Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 345196, 16 pages
doi:10.1155/2009/345196

Research Article

Strong Superconvergence of Finite Element Methods for Linear Parabolic Problems

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Received 30 March 2009; Revised 16 June 2009; Accepted 5 July 2009

Recommended by Thomas Witelski

We study the strong superconvergence of a semidiscrete finite element scheme for linear parabolic problems on $Q = \Omega \times (0, T]$, where Ω is a bounded domain in $\mathcal{R}^d$ $(d \leq 4)$ with piecewise smooth boundary. We establish the global two order superconvergence results for the error between the approximate solution and the Ritz projection of the exact solution of our model problem in $W^{1,p}(\Omega)$ and $L_p(Q)$ with $2 \leq p < \infty$ and the almost two order superconvergence in $W^{1,\infty}(\Omega)$ and $L_\infty(Q)$. Results of the $p = \infty$ case are also included in two space dimensions ($d = 1$ or 2). By applying the interpolated postprocessing technique, similar results are also obtained on the error between the interpolation of the approximate solution and the exact solution.

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1. Introduction

Consider the following initial boundary value parabolic problem:

$$\begin{aligned} u_t + Au &= f, \quad \text{in } Q = \Omega \times J,\ J = (0, T], \\ u(x,t) &= 0, \quad \text{on } \partial\Omega \times J, \\ u(x,0) &= u_0, \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathcal{R}^d$ $(d \leq 4)$ with piecewise smooth boundary $\partial\Omega$, and A is a second-order symmetric positive definite elliptic operator. Coefficients of $A, f(x,t)$ and $u_0(x)$ together with their derivatives up to certain order are bounded in order to guarantee our analysis. Note that our assumptions on u do not have any restrictions, since it will be shown that approximate solutions considered below are uniformly close to the exact solution and thus only depend on the data of (1.1) in a neighborhood of u.

Superconvergence of finite element methods for parabolic problems has been studied in many works. For example, Thomée [1], Chen and Huang [2] studied superconvergence of the gradient in L_2 norm. In 1989, Thomée et al. [3] studied maximum-norm superconvergence of the gradient in piecewise linear finite element approximations of a parabolic problem. An analogous result was also obtained by Chen [4]. Moreover, Li and Wei [5] investigated global strong superconvergence of finite element schemes for a class of Sobolev equations in $\mathcal{R}^d$ $(d \geq 1)$, and two order superconvergence results are proved in $W^{1,p}(\Omega)$ and $L_p(\Omega)$ for $2 \leq p < \infty$. In particular, Kwak et al. [6] studied superconvergence of a semi-discrete finite element scheme for parabolic problems in $\mathcal{R}^2$, in which superconvergence results in $W^{1,p}(\Omega)$ and $L_p(\Omega)$ are established for $2 \leq p \leq \infty$.

In this paper, we extend superconvergence results obtained in [6]. We derive the two order $(2 \leq p < \infty)$ and the almost two order $(p = \infty)$ global superconvergence estimates of $U - R_h u$ in $W^{1,p}(\Omega)$ and in $L_p(Q)$, where U is the approximate solution, and $R_h u$ is the Ritz projection of the exact solution of (1.1). In addition to the results in [6], we establish two order superconvergence estimates in L_p norm for piecewise cubic or higher elements. Moreover, results of the $p = \infty$ case are also included in two space dimensions ($d = 1$ or 2). As an application, by employing the interpolated finite element operators (cf. [7, 8]) to the approximate solution U in the rectangular mesh, we obtain the two order global superconvergence of the error between u and the interpolation of U. For a general domain Ω, we can also apply the optimal partition to most rectangular meshes to derive one and a half-order superconvergence.

The rest of this paper is organized as follows. Section 2 provides some preliminaries. Several useful lemmas are established in Section 3. In Sections 4 and 5, we derive the superconvergence in $W^{1,p}(\Omega)$ and $L_p(Q)$ respectively. Finally, an application is presented in Section 6.

2. Preliminaries

We denote $W^{m,p}(\Omega)$ and $H^m(\Omega)$, $m \geq 0$ and $1 \leq p \leq \infty$, the Sobolev spaces on Ω associated with the norms

$$\|\cdot\|_{m,p} = \|\cdot\|_{W^{m,p}(\Omega)}, \qquad \|\cdot\|_m = \|\cdot\|_{H^m(\Omega)}, \qquad \|\cdot\| = \|\cdot\|_{L_2(\Omega)}. \tag{2.1}$$

If X is a normed space with the norm $\|\cdot\|_X$ and $\phi : J \to X$, then we define

$$\begin{aligned} \|\phi\|^p_{L_p(0,t;X)} &= \int_0^t \|\phi(\tau)\|^p_X \, d\tau, \\ \|\phi\|^p_{L_p(J;X)} &= \int_0^T \|\phi(\tau)\|^p_X \, d\tau, \\ \|\phi\|_{L_\infty(J;X)} &= \operatorname*{ess\,sup}_{t\in J} \|\phi(t)\|_X. \end{aligned} \tag{2.2}$$

Moreover, we denote

$$
\begin{aligned}
(\phi, \psi) &= \int_{\Omega} \phi\psi \, dx, \\
(\phi, \psi)_Q &= \int_Q \phi\psi \, dx \, dt = \int_0^T (\phi, \psi) dt,
\end{aligned}
\tag{2.3}
$$

the inner product in $L_2(\Omega)$ (or $L_2(\Omega)^2$) and $L_2(Q)$ (or $L_2(Q)^2$), respectively. We also use Sobolev spaces $W_p^{2l,l}(Q)$ with norm

$$
\|\phi\|_{W_p^{2l,l}(Q)} = \left(\int_Q \sum_{|\alpha|+2s\le 2l} |D_x^{\alpha} D_t^s \phi(x,t)|^p \; dxdt \right)^{1/p}. \tag{2.4}
$$

We use C to denote a generic positive constant independent of h that can take different values at different occurrences.

Let $\mathcal{T}$ be a family of quasiuniform triangulation of Ω, and let $S_h \subset H_0^1(\Omega)$ be the kth ($k \ge 1$) degree finite element space satisfying the following properties (cf. [9, 10]).

Lemma 2.1. *For all $k \ge 1, 1 \le s \le k+1$ and $1 \le p \le \infty$, we have*

$$
\inf_{\chi\in S_h} \left\{ \|v-\chi\|_{0,p} + h\|v-\chi\|_{1,p} \right\} \le Ch^s \|v\|_{s,p}, \quad \forall v \in W^{s,p}(\Omega) \cap H_0^1(\Omega). \tag{2.5}
$$

Lemma 2.2. *For all $\chi \in S_h$, we have*

$$
\|\chi\|_{1,p'} \le Ch^{-d/p} \|\chi\|_{1,1}, \quad 2 \le p < \infty, \; p' = \frac{p}{p-1}, \tag{2.6}
$$

$$
\|\chi\|_{0,\infty} \le Ch^{-d/p} \|\chi\|_{0,p}, \quad 2 \le p < \infty. \tag{2.7}
$$

Given a function $u \in W^{k+1,p}(\Omega) \cap H_0^1(\Omega)$, we define its Ritz projection $R_h u \in S_h$ that satisfies

$$
A(R_h u - u, \chi) = 0, \quad \forall \chi \in S_h. \tag{2.8}
$$

Then we get the following well-known estimate:

$$
\|R_h u - u\|_{0,p} + h\|R_h u - u\|_{1,p} \le Ch^{k+1} \|u\|_{k+1,p}, \quad 1 < p < \infty. \tag{2.9}
$$

Moreover, by using the duality argument and (2.9), it is true that for $v \in W^{l,p'}(\Omega)$,

$$|(R_h u - u, v)| \le Ch^{k+1+l} \|u\|_{k+1,p} \|v\|_{l,p'}, \quad 0 \le l \le k-1, \tag{2.10}$$

where $1 < p < \infty$ and $1/p + 1/p' = 1$.

We now turn to the finite element scheme of (1.1).

Find a map $U(t) : J \to S_h$ such that

$$\begin{gathered}(U_t, \chi) + A(U, \chi) = (f(t), \chi), \quad \chi \in S_h,\ t \in J, \\ U(0) = U_0 \quad \text{in } \Omega,\end{gathered} \tag{2.11}$$

where $U_0 \in S_h$ is defined by

$$A(U_0, \chi) = (f(0), \chi) - (R_h u_t(0), \chi), \quad \chi \in S_h, \tag{2.12}$$

and $u_t(0) = f(0) - Au_0$ is given by (1.1).

3. Auxiliary Lemmas

To investigate the superconvergence of finite element approaches for parabolic problems, here and throughout the paper, we decompose the error as $U - u = (U - R_h u) + (R_h u - u) = \xi + \eta$ and estimate ξ in a superconvergent order.

We start with the superconvergence of initial value errors.

Lemma 3.1. *Let u and U be solutions of* (1.1) *and* (2.11)*, respectively. Then the following estimates are true:*

$$U_t(0) = R_h u_t(0), \tag{3.1}$$

$$\|\xi(0)\|_1 \le \begin{cases} Ch^{k+2} \|u_t(0)\|_{k+1}, & k > 1, \\ Ch^2 \|u_t(0)\|_2, & k = 1, \end{cases} \tag{3.2}$$

$$\|\xi(0)\|_{0,p} \le Ch^{k+3} \|u_t(0)\|_{k+1,p}, \quad 2 \le p < \infty,\ k > 2. \tag{3.3}$$

Proof. From (2.12) and (2.11), we have for all $\chi \in S_h$

$$(R_h u_t(0), \chi) = (f(0), \chi) - A(U_0, \chi) = (U_t(0), \chi), \tag{3.4}$$

and thus (3.1) holds.

By the definition of ξ, (2.11), (2.8), (1.1), and the definition of η, we obtain that for all $\chi \in S_h$,

$$\begin{aligned}(\xi_t, \chi) + A(\xi, \chi) &= (U_t, \chi) + A(U, \chi) - (R_h u_t, \chi) - A(R_h u, \chi) \\ &= (f, \chi) - (R_h u_t, \chi) - A(u, \chi) \\ &= (u_t, \chi) - (R_h u_t, \chi) \\ &= -(\eta_t, \chi).\end{aligned} \tag{3.5}$$

Let $t = 0$, and note that $\xi_t(0) = 0$, we obtain

$$A(\xi(0), \chi) = -(\eta_t(0), \chi), \quad \chi \in S_h. \tag{3.6}$$

Then by taking $\chi = \xi(0)$, it follows from (2.10) that

$$\|\xi(0)\|_1^2 \le \begin{cases} Ch^{k+2}\|u_t(0)\|_{k+1}\|\xi(0)\|_1, & k > 1, \\ Ch^2\|u_t(0)\|_2\|\xi(0)\|, & k = 1, \end{cases} \tag{3.7}$$

which implies (3.2).

Finally we turn to the proof of (3.3). To do so, we construct an auxiliary problem. Let $\Phi \in W_0^{1,p'}(\Omega)$ satisfy

$$A(v, \Phi) = (v, \phi), \quad v \in H_0^1(\Omega), \tag{3.8}$$

and hence by the regularity estimate, it holds that

$$\|\Phi\|_{2,p'} \le C\|\phi\|_{0,p'}, \tag{3.9}$$

where $p' = p/(p-1)$.

Therefore, it follows from (3.8), (2.8), (3.6), (2.10), (2.9), and (3.9) that for $\phi \in L_{p'}(\Omega)$,

$$\begin{aligned}(\xi(0), \phi) &= A(\xi(0), \Phi) \\ &= A(\xi(0), R_h\Phi) \\ &= (\eta_t(0), \Phi - R_h\Phi) - (\eta_t(0), \Phi) \\ &\le Ch^{k+2}\|u_t(0)\|_{k+1,p}\|\Phi - R_h\Phi\|_{1,p'} + Ch^{k+3}\|u_t(0)\|_{k+1,p}\|\Phi\|_{2,p'} \\ &\le Ch^{k+3}\|u_t(0)\|_{k+1,p}\|\Phi\|_{2,p'} \\ &\le Ch^{k+3}\|u_t(0)\|_{k+1,p}\|\phi\|_{0,p'},\end{aligned} \tag{3.10}$$

which implies (3.3). □

The following lemma gives superconvergence estimates for ξ_{tt} and $\nabla\xi$.

Lemma 3.2. *Let u and U be solutions of* (1.1) *and* (2.11)*, respectively. Then for $k > 1$,*

$$\left(\int_0^t \|\xi_{tt}\|^2 d\tau\right)^{1/2} + \|\xi_t\|_1 \le Ch^{k+2}\left[\|u_{tt}\|_{k+1} + \left(\int_0^t \|u_{ttt}\|_{k+1}^2 d\tau\right)^{1/2}\right], \quad t \in J. \tag{3.11}$$

Proof. By differentiating (3.5) in time, we have

$$(\xi_{tt}, \chi) + A(\xi_t, \chi) = -(\eta_{tt}, \chi), \quad \chi \in S_h. \tag{3.12}$$

Choosing $\chi = \xi_{tt}$, (3.12) becomes

$$\|\xi_{tt}\|^2 + \frac{1}{2}\frac{d}{dt}A(\xi_t, \xi_t) = -(\eta_{tt}, \xi_{tt}). \tag{3.13}$$

Integrating both sides of (3.13) with respect to t and applying the integration by parts argument, we obtain that from (3.1) and (2.10)

$$\begin{aligned}
\int_0^t \|\xi_{tt}\|^2 d\tau + \frac{1}{2}A(\xi_t, \xi_t) &= -\int_0^t (\eta_{tt}, \xi_{tt}) d\tau \\
&= -(\eta_{tt}, \xi_t) + \int_0^t (\eta_{ttt}, \xi_t) d\tau \\
&\le Ch^{k+2}\|u_{tt}\|_{k+1}\|\xi_t\|_1 + Ch^{k+2}\int_0^t \|u_{ttt}\|_{k+1}\|\xi_t\|_1 d\tau \\
&\le \varepsilon\|\xi_t\|_1^2 + C\left[h^{2k+4}\left(\|u_{tt}\|_{k+1}^2 + \int_0^t \|u_{ttt}\|_{k+1}^2 d\tau\right) + \int_0^t \|\xi_t\|_1^2 d\tau\right].
\end{aligned} \tag{3.14}$$

Therefore, the proof is completed by eliminating $\varepsilon\|\xi_t\|_1^2$ and applying the Gronwall inequality. □

Furthermore, the result below for $k = 1$ can then be obtained by replacing (2.10) by (2.9) in the proof of Lemma 3.2.

Lemma 3.3. *It holds that*

$$\left(\int_0^t \|\xi_{tt}\|^2 d\tau\right)^{1/2} + \|\xi_t\|_1 \le Ch^2\left[\|u_{tt}\|_2 + \left(\int_0^t \|u_{ttt}\|_2^2 d\tau\right)^{1/2}\right], \quad t \in J. \tag{3.15}$$

4. Superconvergence in $W^{1,p}(\Omega)$

In this Section, we derive the two order global superconvergence ($2 \le p < \infty$) and the almost two order global superconvergence ($p = \infty$) estimates on ξ in $W^{1,p}(\Omega)$.

Theorem 4.1. *Under the assumptions that* $u_t \in W^{k+1,p}(\Omega), u_{tt} \in H^{k+1}(\Omega)$, *and* $u_{ttt} \in L_2(0,t;H^{k+1}(\Omega))$, *we have for* $d \leq 4$ *and* $2 \leq p < \infty$,

$$\|\xi\|_{1,p} \leq Ch^{k+2}, \quad k > 1. \tag{4.1}$$

Proof. We first introduce an auxiliary problem.

For $\varphi \in L_{p'}(\Omega)$, let φ_x be an arbitrary component of $\nabla\varphi$, and let $\Psi \in W_0^{1,p'}(\Omega)$ be the solution of

$$A(v,\Psi) = -(v,\varphi_x), \quad \forall v \in H_0^1(\Omega). \tag{4.2}$$

The following priori estimate holds:

$$\|\Psi\|_{1,p'} \leq C\|\varphi\|_{0,p'}. \tag{4.3}$$

Let $v = \xi$ in (4.2), it follows from the integration by parts argument, (2.8) and (3.5) that

$$\begin{aligned}(\xi_x,\varphi) &= A(\xi,\Psi)\\ &= A(\xi,R_h\Psi)\\ &= -(\eta_t+\xi_t,R_h\Psi).\end{aligned} \tag{4.4}$$

From (2.10), the stability of R_h and (4.3), we obtain

$$\begin{aligned}-(\eta_t,R_h\Psi) &\leq Ch^{k+2}\|u_t\|_{k+1,p}\|R_h\Psi\|_{1,p'}\\ &\leq Ch^{k+2}\|u_t\|_{k+1,p}\|\Psi\|_{1,p'}\\ &\leq Ch^{k+2}\|u_t\|_{k+1,p}\|\varphi\|_{0,p'}.\end{aligned} \tag{4.5}$$

On the other hand, for $s = s' = 2$ and $d = 1$ or 2, or $s = 2d/(d-2)$, $s' = s/(s-1)$, and $d = 3$ or 4, Sobolev embedding inequalities (cf. [11]), Lemma 3.2, the stability of R_h, and (4.3) imply that

$$\begin{aligned}-(\xi_t,R_h\Psi) &\leq C\|\xi_t\|_{0,s}\|R_h\Psi\|_{0,s'}\\ &\leq C\|\xi_t\|_1\|R_h\Psi\|_{1,p'}\\ &\leq Ch^{k+2}\|\varphi\|_{0,p'}.\end{aligned} \tag{4.6}$$

Combining (4.4), (4.5), and (4.6), we have

$$\|\xi_x\|_{0,p} = \sup_{\varphi\in L_{p'}(\Omega)} \frac{(\xi_x,\varphi)}{\|\varphi\|_{0,p'}} \le Ch^{k+2}. \tag{4.7}$$

Therefore, (4.1) follows from summing up all components ξ_x of $\nabla\xi$. □

The following theorem can then be obtained immediately by using Lemma 3.2 and Theorem 4.1.

Theorem 4.2. *Under the assumptions of Lemma 3.2 and Theorem 4.1, we have that for $d \le 4$,*

$$\|\xi\|_{1,Q} \le Ch^{k+2}, \quad k > 1. \tag{4.8}$$

We now turn to the case of $p = \infty$.

Theorem 4.3. *Assume that $u_t \in L_\infty(J; W^{k+1,p}(\Omega)), u_{tt} \in L_\infty(J; H^{k+1}(\Omega))$, and $u_{ttt} \in L_2(J; H^{k+1}(\Omega))$. Then for $d = 1$ and 2,*

$$\|\xi\|_{L_\infty(J;W^{1,\infty}(\Omega))} \le Ch^{k+2-\varepsilon}, \quad k > 1, \tag{4.9}$$

where p is large enough and $\varepsilon > d/p$.

Proof. We first define the Green functions associated with the bilinear form $A(\cdot,\cdot)$.

Let $G_z^* \in H_0^1(\Omega)$ be the pre-Green function, and let $\partial_z G_z^*$ be the directional derivative of G_z^* along some direction with respect to z. Let $G_z^h, \partial_z G_z^h \in S_h$ be the finite element approximations of G_z^* and $\partial_z G_z^*$, respectively. Then we know that (cf. [1, 12])

$$\left\|G_z^h\right\| + \left\|G_z^h\right\|_{1,q} \le C, \quad q < 2, \tag{4.10}$$

$$\left\|\partial_z G_z^h\right\|^2 + \left\|\partial_z G_z^h\right\|_{1,1} \le C\log\frac{1}{h}. \tag{4.11}$$

Now by definitions of Green functions, (3.5), Hölder's inequalities, (2.10), and (3.11), it is true that for all $(z,t) \in Q$,

$$\begin{aligned}
\xi(z,t) &= A\left(\xi, G_z^h\right) \\
&= -\left(\eta_t, G_z^h\right) - \left(\xi_t, G_z^h\right) \\
&\le Ch^{k+2}\|u_t\|_{k+1,p}\left\|G_z^h\right\|_{1,p'} + \|\xi_t\|\left\|G_z^h\right\| \\
&\le Ch^{k+2}\left(\left\|G_z^h\right\|_{1,p'} + \left\|G_z^h\right\|\right),
\end{aligned} \tag{4.12}$$

which together with (4.10) yields

$$\|\xi\|_{0,\infty} \le Ch^{k+2}. \tag{4.13}$$

Similarly, by the inverse property (2.6) and (4.11), we have

$$\begin{aligned}\partial_z \xi(z,t) &= A\left(\xi, \partial_z G_z^h\right)\\ &\le Ch^{k+2}\left(\|\partial_z G_z^h\|_{1,p'} + \|\partial_z G_z^h\|\right)\\ &\le Ch^{k+2}\left(h^{-d/p}\|\partial_z G_z^h\|_{1,1} + \|\partial_z G_z^h\|\right)\\ &\le Ch^{k+2-d/p}\log\frac{1}{h},\end{aligned} \tag{4.14}$$

which implies that, for p large enough and h sufficiently small,

$$\|\nabla\xi\|_{0,\infty} \le Ch^{k+2-\varepsilon}, \quad \varepsilon > \frac{d}{p}. \tag{4.15}$$

Inequality (4.9) then follows from (4.13) and (4.15). □

By the similar arguments used in the proof of Theorems 4.1–4.3 and Lemma 3.3, we obtain the following results.

Theorem 4.4. *Under the assumptions of Theorems 4.1–4.3 with $k = 1$, we have, for $d \le 4$,*

$$\|\xi\|_{1,p} \le Ch^2, \tag{4.16}$$

$$\|\xi\|_{1,Q} \le Ch^2. \tag{4.17}$$

Moreover, for $d = 1, 2$,

$$\|\xi\|_{L_\infty(J;W^{1,\infty}(\Omega))} \le Ch^2. \tag{4.18}$$

5. Superconvergence in $L_p(Q)$

In this section, we establish the strong superconvergence for ξ in $L_p(Q)$ with $2 \le p \le \infty$.

We start with the following two order global superconvergence for $2 \le p < \infty$.

Theorem 5.1. *Assume that $u_t(0) \in W^{k+1,p}(\Omega)$, $u_t \in L_p(J; W^{k+1,p}(\Omega))$, $u_{tt} \in L_p(J; H^{k+1}(\Omega))$, and $u_{ttt} \in L_p(J; L_2(0,t; H^{k+1}(\Omega)))$. Then, for $d \le 4$ and $2 \le p < \infty$, it holds that*

$$\|\xi\|_{0,p,Q} \le Ch^{k+3}, \quad k > 2. \tag{5.1}$$

Proof. First, we construct an adjoint problem of (1.1).

Let $W \in H_0^1(\Omega)$ satisfy

$$(v, W_t) - A(v, W) = (v, w), \quad v \in H_0^1(\Omega), \tag{5.2}$$

$$W(T) = 0 \quad \text{in } \Omega. \tag{5.3}$$

By taking $s = T - t$, (5.2) and (5.3) can then be reduced to the weak form of (1.1) and thus we have the regularity estimate (cf. [2])

$$\|W\|_{W_{p'}^{2,1}(Q)} \le C\|w\|_{0,p',Q}. \tag{5.4}$$

Let $v = \xi$ in (5.2), it follows from (2.8) and (3.12) that

$$\begin{aligned}
(\xi, w) &= (\xi, W_t) - A(\xi, W) \\
&= \frac{d}{dt}(\xi, W) - [(\xi_t, W) + A(\xi, W)] \\
&= \frac{d}{dt}(\xi, W) - [(\xi_t, W) + A(\xi, R_h W)] \\
&= \frac{d}{dt}(\xi, W) - \left[(\xi_t, W) - \left(\eta_t, R_h W\right) - (\xi_t, R_h W)\right] \\
&= \frac{d}{dt}(\xi, W) + \left(\eta_t, R_h W\right) + (\xi_t, R_h W - W).
\end{aligned} \tag{5.5}$$

After integrating in t, we have

$$(\xi, w)_Q = -(\xi(0), W(0)) + \int_0^T \left(\eta_t, R_h W\right)\, dt + \int_0^T (\xi_t, R_h W - W) dt. \tag{5.6}$$

Here the fact that $W(T) = 0$ was used.

Now we estimate the right-hand side of (5.6) term by term.

First of all, by Hölder's inequalities, (3.3), and the Sobolev embedding inequality, we obtain that

$$\begin{aligned}
-(\xi(0), W(0)) &\le \|\xi(0)\|_{0,p}\|W(0)\|_{0,p'} \\
&\le Ch^{k+3}\|u_t(0)\|_{k+1,p}\|W\|_{L_\infty(J;L_{p'}(\Omega))} \\
&\le Ch^{k+3}\|u_t(0)\|_{k+1,p}\|W\|_{W^{1,1}(J;L_{p'}(\Omega))} \\
&\le Ch^{k+3}\|u_t(0)\|_{k+1,p}\|W\|_{W_{p'}^{2,1}(Q)} \\
&\le Ch^{k+3}\|W\|_{W_{p'}^{2,1}(Q)},
\end{aligned} \tag{5.7}$$

where

$$\|W\|_{W^{1,1}(J;L_{p'}(\Omega))} = \int_0^T \left(\|W\|_{0,p'} + \|W_t\|_{0,p'}\right) dt. \tag{5.8}$$

Secondly, it follows from (2.9), (2.10) and Hölder's inequalities that

$$\begin{aligned}
\int_0^T (\eta_t, R_h W) dt &= \int_0^T (\eta_t, W) dt + \int_0^T (\eta_t, R_h W - W) dt \\
&\leq Ch^{k+3} \int_0^T \|u_t\|_{k+1,p} \|W\|_{2,p'}\, dt + Ch^{k+2} \int_0^T \|u_t\|_{k+1,p} \|R_h W - W\|_{1,p'}\, dt \\
&\leq Ch^{k+3} \int_0^T \|u_t\|_{k+1,p} \|W\|_{2,p'}\, dt \\
&\leq Ch^{k+3} \left(\int_0^T \|u_t\|_{k+1,p}^p\, dt \right)^{1/p} \|W\|_{W_{p'}^{2,1}(Q)} \\
&\leq Ch^{k+3} \|W\|_{W_{p'}^{2,1}(Q)}.
\end{aligned} \tag{5.9}$$

Finally, by Hölder's inequalities, Sobolev embedding inequalities and Lemma 3.2, we have

$$\begin{aligned}
\int_0^T (\xi_t, R_h W - W) dt &\leq \int_0^T \|\xi_t\|_{0,4} \|R_h W - W\|_{0,4/3}\, dt \\
&\leq Ch \int_0^T \|\xi_t\|_1 \|W\|_{1,4/3}\, dt \\
&\leq Ch \int_0^T \|\xi_t\|_1 \|W\|_{2,p'}\, dt \\
&\leq Ch \left(\int_0^T \|\xi_t\|_1^p\, dt \right)^{1/p} \|W\|_{W_{p'}^{2,1}(Q)} \\
&\leq Ch^{k+3} \|W\|_{W_{p'}^{2,1}(Q)}.
\end{aligned} \tag{5.10}$$

Therefore, (5.1) holds by combining all estimates together with (5.4) and (5.6). □

We finally establish the almost two order global superconvergence in $L_\infty(Q)$. We define a function $g(t) \in H_0^1(\Omega)$, and its finite element approximation $g_h(t) \in S_h$ satisfy that

$$(v, g_t) - A(v, g) = 0, \quad v \in H_0^1(\Omega), \tag{5.11}$$

$$g(T) = \delta_h \quad \text{in } \Omega, \tag{5.12}$$

$$(\chi, g_{ht}) - A(\chi, g_h) = 0, \quad \chi \in S_h, \tag{5.13}$$

$$g_h(T) = \delta_h \quad \text{in } \Omega, \tag{5.14}$$

where $\delta_h = \delta_h^z(x)$ is the discrete Delta function which satisfies

$$(\delta_h, \chi) = \chi(z), \quad \forall \chi \in S_h. \tag{5.15}$$

Then the following estimate holds (cf. [2]):

$$\|g_h\|_{0,1,Q} + \|g_{ht}\|_{0,1,Q} + \left\|\phi^2 D_x^2 g\right\|_{0,2,Q}^2 \le C \log \frac{1}{h}, \tag{5.16}$$

where ϕ is the weight function defined by

$$\phi(t) = \left(|x-z|^2 + t + \beta^2\right)^{1/2}, \quad \beta = \gamma h, \ \gamma \gg 1. \tag{5.17}$$

Furthermore, we have the following estimate.

Lemma 5.2. *For $1 < q < 2$ and its conjugate index q', we have*

$$\|g\|_{L_q(J;W^{2,q}(\Omega))} \le C h^{-4/q'} \left(\log \frac{1}{h}\right)^{1/2}. \tag{5.18}$$

Proof. Using the Hölder inequality, it is easy to see that

$$\left\|D_x^2 g\right\|_{0,q,Q} \le \left(\int_Q \phi^{-4q/(2-q)} \, dx \, dt\right)^{(2-q)/2q} \|\phi^2 D_x^2 g\|_{0,2,Q}. \tag{5.19}$$

Note that (cf. [2])

$$\int_Q \phi^{-\lambda} \, dx \, dt \le C \frac{\beta^{4-\lambda}}{\lambda - 4}, \quad \lambda > 4, \tag{5.20}$$

the proof is then completed by the norm equivalence in $H_0^1(\Omega) \cap W^{2,q}(\Omega)$. □

The following theorem gives the superconvergence of ξ in $L_\infty(Q)$.

Theorem 5.3. *Assume that $u_t(0) \in W^{k+1,p}(\Omega)$ and $u_t \in L_p(J; W^{k+1,p}(\Omega))$. Then, for $d = 1, 2$,*

$$\|\xi\|_{0,\infty,Q} \leq Ch^{k+3-\varepsilon}, \quad k > 2, \tag{5.21}$$

with $\varepsilon > 4/p$ and p large enough.

Proof. (5.13) and (3.5) yield that

$$\begin{aligned} D_t(\xi, g_h) &= (\xi_t, g_h) + (\xi, g_{ht}) \\ &= (\xi_t, g_h) + A(\xi, g_h) \\ &= -(\eta_t, g_h). \end{aligned} \tag{5.22}$$

Then by integrating in t, it follows from (5.15) and (5.14) that

$$\begin{aligned} \xi(z, T) &= (\xi(T), \delta_h) \\ &= (\xi(T), g_h(T)) - (\xi(0), g_h(0)) + (\xi(0), g_h(0)) \\ &= -\int_0^T (\eta_t, g_h)\, dt + (\xi(0), g_h(0)). \end{aligned} \tag{5.23}$$

On the one hand, (2.10) and Lemma 5.2 imply that, for $p > 2$,

$$\begin{aligned} -\int_0^T (\eta_t, g_h) dt &= \int_0^T (\eta_t, g - g_h) dt - \int_0^T (\eta_t, g) dt \\ &\leq Ch^{k+2} \int_0^T \|u_t\|_{k+1,p} \|g - g_h\|_{1,p'}\, dt + Ch^{k+3} \int_0^T \|u_t\|_{k+1,p} \|g\|_{2,p'}\, dt \\ &\leq Ch^{k+3} \int_0^T \|u_t\|_{k+1,p} \|g\|_{2,p'}\, dt \\ &\leq Ch^{k+3-4/p} \left(\int_0^T \|u_t\|_{k+1,p}^p\, dt \right)^{1/p} \|g\|_{L_{p'}(J; W^{2,p'}(\Omega))} \\ &\leq Ch^{k+3-4/p} \left(\log \frac{1}{h} \right)^{1/2}. \end{aligned} \tag{5.24}$$

On the other hand, it follows from (2.7), the Sobolev embedding inequality, (3.3), and (5.16) that

$$\begin{aligned}
\left(\xi(0), g_h(0)\right) &\leq \|\xi(0)\|_{0,\infty} \left\|g_h(0)\right\|_{0,1} \\
&\leq Ch^{-d/p} \|\xi(0)\|_{0,p} \left\|g_h\right\|_{L_\infty(J;L_1(\Omega))} \\
&\leq Ch^{-d/p} \|\xi(0)\|_{0,p} \left\|g_h\right\|_{W^{1,1}(J;L_1(\Omega))} \\
&\leq Ch^{k+3-d/p} \log \frac{1}{h}.
\end{aligned} \tag{5.25}$$

Therefore, (5.21) follows from (5.23), (5.24), and (5.25). □

6. An Application

In this section, we apply the interpolation postprocessing technique to improve the accuracy of the approximate solution U. Let $\mathcal{T}_h$ be a quasi-uniform rectangular partition of $\Omega \subset \mathcal{R}^2$, and let S_h be the space of continuous piecewise polynomial:

$$S_h = \left\{ v \in H_0^1(\Omega) : v \in Q^k(T),\ T \in \mathcal{T}_h \right\}, \tag{6.1}$$

where

$$Q^k = \text{ span } \left\{ x_1^i x_2^j : 0 \leq i,\ j \leq k \right\}, \quad k \geq 1. \tag{6.2}$$

We introduce the higher interpolation operator I_{2h}^{k+2}, which satisfies the following properties (cf. [7, 8]), for $k \geq 1,\ 2 \leq p \leq \infty,$ and $l = 0, 1,$

$$\left\| u - I_{2h}^{k+2} u \right\|_{l,p} \leq Ch^{k+3-l} \|u\|_{k+3,p}, \tag{6.3}$$

$$I_{2h}^{k+2} i_h^k = I_{2h}^{k+2}, \tag{6.4}$$

$$\left\| I_{2h}^{k+2} \chi \right\|_{l,p} \leq C \|\chi\|_{l,p}, \quad \chi \in S_h, \tag{6.5}$$

where i_h^k is the finite element interpolation operator.

In addition, we assume that $Au = -\Delta u$ in (1.1). By replacing the approximate solution U by its interpolation $I_{2h}^{k+2} U$, we can then establish the two order and the almost two order global superconvergence of $u - I_{2h}^{k+2} U$ in $W^{1,p}(\Omega)$ and $L_p(Q)$ for $2 \leq p \leq \infty$.

Theorem 6.1. *Under the assumptions of Theorems 4.1–4.4, 5.1 and 5.3, we have for $u \in W^{k+3,p}(\Omega) \cap L_p(J; W^{k+3,p}(\Omega))$ with $k > 1$ and $u \in W^{3,p}(\Omega) \cap L_p(J; W^{3,p}(\Omega))$ with $k = 1$,*

$$\begin{aligned}
\left\| u - I_{2h}^{k+2} U \right\|_{1,p} &\leq C h^{k+2}, \quad 2 \leq p < \infty,\ k > 1, \\
\left\| u - I_{2h}^{k+2} U \right\|_{1,Q} &\leq C h^{k+2}, \quad k > 1, \\
\left\| u - I_{2h}^{k+2} U \right\|_{L_\infty(J; W^{1,\infty}(\Omega))} &\leq C h^{k+2-\varepsilon}, \quad k > 1, \\
\left\| u - I_{2h}^{2} U \right\|_{1,p} &\leq C h^{2}, \quad k = 1, \\
\left\| u - I_{2h}^{2} U \right\|_{1,Q} &\leq C h^{2}, \quad k = 1, \\
\left\| u - I_{2h}^{2} U \right\|_{L_\infty(J; W^{1,\infty}(\Omega))} &\leq C h^{2}, \quad k = 1, \\
\left\| u - I_{2h}^{k+2} U \right\|_{0,p,Q} &\leq C h^{k+3}, \quad 2 \leq p < \infty,\ k > 2, \\
\left\| u - I_{2h}^{k+2} U \right\|_{0,\infty,Q} &\leq C h^{k+3-\varepsilon}, \quad k > 2,
\end{aligned} \tag{6.6}$$

with $\varepsilon > 2/p$ and p large enough.

Proof. From (6.4), we have

$$u - I_{2h}^{k+2} U = u - I_{2h}^{k+2} u + I_{2h}^{k+2}\left(i_h^k u - R_h u \right) + I_{2h}^{k+2}(R_h u - U), \tag{6.7}$$

which together with the triangular inequality and (6.5) yields that

$$\left\| u - I_{2h}^{k+2} U \right\|_{l,p} \leq \left\| u - I_{2h}^{k+2} u \right\|_{l,p} + C\left(\left\| i_h^k u - R_h u \right\|_{l,p} + \| R_h u - U \|_{l,p} \right). \tag{6.8}$$

Moreover, for $2 \leq p \leq \infty$, the following estimates hold (cf. [7, 8]):

$$\begin{aligned}
\left\| i_h^k u - R_h u \right\|_{0,p} &\leq C h^{k+3} \| \mathrm{u} \|_{k+3,p} \left(\log \frac{1}{h} \right)^{\overline{p}}, \quad k > 2, \\
\left\| i_h^k u - R_h u \right\|_{1,p} &\leq C h^{k+2} \| u \|_{k+3,p}, \quad k > 1, \\
\left\| i_h^1 u - R_h u \right\|_{1,p} &\leq C h^{2} \| u \|_{3,p},
\end{aligned} \tag{6.9}$$

where

$$\overline{p} = \begin{cases} 0, & \text{when } p < \infty, \\ 1, & \text{when } p = \infty. \end{cases} \tag{6.10}$$

Hence the proof is completed by (6.3) and the estimates for ξ in Theorems 4.1–4.4, 5.1, and 5.3. $\square$

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