhere in $(0,1)$ and that $u^i(1) = 0$, $i = 1,2$. Denote $t_1 = u^{1'}(1)$, $t_2 = u^{1''}(1)$, $s_1 = u^{2'}(1)$, and $s_2 = u^{2''}(1)$. From (2.4),

$$t_2 = -2t_1, \qquad s_2 = -2s_1. \tag{2.12}$$

In a neighborhood of the point $r = 1$, one has

$$\frac{u^{1'}(r)}{u^1(r)} = \frac{t_1 - 2t_1(r-1) + O\left((r-1)^2\right)}{t_1(r-1) - t_1(r-1)^2 + O\left((r-1)^3\right)} = \frac{1}{r-1} - 1 + O(r-1), \tag{2.13}$$

and by analogy

$$\frac{u^{2'}(r)}{u^2(r)} = \frac{1}{r-1} - 1 + O(r-1). \tag{2.14}$$

So, we see that the difference

$$r^{-1}\left(\frac{u^{1'}(r)}{u^1(r)} - \frac{u^{2'}(r)}{u^2(r)}\right) \tag{2.15}$$

goes to 0 as $r \to 1-0$. But then $(u^{1'}(r)/u^1(r))' - (u^{2'}(r)/u^2(r))' > 0$ in a left half-neighborhood of the point $r = 1$ (because by (2.9) $(u^{1'}(r)/u^1(r))^2 - (u^{2'}(r)/u^2(r))^2 \leq 0$ in $(0,1)$), and since in addition $u^{1'}(r)/u^1(r) > u^{2'}(r)/u^2(r)$ in $(0,1)$, and we arrive at a contradiction as earlier. Thus, claim (c) of Theorem 1.1 is proved.

Now, we turn to proving claim (d) of Theorem 1.1. Let us prove the following.

Lemma 2.4. *There exist $\overline{p} > 0$ and $C > 0$ such that for any $p \in (0, \overline{p}]$, $a \in (0,1]$, $\lambda \in \mathbb{R}$, and the corresponding solution $u(r)$ of* (2.4)-(2.5) *which satisfies $u(a) = 0$, one has*

$$\|u_\lambda\|^2_{L_{2,r^2}(0,a)} \leq C\|u\|^2_{L_{2,r^2}(0,a)}. \tag{2.16}$$

Proof. First, multiply (2.6) by u'_λ and integrate the result from 0 to $r \in (0,a]$. Then, after integration by parts, we obtain

$$\begin{aligned}\frac{1}{2}r^2 u'^2_\lambda(r) + \frac{\lambda}{2}r^2 u^2_\lambda(r) + \int_0^r s u'^2_\lambda(s)ds \\ = \lambda\int_0^r s u^2_\lambda(s)ds + \int_0^r s^2 f'(u(s))u_\lambda(s)u'_\lambda(s)ds - \int_0^r s^2 u(s)u'_\lambda(s)ds.\end{aligned} \tag{2.17}$$

Second, multiply (2.6) by $r^{-1}u_\lambda(r)$ and integrate the result from 0 to r. Then, after integration by parts,

$$\begin{aligned}\frac{1}{2}u^2_\lambda(r) + \lambda\int_0^r s u^2_\lambda(s)ds = \int_0^r s u'^2_\lambda(s)ds - r u_\lambda(r)u'_\lambda(r) \\ + \int_0^r s f'(u(s))u^2_\lambda(s)ds - \int_0^r s u(s)u_\lambda(s)ds.\end{aligned} \tag{2.18}$$

Now, add (2.18) to (2.17). Then, we obtain the following identity:

$$\begin{aligned}
&\frac{1}{2}r^2u_\lambda'^2(r)+\frac{\lambda}{2}r^2u_\lambda^2(r)+\frac{1}{2}u_\lambda^2(r)\\
&\quad=-ru_\lambda(r)u_\lambda'(r)+\int_0^r sf'(u(s))u_\lambda^2(s)ds-\int_0^r s^2u(s)u_\lambda'(s)ds\\
&\qquad+\int_0^r s^2f'(u(s))u_\lambda(s)u_\lambda'(s)ds-\int_0^r su(s)u_\lambda(s)ds.
\end{aligned}\tag{2.19}$$

Now, take an arbitrary $\overline{p}>0$ and integrate (2.19) for one more time from 0 to r. Then, applying the inequality $ab\leq(1/2t)a^2+(t/2)b^2$, $t>0$,

$$\begin{aligned}
&\frac{1}{2}\int_0^r s^2\left[u_\lambda'^2(s)+\lambda u_\lambda^2(s)\right]ds+\frac{1}{2}ru_\lambda^2(r)\\
&\quad\leq C_1(\overline{p})\int_0^r dt\left(\int_0^t\left(s+s^2\right)u_\lambda^2(s)ds\right)+\frac{1}{2}\int_0^r dt\left(\int_0^t s^2u_\lambda'^2(s)ds\right)\\
&\qquad+\frac{1}{2}\int_0^r dt\left(\int_0^t u_\lambda^2(s)ds\right)+\frac{3}{2}\int_0^r dt\left(\int_0^t s^2u^2(s)ds\right)\\
&\quad\leq C_2(\overline{p})r\int_0^r su_\lambda^2(s)ds+\frac{r}{2}\int_0^r s^2u_\lambda'^2(s)ds\\
&\qquad+\frac{r}{2}\int_0^r u_\lambda^2(s)ds+\frac{3}{2}r\int_0^r s^2u^2(s)ds
\end{aligned}\tag{2.20}$$

for constants $C_1=C_1(\overline{p})>0$ and $C_2=C_2(\overline{p})>0$ independent of $p\in(0,\overline{p}]$, a, r, and λ. Consequently, since $\lambda>0$ for $\overline{p}>0$ sufficiently small by the comparison theorem, for $\overline{p}>0$ sufficiently small

$$u_\lambda^2(r)\leq C_3(\overline{p})\int_0^r u_\lambda^2(s)ds+C_4\int_0^r s^2u^2(s)ds,\tag{2.21}$$

where the constants $C_3(\overline{p})>0$ and $C_4>0$ do not depend on $p\in(0,\overline{p}]$, a, r, and λ. Now, the statement of Lemma 2.4 follows by the Gronwell's lemma. □

Lemma 2.5. *There exists $\widehat{p}>0$ such that for any $p\in(0,\widehat{p}]$ there is no $a\in(0,1]$, and $\lambda\in\mathbb{R}$ for which $u(a)=u_\lambda(a)=0$.*

Proof. On the contrary, suppose that there exist arbitrary small $p>0$, $a\in(0,1]$, and $\lambda\in\mathbb{R}$ for which $u(a)=u_\lambda(a)=0$. Multiply (2.4) by $u_\lambda(r)$, (2.6) by $u(r)$, subtract the results from each other and integrate the obtained identity from 0 to a. Then, after integration by parts,

$$\begin{aligned}\int_0^a s^2 u^2(s)ds &= \int_0^a s^2\left[f'(u(s)) - \frac{f(u(s))}{u(s)}\right]u(s)u_\lambda(s)ds \\ &\le C_5(p)\left[\|u\|^2_{L_{2,r^2}(0,a)} + \|u_\lambda\|^2_{L_{2,r^2}(0,a)}\right],\end{aligned} \tag{2.22}$$

where $C_5(p) \to +0$ as $p \to +0$ because $f(0) = f'(0) = 0$. Therefore

$$\|u\|^2_{L_{2,r^2}(0,a)} \le 2C_5(p)\|u_\lambda\|_{L_{2,r^2}(0,a)} \tag{2.23}$$

for $p > 0$ sufficiently small, which contradicts Lemma 2.4. □

Let us prove Theorem 1.1(d). Let now $p \in (0, \hat{p}]$. Take an arbitrary integer $n \ge 0$ and let $\lambda_n = \inf \Lambda_n$, as earlier. Then, a simple corollary of Lemma 2.5 is that there exists a right half-neighborhood I_n of λ_n belonging to Λ_n in which the $(n+1)$st zero r_{n+1} of the solution $u(r)$ of problem (2.4)-(2.5) is a strictly decreasing function of $\lambda \in I_n$ (so that in particular $r_{n+1} \in (0,1)$). Letting λ increase further, by Lemma 2.5, again, one sees that the $(n+1)$st zero of the corresponding solution $u(r)$ of (2.4)-(2.5) continue to decrease strictly so that in the half-line $(\lambda_n, +\infty)$ there is no value λ for which $u(r)$ has precisely n zeros in the interval $(0,1)$ and $u(1) = 0$. Claim (d) of Theorem 1.1 is proved. Our proof of Theorem 1.1 is complete.

3. Proof of Theorem 1.2

Denote $w_n(r) = u_n(r) - v_n(r)$. Then

$$\left(-r^2 w_n'\right)' = \mu_n r^2 w_n + r^2 g_n(r), \quad r \in (0,1), \tag{3.1}$$

$$w_n(0) = w_n'(0) = w_n(1) = 0, \tag{3.2}$$

where $g_n(r) = (\lambda_n - \mu_n)r^2 u_n(r) - r^2 f(u_n(r))$. By the comparison theorem and Theorem 1.1(b)

$$\left|\lambda_n - \mu_n\right| \le C_1(p), \tag{3.3}$$

where $C_1(p) > 0$ goes to 0 as $p \to +0$ and C_1 does not depend on n. In addition, by the standard comparison theorem, again, there exists $C_2 > 0$ such that

$$C_2^{-1}(n+1)^2 \le \mu_n \le C_2(n+1)^2 \tag{3.4}$$

for all n. By (3.3) and Theorem 1.1(b),

$$\left|g_n(r)\right| \le C_3(p)|u_n(r)|, \tag{3.5}$$

where $C_3(p) > 0$ goes to 0 as $p \to +0$ uniformly in n.

Now, we proceed as when proving Lemma 2.4. First, multiply (3.1) by $w_n'(r)$ and integrate the result from 0 to r. Then, after integration by parts,

$$\frac{1}{2}r^2 w_n'^2(r) + \frac{\mu_n}{2}r^2 w_n^2(r) + \int_0^r s w_n'^2(s)ds = \mu_n \int_0^r s w_n^2(s)ds - \int_0^r s^2 g_n(s) w_n'(s)ds. \tag{3.6}$$

Second, multiply (3.1) by $r^{-1}w_n(r)$ and integrate the result from 0 to r. Then, in view of the boundary conditions (3.2), after integration by parts,

$$\frac{1}{2}w_n^2(r) + \mu_n \int_0^r s w_n^2(s)ds = \int_0^r s w_n'^2(s)ds - r w_n(r) w_n'(r) - \int_0^r s g_n(s) w_n(s)ds. \tag{3.7}$$

Add (3.7) to (3.6). Then, we obtain

$$\begin{aligned} &\frac{1}{2}r^2 w_n'^2(r) + \frac{\mu_n}{2}r^2 w_n^2(r) + \frac{1}{2}w_n^2(r) \\ &\quad = -r w_n(r) w_n'(r) - \int_0^r s^2 g_n(s) w_n'(s)ds - \int_0^r s g_n(s) w_n(s)ds. \end{aligned} \tag{3.8}$$

Integrate (3.8) for one more time from 0 to r. Then, as when deriving (2.20),

$$\begin{aligned} &\frac{1}{2}\int_0^r s^2 \left[w_n'^2(s) + \mu_n w_n^2(s)\right]ds + \frac{1}{2} r w_n^2(r) \\ &\quad \le \frac{r}{2}\int_0^r w_n^2(s)ds + \frac{r}{2}\int_0^r s^2 w_n'^2(s)ds + r\int_0^r s^2 g_n^2(s)ds. \end{aligned} \tag{3.9}$$

Hence, in view of (3.3) and (3.5),

$$w_n^2(r) \le \int_0^r w_n^2(s)ds + C_4(p)\int_0^r s^2 u_n^2(s)ds, \tag{3.10}$$

where $C_4(p) > 0$ is a constant independent of n and r and $C_4(p) \to 0$ as $p \to +0$. Now, we obtain from (3.10) by the Gronwell's lemma the following:

$$w_n^2(r) \le C_5(p)\|u_n\|^2_{L_{2,r^2}(0,r)}, \quad r \in [0,1], \tag{3.11}$$

where $C_5(p) > 0$ does not depend on n and r and goes to 0 as $p \to +0$. Thus, finally from (3.4) and (3.9), we have

$$\|w_n\|^2_{L_2} \le C_6(p)(n+1)^{-2}\|u_n\|^2_{L_2} \tag{3.12}$$

for a constant $C_6 > 0$ independent of n and going to 0 as $p \to +0$.

Now, it follows from (3.12) that

$$\sum_{n=0}^{\infty}\left\|\frac{u_n}{\|u_n\|_{L_2}}-\frac{v_n}{\|u_n\|_{L_2}}\right\|_{L_2}^2<\overline{C}(p), \tag{3.13}$$

where $\overline{C}(p)>0$ is defined for all $p>0$ and goes to 0 as $p\to+0$. So, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left\|\frac{u_n}{\|u_n\|_{L_2}}-\frac{v_n}{\|v_n\|_{L_2}}\right\|_{L_2}^2&\le 2\sum_{n=0}^{\infty}\left\{\left\|\frac{u_n}{\|u_n\|_{L_2}}-\frac{v_n}{\|u_n\|_{L_2}}\right\|_{L_2}^2+\left\|\frac{v_n}{\|u_n\|_{L_2}}-\frac{v_n}{\|v_n\|_{L_2}}\right\|_{L_2}^2\right\}\\&=2\left\{\overline{C}(p)+\sum_{n=0}^{\infty}\left(\frac{\|v_n\|_{L_2}}{\|u_n\|_{L_2}}-1\right)^2\right\}.\end{aligned} \tag{3.14}$$

In view of this estimate, to prove Theorem 1.2, it suffices to show that

$$\sum_{n=0}^{\infty}\left(\frac{\|v_n\|_{L_2}}{\|u_n\|_{L_2}}-1\right)^2\le\widehat{C}(p), \tag{3.15}$$

where $\widehat{C}(p)>0$ goes to 0 as $p\to+0$. But we have

$$\sum_{n=0}^{\infty}\left(\frac{\|v_n\|_{L_2}}{\|u_n\|_{L_2}}-1\right)^2\le\sum_{n=0}^{\infty}\left\|\frac{u_n}{\|u_n\|_{L_2}}-\frac{v_n}{\|u_n\|_{L_2}}\right\|_{L_2}^2\le\overline{C}(p) \tag{3.16}$$

by the proved part of Theorem 1.2, where $\overline{C}(p)\to+0$ as $p\to+0$. Thus, (3.15) follows. Our proof of Theorem 1.2 is complete.

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