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Research Article

A "v-Operation Free" Approach to Prüfer v-Multiplication Domains

Marco Fontana[1] and Muhammad Zafrullah[2]

[1] *Dipartimento di Matematica, Università degli Studi "Roma Tre", 00146 Rome, Italy*
[2] *Department of Mathematics, Idaho State University, Pocatello, ID 83209, USA*

Correspondence should be addressed to Marco Fontana, fontana@mat.uniroma3.it

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The so-called Prüfer v-multiplication domains (PvMDs) are usually defined as domains whose finitely generated nonzero ideals are t-invertible. These domains generalize Prüfer domains and Krull domains. The PvMDs are relatively obscure compared to their very well-known special cases. One of the reasons could be that the study of PvMDs uses the jargon of star operations, such as the v-operation and the t-operation. In this paper, we provide characterizations of and basic results on PvMDs and related notions without star operations.

1. Introduction and Preliminaries

Prüfer v-multiplication domains, explicitly introduced in [1] under the name of v-multiplication rings, have been studied a great deal as a generalization of Prüfer domains and Krull domains. One of the attractions of Prüfer v-multiplication domains is that they share many properties with Prüfer domains and, furthermore, they are stable in passing to polynomials, unlike Prüfer domains (since a polynomial ring $D[X]$ is a Prüfer domain only in the trivial case, i.e., when D is a field). On the other hand, Prüfer v-multiplication domains are a special case of v-domains, a class of integrally closed domains which has recently attracted new attention [2–4]. The paper [5] provides a clue to where v-domains arose as a separate class of rings, though they were not called v-domains there.

The notions of v-domain and of several of its specializations may be obscured by the jargon of Krull's star operations used in the "official" definitions and standard characterizations (the best source available for star operations and for this type of approach to v-domains is Sections 32 and 34 of [6]). The overhanging presence of star operations seems to have limited the popularization of these distinguished classes of integral domains and, perhaps, has prevented the use of other powerful techniques, such as those of homological algebra, in their study.

The aim of this note is to provide "star operation free" definitions and characterizations of the above-mentioned classes of integral domains. In particular, we prove statements that, when used as definitions, do not mention any star operations, leading to new characterizations of various special classes of v-domains.

Let D be an integral domain with quotient field K. Let $\overline{\mathbf{F}}(D)$ be the set of all nonzero D-submodules of K and let $\mathbf{F}(D)$ be the set of all nonzero fractional ideals of D, that is, $A \in \mathbf{F}(D)$ if $A \in \overline{\mathbf{F}}(D)$ and there exists an element $0 \neq d \in D$ with $dA \subseteq D$. Let $\mathbf{f}(D)$ be the set of all nonzero finitely generated D-submodules of K. Then, obviously $\mathbf{f}(D) \subseteq \mathbf{F}(D) \subseteq \overline{\mathbf{F}}(D)$.

For D-submodules $A, B \in \overline{\mathbf{F}}(D)$, we use the notation $(A : B)$ to denote the set $\{x \in K \mid xB \subseteq A\}$. If $(A : B) \neq (0)$, clearly, $(A : B) \in \overline{\mathbf{F}}(D)$ and if $A \in \mathbf{F}(D)$, then $(A : B) \in \mathbf{F}(D)$. Denote $(D : A)$ by A^{-1}, which belongs to $\mathbf{F}(D)$) whenever A does, and $(D : A) = (0)$ if $A \in \overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$. If $A \subseteq B$ then $A^{-1} \supseteq B^{-1}$. Moreover, from the definition, it follows that $AA^{-1} \subseteq D$ and $D^{-1} = D$. Recall that, for $A \in \overline{\mathbf{F}}(D)$, $A^v := (A^{-1})^{-1} = (D : (D : A))$ and note that, if $A \in \overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$, then $A^v = K$, since $(D : A) = (0)$. Set $A^t := \bigcup\{F^v \mid F \subseteq A \text{ and } F \in \mathbf{f}(D)\}$. It can be easily shown that $(A^v)^{-1} = A^{-1} = (A^{-1})^v$. If $A \in \mathbf{F}(D)$) is such that $A = A^v$ (resp., $A = A^t$) we say that A is a *fractional v-ideal* (resp., a *fractional t-ideal*) *of* D. Note that, if $A \in \overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$, then $A = A^v$ if and only if $A = K$; on the other hand, it is possible that $A = A^t \subsetneq K$ for $A \in \overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$ (e.g., if D is a fgv-domain, i.e., an integral domain such that every nonzero finitely generated ideal is a v-ideal [7], then $A = A^t$ for every $A \in \overline{\mathbf{F}}(D)$).

A fractional v-ideal is also called a *fractional divisorial ideal*. If $A \in \mathbf{F}(D)$, A^{-1} is a fractional v-ideal, and every *fractional invertible ideal* (i.e., every fractional ideal A such that $AA^{-1} = D$) is both a fractional v-ideal and a fractional t-ideal. If there is a finitely generated fractional ideal F such that $A^v = F^v$, we say that A^v is a *fractional v-ideal of finite type*. Note that, in this definition, we do not require that $F \subseteq A$; if there is a finitely generated fractional ideal F such that $A^v = F^v$ and $F \subseteq A$, we say that A^v is a *fractional v-ideal of strict finite type*. Examples of v-ideals of finite type that are not v-ideals of strict finite type are given in [8, Section (4c)]. If $*$ provides here a general notation for the v- and t-operation, then call $A \in \mathbf{F}(D)$*$*$-invertible* if there is $B \in \mathbf{F}(D)$ such that $(AB)^* = D$. It can be shown that in this case $B^* = A^{-1}$. It is obvious that an invertible ideal is t-invertible and a t-invertible ideal is also v-invertible. So, D is called a *v-domain* (resp., a *Prüfer v-multiplication domain* (for short, *PvMD*)) if every $F \in \mathbf{f}(D)$ is v-invertible (resp., t-invertible). Both these notions generalize the concept of Prüfer domain, since a Prüfer domain can be characterized by the fact that every $F \in \mathbf{f}(D)$ is invertible, and, at the same time, the concept of Krull domain because, as we mention later, a domain D is a Krull domain if and only if every nonzero ideal of D is t-invertible.

It can be shown that $F \in \mathbf{f}(D)$ is t-invertible if and only if F is v-invertible and F^{-1} is a v-ideal of finite type [9, Theorem 1.1(c)]. In particular, from the previous considerations, we deduce

$$\text{Prüfer domain} \Longrightarrow \text{P}v\text{MD} \Longrightarrow v\text{-domain}. \tag{1.1}$$

It is well known that the converse of each of the previous implications does not hold in general. For instance, a Krull domain which is not Dedekind (e.g., the polynomial ring $\mathbb{Z}[X]$) shows the irreversibility of the first implication. An example of a v-domain which is not a PvMD was given in [5].

2. Results

The following result maybe in the folklore. We have taken it from [10], where the second-named author of the present paper made a limited attempt to define PvMDs without the v-operation.

Lemma 2.1. *Given an integral domain D, a fractional ideal $A \in \mathbf{F}(D)$ is v-invertible if and only if $(A^{-1} : A^{-1}) = D$.*

Proof. Suppose that $(A^{-1} : A^{-1}) = D$. Let $x \in (AA^{-1})^{-1} \supseteq D$. Then, $x(AA^{-1}) \subseteq D$ or $xA^{-1} \subseteq A^{-1}$ or $x \in (A^{-1} : A^{-1}) = D$. So, $(AA^{-1})^{-1} \subseteq D$ and we have $(AA^{-1})^{-1} = D$. This gives $(AA^{-1})^{v} = D$.

Conversely, if A is v-invertible, then $(AA^{-1})^{-1} = D$. Let $x \in (A^{-1} : A^{-1}) \supseteq D$. Then, $xA^{-1} \subseteq A^{-1}$. Multiplying both sides by A and applying the v-operation, we get $x \in D$. So, $D \subseteq (A^{-1} : A^{-1}) \subseteq D$ and the equality follows. □

Theorem 2.2. *The following are equivalent for an integral domain D:*

(i) *D is a v-domain,*

(ii) *$(F^{-1} : F^{-1}) = D$ for each $F \in \mathbf{f}(D)$,*

(iii) *$(F^{v} : F^{v}) = D$ for each $F \in \mathbf{f}(D)$,*

(iv) *$((a,b)^{-1} : (a,b)^{-1}) = D$ for each two generated fractional ideals $(a,b) \in \mathbf{f}(D)$,*

(v) *$((a) \cap (b)) : ((a) \cap (b)) = D$ for all $a,b \in D \setminus \{0\}$.*

Proof. (i)⇔(ii) follows from Lemma 2.1 and from the definition of a v-domain.

(i)⇒(iii). Let $F \in \mathbf{f}(D)$ and $x \in (F^{v} : F^{v}) \supseteq D$. Then, $xF^{v} \subseteq F^{v}$. Multiplying both sides by F^{-1} and applying the v-operation, we get $x(F^{v}F^{-1})^{v} \subseteq (F^{v}F^{-1})^{v}$. But, by (i), $(F^{v}F^{-1})^{v} = (FF^{-1})^{v} = D$ and so $x \in D$. This forces $D \subseteq (F^{v} : F^{v}) \subseteq D$.

(iii)⇒(i). Let $F \in \mathbf{f}(D)$ and $x \in (F^{v}F^{-1})^{-1} \supseteq D$. Then, $x(F^{v}F^{-1}) \subseteq D$. But then $xF^{v} \subseteq F^{v}$, which gives $x \in (F^{v} : F^{v}) = D$. Therefore $D \subseteq (F^{v}F^{-1})^{-1} \subseteq D$, which means that every $F \in \mathbf{f}(D)$ is v-invertible.

(ii)⇒(iv) is obvious.

(iv)⇒(v). Let $a,b \in D$ be two nonzero elements and, by (iv), let $((a,b)^{-1} : (a,b)^{-1}) = D$. Since $(a,b)^{-1} = (D : (a,b)) = (D : (a)) \cap (D : (b)) = (a^{-1}) \cap (b^{-1}) = a^{-1}b^{-1}((a) \cap (b))$, then from the assumption we have $(a^{-1}b^{-1}((a) \cap (b)) : a^{-1}b^{-1}((a) \cap (b))) = D$ which is the same as $((a) \cap (b)) : ((a) \cap (b)) = D$, for all $a,b \in D \setminus \{0\}$.

(v)⇒(i). Recall that D is a v-domain if and only if every two generated nonzero ideal of D is v-invertible [11, Lemma 2.6]. (Note that H. Prüfer proved that every $F \in \mathbf{f}(D)$ is invertible if and only if every two generated nonzero ideal of D is invertible [12, page 7]; a similar result, for the t-invertibility case, was proved in [11, Lemma 1.7].) Now, let $a,b \in D \setminus \{0\}$ and $x \in ((a,b)(a,b)^{-1})^{-1} \supseteq D$. Then $x(a,b)(a,b)^{-1} \subseteq D$, or $x(a,b)^{-1} \subseteq (a,b)^{-1}$, or $xa^{-1}b^{-1}((a) \cap (b)) \subseteq a^{-1}b^{-1}((a) \cap (b))$. This is equivalent to $x((a) \cap (b)) \subseteq (a) \cap (b)$ or $x \in (((a) \cap (b)) : ((a) \cap (b))) = D$. This forces $D \subseteq ((a,b)(a,b)^{-1})^{-1} \subseteq D$. □

Call an integral domain D a *v-finite conductor* (for short, a *v-FC-*) *domain* if $(a) \cap (b)$ is a v-ideal of finite type, for every pair $a,b \in D \setminus \{0\}$ [13, Section 2].

The above definition of v-FC-domain makes use of the v-operation. We have a somewhat contrived solution for this, in the form of the following characterization of v-FC-domains.

Proposition 2.3. *An integral domain D with quotient field K is a v-FC-domain if and only if for each pair a, b in $D \setminus \{0\}$ there exist $y_1, y_2, \ldots, y_n \in K \setminus \{0\}$, with $n \geq 1$, such that $(a, b)^v = \bigcap\{y_i D \mid 1 \leq i \leq n\}$. Consequently, D is a v-FC-domain if and only if for each pair a, b in $D \setminus \{0\}$ there exist $z_1, z_2, \ldots, z_m \in K \setminus \{0\}$, with $m \geq 1$, such that $((a) \cap (b))^{-1} = \bigcap\{z_j D \mid 1 \leq j \leq m\}$.*

Proof. Let D be a v-FC-domain and let $a, b \in D \setminus \{0\}$. Then, there are $a_1, a_2, \ldots, a_n \in D$ such that $(a) \cap (b) = (a_1, a_2, \ldots, a_n)^v$. Dividing both sides by ab, we get $(a, b)^{-1} = a^{-1}b^{-1}((a) \cap (b)) = (a_1/ab, a_2/ab, \ldots, a_n/ab)^v$. This gives

$$(a, b)^v = \left(\left(\frac{a_1}{ab}, \frac{a_2}{ab}, \ldots, \frac{a_n}{ab}\right)^v\right)^{-1} = \bigcap\left\{\frac{ab}{a_i} D \mid 1 \leq i \leq n\right\}. \tag{2.1}$$

Conversely, if for each pair a, b in $D \setminus \{0\}$ there exist $y_1, y_2, \ldots, y_n \in K \setminus \{0\}$ such that $(a, b)^v = \bigcap\{y_i D \mid 1 \leq i \leq n\}$, then

$$a^{-1}b^{-1}((a) \cap (b)) = (a, b)^{-1} = ((a, b)^v)^{-1} = \left(\bigcap\{y_i D \mid 1 \leq i \leq n\}\right)^{-1}. \tag{2.2}$$

On the other hand, $(\bigcap\{y_i D \mid 1 \leq i \leq n\})^{-1} = (y_1^{-1}D, y_2^{-1}D, \ldots, y_n^{-1}D)^v$ [14, Lemma 1.1], and this gives $(a) \cap (b) = ((ab/y_1)D, (ab/y_2)D, \ldots, (ab/y_n)D)^v$.

For the "consequently" part, note that $((a) \cap (b))^{-1} = a^{-1}b^{-1}(a, b)^v$. □

An immediate consequence of the above results is the following characterization of PvMDs, in which statements (iii) and (iv) are "v-operation free."

Corollary 2.4. *The following are equivalent for an integral domain D,*

(i) *D is a PvMD,*

(ii) *D is a v-domain and a v-FC-domain,*

(iii) *for all $a, b \in D \setminus (0)$, $((a) \cap (b))^{-1}$ is a finite intersection of principal fractional ideals and $((a) \cap (b)) : ((a) \cap (b)) = D$,*

(iv) *for all $a, b \in D \setminus (0)$, $((a) \cap (b))^{-1}$ is a finite intersection of principal fractional ideals and $((a, b)^{-1} : (a, b)^{-1}) = D$.*

Proof. (i)⇔(ii) stems from the fact that D is a PvMD (resp., a v-domain) if and only if every two generated nonzero ideal of D is t-invertible (resp., v-invertible) [15, Lemma 1.7] (resp., [11, Lemma 2.6]). Moreover, every two generated ideal of D is t-invertible if and only if every two-generated ideals (a, b) of D is v-invertible and such that $(a, b)^{-1} = (x_1, x_2, \ldots, x_r)^v$ where $r \geq 1$ and $x_1, x_2, \ldots, x_r \in K$ [9, Theorem 1.1(c)]. Finally, since $(a, b)^{-1} = a^{-1}b^{-1}((a) \cap (b))$, $(a, b)^{-1}$ is a fractional v-ideal of finite type if and only if $(a) \cap (b)$ is a v-ideal of finite type.

(ii)⇔(iii) and (ii)⇔(iv) are straightforward consequences of Theorem 2.2 and Proposition 2.3. □

Recall that an integral domain D is called a *finite conductor* (for short, *FC-*) *domain* if $((a) \cap (b))$ is finitely generated for each pair $a, b \in D$. Just to show how far we have traveled since 1978, when this notion was introduced, we state and provide an easy proof to the following statement, which appeared as the main result in [16, Theorem 2].

Corollary 2.5. *An integrally closed FC-domain is a PvMD.*

Proof. First note that, since D is integrally closed $(F : F) = D$ for every finitely generated ideal F of D [6, Theorem 34.7]. So, for each pair $a, b \in D \setminus \{0\}$, since D is a FC-domain, $((a) \cap (b)) : ((a) \cap (b)) = D$. But this makes D a v-domain by Theorem 2.2 and, so, a PvMD by Corollary 2.4. □

Lemma 2.1 can also be instrumental in characterizing completely integrally closed (for short, CIC-) domains (see, e.g., [6, Theorem 34.3]). Also the previous approach leads to a characterization of Krull domains in a manner similar to the characterization of v-domains leading to the characterization of PvMDs.

Proposition 2.6. *The following are equivalent for an integral domain D:*

(i) *D is a CIC-domain,*

(ii) $(A^{-1} : A^{-1}) = D$ *for all* $A \in \mathbf{F}(D)$.

In particular, a CIC-domain is a v-domain.

Proof. Note that D is CIC if and only if every $A \in \mathbf{F}(D)$ is v-invertible [6, Proposition 34.2 and Theorem 34.3]. Now, the equivalence (i)⇔(ii) is an immediate consequence of Lemma 2.1. The last statement is a straightforward consequence of the equivalence (i)⇔(ii) of Theorem 2.2. □

Remark 2.7. We have been informed by the referee that he/she has used Proposition 2.6 while teaching a course on multiplicative ideal theory. So, like Lemma 2.1, this is another folklore result in need of a standard reference.

Theorem 2.8. *The following are equivalent for an integral domain D:*

(i) *D is a Krull domain,*

(ii) *D is a Mori v-domain,*

(iii) *for each $A \in \mathbf{F}(D)$, there exist $y_1, y_2, \ldots, y_n \in A$ such that $A^{-1} = \bigcap\{y_i^{-1}D \mid 1 \leq i \leq n\}$ and, for all $a, b \in D \setminus \{0\}$, $((a) \cap (b)) : ((a) \cap (b)) = D$,*

(iv) *for each $A \in \mathbf{F}(D)$, there exist $x, y \in A$ such that $A^{-1} = x^{-1}D \cap y^{-1}D$ and for all $a, b \in D \setminus \{0\}$, $((a) \cap (b)) : ((a) \cap (b)) = D$.*

Before we prove Theorem 2.8, it seems pertinent to give some introduction. For a quick review of Krull domains, the reader may consult the first few pages of [17]. A number of characterizations of Krull domains can be also found in [14, Theorem 2.3]. The one that we can use here is: *D is a Krull domain if and only if each $A \in \mathbf{F}(D)$ is t-invertible*. Which means, as observed above, that *D is a Krull domain if and only if for each $A \in \mathbf{F}(D)$, A is v-invertible and A^{-1} is a fractional v-ideal of finite type*. In particular, we reobtain that a Krull domain is a PvMD (and so, in particular, a v-domain).

An integral domain D is called a *Mori domain* if D satisfies the ascending chain condition on integral divisorial ideals (see, e.g., [18]). Different aspects of Mori domains were studied by Toshio Nishimura in a series of papers. For instance, in [19, Theorem, page 2], he showed that *a domain D is a Krull domain if and only if D is a Mori domain and completely integrally closed*. For another proof of this result, see [20, Corollary 2.2].

On the other hand, an integral domain D is a Mori domain if and only if, for each $A \in \mathbf{F}(D)$, A^v is a fractional v-ideal of strict finite type [21, Lemma 1]. A variation of this characterization is given next.

Lemma 2.9. *Let D be an integral domain. Then, D is Mori if and only if for each $A \in \mathbf{F}(D)$ there exist $y_1, y_2, \dots, y_n \in A \setminus \{0\}$, with $n \geq 1$, such that $A^{-1} = \bigcap\{{y_i}^{-1}D \mid 1 \leq i \leq n\}$.*

Proof. As we observed above, D is a Mori domain if and only if for each $A \in \mathbf{F}(D)$ there exist $y_1, y_2, \dots, y_n \in A \setminus \{0\}$ such that $A^v = (y_1, y_2, \dots, y_n)^v$. This last equality is equivalent to $A^{-1} = (y_1, y_2, \dots, y_n)^{-1} = \bigcap\{{y_i}^{-1}D \mid 1 \leq i \leq n\}$, since, by [14, Lemma 1.1], we have

$$\left(\bigcap\{y_i^{-1}D \mid 1 \leq i \leq n\}\right)^{-1} = \left(\left(y_1^{-1}\right)^{-1}, \left(y_2^{-1}\right)^{-1}, \dots, \left(y_n^{-1}\right)^{-1}\right)^v = (y_1, y_2, \dots, y_n)^v. \tag{2.3}$$

□

Proof of Theorem 2.8. (i)⇒(ii) because we already observed that a Krull domain is a CIC Mori domain. Moreover, a CIC-domain is a v-domain (Proposition 2.6).

(ii)⇒(i) We want to prove that, for each $A \in \mathbf{F}(D), A$ is v-invertible and A^{-1} is a fractional v-ideal of finite type. The second property is a particular case of the assumption that every fractional divisorial ideal of D is a v-ideal of finite type. For the first property, we have that, for each $A \in \mathbf{F}(D)$, there exists $F \in \mathbf{f}(D)$, with $F \subseteq A$, such that $A^v = F^v$ (or, equivalently, $A^{-1} = F^{-1}$). Since D is a v-domain, we have $D = (FF^{-1})^v = (F^vF^{-1})^v = (A^vF^{-1})^v = (AA^{-1})^v$.

(ii)⇔(iii) is a straightforward consequence of Lemma 2.9 and Theorem 2.2 ((i)⇔(v)).

(iii)⇒(iv) follows form the fact that (iii)⇔(i) and, if D is a Krull domain, then for every $A \in \mathbf{F}(D)$ there exist $x, y \in A$ such that $A^v = (x, y)^v$ [22, Proposition 1.3]. Therefore, $A^{-1} = (x, y)^{-1} = x^{-1}D \cap y^{-1}D$.

(iv)⇒(iii) is trivial. □

Remark 2.10. (1) In (iii) of Theorem 2.8, we cannot say that for every $A \in \mathbf{F}(D)$ the inverse A^{-1} is expressible as a finite intersection of principal fractional ideals, because this would be equivalent to A^v being of finite type for each $A \in \mathbf{F}(D)$. But there do exist non-Mori domains D such that A^v is of finite type for all $A \in \mathbf{F}(D)$. For a discussion of those examples you may consult [23, Section 2] and [8, Section (4c)].

(2) Note that a Mori domain is obviously a v-FC-domain, since in a Mori domain every divisorial ideal is a v-ideal of (strict) finite type. Therefore, the equivalences (i)⇔(ii) of Theorem 2.8 and of Corollary 2.4 shed new light on the relations between PvMDs and Krull domains in the class of v-domains.

While several of the above results provide characterizations of Prüfer v-multiplication domains, v-domains, Mori and Krull domains, without using Krull's theory of star operations, they do not diminish the importance of star operations in any way. After all, it was the star operations that developed the notions mentioned above this far. An interested reader will have to extend this work further so that mainstream techniques could be used. To make a start in that direction, we give below some further "star operation free" characterizations of PvMDs, besides the ones we have already given above.

Given an integral domain D, a *prime ideal* P is called *essential for* D if D_P is a valuation domain and the *domain* D is called *essential* if there is a family of essential primes $\{P_\alpha\}$ for D such that $D = \bigcap D_{P_\alpha}$. Also, call a prime ideal P of D an *associated prime of a principal ideal* if P is a minimal prime over a proper nonzero ideal of the type $((a) :_D (b))$, for some

$a, b \in D$. The associated primes of principal ideals have been discussed in [24], where it was also shown that if S is a multiplicative set in D then $D_S = \bigcap D_P$ where P ranges over the associated primes of principal ideals disjoint from S [24, Proposition 4]. For brevity, we call here an associate prime of a principal ideal of D simply an *associated prime of* D.

Following [25], call D a *P-domain* if every associated prime of D is essential. It is easy to see that a *P-domain is an essential domain*. More precisely, it was shown in [25, Proposition 1.1] that D is a *P-domain if and only if* D *is essential and every quotient ring of* D *is essential*. Also, if D is a P-domain then so are the rings of fractions of D and the rings of polynomials over D [25, Corollary 1.2]. From [25, Corollary 1.4 and Example 2.1] one can also get the information that a PvMD is a P-domain, but not conversely.

We now state a result that is already known but that can be of use if someone wants to deal with PvMDs without having to use, in statements (ii) and (iii), the star operations.

Proposition 2.11. *The following are equivalent for an integral domain* D*:*

(i) D *is a P*v*MD,*

(ii) D *is a P-domain such that, for every pair* $a, b \in D \setminus \{0\}$, $((a) \cap (b))^{-1}$ *is a finite intersection of principal fractional ideals,*

(ii′) D *is a P-domain and a* v*-FC-domain,*

(iii) D *is an essential domain such that, for every pair* $a, b \in D \setminus \{0\}$, $((a) \cap (b))^{-1}$ *is a finite intersection of principal fractional ideals,*

(iii′) D *is an essential* v*-FC-domain.*

Proof. As we already mentioned above, from [25] we know that a PvMD is a P-domain and that a P-domain is essential. Moreover, from Corollary 2.4, if D is a PvMD, we have, for every pair $a, b \in D \setminus \{0\}$, that $((a) \cap (b))^{-1}$ is a finite intersection of principal fractional ideals (or, equivalently, D is a v-FC-domain, by Proposition 2.3). Therefore, (i)⇒(ii)⇒(iii), (ii)⇔(ii′), and (iii)⇔(iii′).

(iii)⇒(ii). Recall that, from [26, Lemma 3.1], we have that an essential domain is a v-domain (the reader may also want to consult the survey paper [4, Proposition 2.1] and, for strictly related results, [27, Lemma 4.5] and [28, Theorem 3.1 and Corollary 3.2]). The conclusion follows from Corollary 2.4 ((ii)⇒(i)) (and Proposition 2.3). □

Remark 2.12. Note that, from the proof of Proposition 2.11 ((iii)⇒(ii)), we have that each of the statements of Proposition 2.11 is equivalent to

(iv) D *is a* v*-domain such that, for every pair* $a, b \in D \setminus \{0\}$, $((a) \cap (b))^{-1}$ *is a finite intersection of principal fractional ideals*

which is obviously also equivalent to (ii) of Corollary 2.4.

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