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Research Article

Lattice Operators and Topologies

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Working within a complete (not necessarily atomic) Boolean algebra, we use a sublattice to define a topology on that algebra. Our operators generalize *complement* on a lattice which in turn abstracts the set theoretic operator. Less restricted than those of Banaschewski and Samuel, the operators exhibit some surprising behaviors. We consider properties of such lattices and their interrelations. Many of these properties are abstractions and generalizations of topological spaces. The approach is similar to that of Bachman and Cohen. It is in the spirit of Alexandroff, Frolík, and Nöbeling, although the setting is more general. Proceeding in this manner, we can handle diverse topological theorems systematically before specializing to get as corollaries as the topological results of Alexandroff, Alo and Shapiro, Dykes, Frolík, and Ramsay.

1. Introduction

We begin with $\mathcal{L}$, a sublattice of a complete (not necessarily atomic) Boolean algebra $\mathcal{B}$. If $\mathcal{L}$ is closed under arbitrary meets, it abstracts the closed sets of a topological space. If not, we introduce a Kurotowski closure operator to define the associated topological lattice. The operators we define generalize *complement* on a lattice which in turn abstracts the set theoretic operator. Less restricted than those of Banaschewski [1] and Samuel [2], the operators exhibit some surprising behaviors. We consider certain properties of such lattices and the implications for the properties of one lattice from those of another, when one is a sublattice of the other. Many of these properties are abstractions and generalizations of topological spaces.

The approach is similar to that of Bachman and Cohen [3, 4]. It is in the spirit of Alexandroff [5], Frolík [6], and Nöbeling [7], although the setting is more general. We generalize a variety of filter arguments used in paved space [8, 9] and in the theory of realcompactness [10, 11]. Proceeding in this manner, we can handle diverse topological theorems systematically before specializing to get as corollaries as the topological results of [5, 8, 12–14].

Section 2 provides some background material and generates a topology on an algebra by means of a sublattice. Section 3 defines operators and topological type properties for a lattice. Section 4 examines filter and measure behavior with respect to the operators. Section 5 looks at covering properties. Section 6 investigates the relationships between two lattices.

2. Background, Terminology, and Notation

We work within a complete Boolean algebra $\mathcal{B}$ with minimal element $\mathbf{0}$ and maximal element $\mathbf{e}$. The usual operators are denoted by $\vee$, $\wedge$, and $'$. $\mathcal{B}$ is not necessarily atomic; equivalently, $\mathcal{B}$ is not necessarily completely distributive [15].

(i) μ, μ_f denote finitely additive zero-one measures on $\mathcal{B}$.

(ii) $\mathcal{L}$, $\mathcal{L}_1$, and $\mathcal{L}_2$ denote sublattices of $\mathcal{B}$ containing $\mathbf{0}$ and $\mathbf{e}$.

(iii) $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$.

(iv) $\wp(S)$ is the power set of the set S.

(v) The indices i, j, k, and n index countable (finite or countably infinite) collections, while α, β, and γ index arbitrary ones.

Definition 2.1. (i) $F \subseteq \mathcal{L}$ is an **$\mathcal{L}$-filter** if and only if for all $a, b \in \mathcal{L}$:

(a) $a, b \in F \Rightarrow a \wedge b \in F$,

(b) $a \in F$ and $a \leq b \Rightarrow b \in F$.

When there is no ambiguity, we simply say that F is a **filter**.

(ii) An $\mathcal{L}$-filter F is a **prime** $\mathcal{L}$-filter if and only if for all $a, b \in \mathcal{L}$, $a \vee b \in F \Rightarrow$ either $a \in F$ or $b \in F$.

(iii) An $\mathcal{L}$-filter F is an **$\mathcal{L}$-ultrafilter** (or **ultra**) if and only if F is a maximal $\mathcal{L}$-filter.

(iv) A filter F is **fixed** if and only if $\wedge F \neq \mathbf{0}$. Otherwise F is **free**.

(v) A filter F has **cmp** (**countable meet property** or **countable intersection property**) if and only if for any *countable* subset of F, $\bigwedge_i a_i \neq \mathbf{0}$.

Remark 2.2. It follows that

(i) $\mathbf{0} \notin F$ if and only if $F \neq \mathcal{L}$,

(ii) every ultrafilter is prime.

In this paper, $\mathbf{0}$ is not in any filter.

Definition 2.3. A measure μ on an algebra $\mathcal{B}$ containing $\mathcal{L}$ is **$\mathcal{L}$-regular** if and only if for all $b \in \mathcal{B}$, $\mu(b) = \sup\{\mu(a) : a \in \mathcal{L}, a \leq b\}$.

Lemma 2.4. *There exist one-to-one correspondences between*

(i) *zero-one measures on $\mathcal{L}$ and prime $\mathcal{L}$-filters,*

(ii) *$\mathcal{L}$-regular zero-one measures on $\mathcal{A}(\mathcal{L})$, and $\mathcal{L}$-ultrafilters [3, 4].*

Thus, analogous measure theoretic results may easily be derived from our filter statements.

We now topologize $\mathcal{B}$ by means of a sublattice $\mathcal{L}$. The lattice elements themselves may not be sufficient to be used as open or closed sets. However, we will generate a topology.

Definition 2.5. Let $\mathcal{B}$ be an algebra, $\mathcal{L}$ a sublattice of $\mathcal{B}$, and $a, b \in \mathcal{B}$.

(i) $\overline{\mathbf{b}} = \bigwedge\{a \in \mathcal{L} : a \geq b\}$.

(ii) $\mathbf{b^0} = \bigvee\{a' : a \in \mathcal{L} \text{ and } a' \leq b\}$.

(iii) $b \in \mathcal{B}$ is **closed** if and only if $b = \overline{b}$.

(iv) $\tau(\mathcal{L})$ denotes the set of closed elements of $\mathcal{B}$.

Remark 2.6. (i) $b \mapsto \overline{b}$ is a Kurotowski closure operator.

(ii) $b \mapsto b^0$ is an interior operator.

(iii) $\tau(\mathcal{L}) = \{b \in \mathcal{B} : b = \bigwedge_\alpha a_\alpha,\ a_\alpha \in \mathcal{L}\}$.

(iv) $\mathcal{L} \subseteq \tau(\mathcal{L})$.

(v) $\tau(\mathcal{L})$ is a lattice which is closed under arbitrary meets.

We observe that $\tau(\mathcal{L})$ is an obvious abstraction of the closed sets in a topological space.

Example 2.7. Let $\mathcal{L}$ be the lattice of zero sets in a T_2 completely regular topological space. Then $\tau(\mathcal{L})$ is the lattice of closed sets [11].

Definition 2.8. Let $\mathcal{M} \subseteq \mathcal{B}$ be a meet semilattice (i.e., a subset of $\mathcal{B}$ closed under finite meets). Consider the generated lattice $\mathcal{L}(\mathcal{M}) = \{\bigvee_1^n\{a_i : a_i \in \mathcal{M}\} : n \in \mathbb{N}\}$. All terminology remains the same except that F is a **prime $\mathcal{M}$-filter** if and only if $\bigvee_1^n a_i \in F$ (for $a_i \in \mathcal{M}$) implies that one of the $a_i \in F$.

Lemma 2.9. (i) *There exist one-to-one correspondences between the prime, ultra-, fixed, and free filters on $\mathcal{M}$ and those on $\mathcal{L}(\mathcal{M})$.*

(ii)*$\mathcal{M}$-filters with cmp correspond with $\mathcal{L}$-filters with cmp.*

Thus we lose no generality in "treating" $\mathcal{M}$ like a lattice.

Example 2.10. Let $\mathcal{M} = \{b \in \mathcal{B} : b = (\overline{b})^0\}$. $\mathcal{M}$ is a meet semi-lattice since $\overline{a \wedge b} \leq \overline{a} \wedge \overline{b}$ implies that $(\overline{a \wedge b})^0 \leq (\overline{a} \wedge \overline{b})^0 = (\overline{a})^0 \wedge (\overline{b})^0 = a \wedge b$, for all $a, b \in \mathcal{M}$. "An open subset G in a topological space is **regularly open** if and only if G is the interior of its closure" [16]. Thus the regularly open sets in a topological space form a meet semi-lattice.

Table 1 summarizes the notation used in this paper.

3. Lattice Operators and Properties

In this section we define certain operators and lattice properties. These properties reduce to the conventional topological ones when the operator is taken to be *complement*.

Definition 3.1. Let $\mathcal{M}$ be a meet semi-lattice containing $\mathbf{0}$ and $\mathbf{e}$. We define $\mathbf{T}$ to be a one-to-one operator on $\mathcal{M}$ such that $T(a \vee b) = T(a) \wedge T(b)$ and $T(\mathbf{e}) = \mathbf{0}$. We define $\mathbf{a}^* = T(a)$ and $\mathcal{L}^* = \{a^* : a \in \mathcal{L}\}$.

Table 1: Summary of notation.

$\mathcal{B}$	Complete Boolean algebra with zero ($\mathbf{0}$) and unit ($\mathbf{e}$)
$\mathcal{L},\ \mathcal{L}_n$	Lattices with $\mathbf{0}$ and $\mathbf{e}$
$F,\ F_n,\ G,\ H$	Filters without $\mathbf{0}$
$\mathcal{A}(\mathcal{L})$	The Algebra generated by $\mathcal{L}$
$\mu,\ \mu_f$	Finitely additive zero-one measures on $\mathcal{B}$
$\tau(\mathcal{L})$	Closed elements of $\mathcal{B}$
$\mathcal{M}$	Meet semi-lattice
$\wp(S)$	Power set of the set S

Proposition 3.2. (a) $a \le b \Leftrightarrow T(a) \ge T(b)$.

(b) $\bigvee_1^n T(a_i) \le T(\bigwedge_1^n a_i)$.

(c) $\bigvee_1^n T(a_i) = \mathbf{e} \Rightarrow \bigwedge_1^n a_i = \mathbf{0}$.

(d) $T(\bigvee_1^n a_i) = \mathbf{0} \Rightarrow \bigvee_1^n a_i = \mathbf{e}$.

Proof. (a) $a \le b \Leftrightarrow a \vee b = b \Leftrightarrow T(a) \wedge T(b) = T(b) \Leftrightarrow T(a) \ge T(b)$,

(b) For all $j,\ a_j \ge \bigwedge_1^n a_i \Rightarrow$ for all $j,\ T(a_j) \le T(\bigwedge_1^n a_i) \Rightarrow \bigvee_1^n T(a_j) \le T(\bigwedge_1^n a_i)$.
The proof of (c) and (d) follows readily. □

From now on, we assume that T is defined on a lattice $\mathcal{L}$. Then $\mathcal{L}^*$ is a meet semi-lattice. By Lemma 2.9, we may "treat" $\mathcal{L}^*$ like a lattice.

Corollary 3.3. *When T is defined on $\tau(\mathcal{L})$, one has the following.*

(a) $\bigvee_\alpha T(a_\alpha) \le T(\bigwedge_\alpha a_\alpha)$.

(b) $\bigvee_\alpha T(a_\alpha) = \mathbf{e} \Rightarrow \ \bigwedge_\alpha a_\alpha = \mathbf{0}$.

Example 3.4. Let $S = \{1,2,3,4\}$ and $\mathcal{B} = \wp(S)$ (the power set of S) with set union and intersection as the join and meet operations. Let $\mathcal{L} = \{\emptyset, a, b, S\}$ with $a = \{1,2\}$ and $b = \{3,4\}$ as in Figure 1. The operator T is defined by $T(S) = \emptyset$, $T(a) = \{4\}$, $T(b) = \{2\}$, and $T(\emptyset) = \{2,4\}$. We have $T(a \vee b) = T(a) \wedge T(b)$ and $T(S) = \emptyset$. In addition $T(a \wedge b) = T(a) \vee T(b)$ and $T(a) \wedge a = \emptyset$. Note that, unlike in [2], $T(a)$ is not the maximal element disjoint from a, and although $\mathcal{L}$ is complemented (that is, $a \in \mathcal{L} \Rightarrow a' \in \mathcal{L}$), $T(a) \ne a'$.

We now define various properties for $\mathcal{L}$. They generalize some of the definitions in point set topology, reducing to the conventional properties when $\mathcal{B}$ is the power set of a set X and T is complement.

Definition 3.5. (i) $\mathcal{L}$ is **compact** if and only if every $\mathcal{L}$-filter is fixed.

(ii) $\mathcal{L}$ is $\aleph_0$**-compact** if and only if every $\mathcal{L}$-filter has cmp.

(iii) $\mathcal{L}$ is an **I-lattice (P-lattice)** if and only if every (prime) $\mathcal{L}$-filter with cmp is contained in an ultrafilter with cmp.

(iv) $\mathcal{L}$ is an **R-lattice** if and only if every filter which contains a fixed prime filter is also fixed.

(v) A topological space is an **I-space** if and only if the lattice of closed sets is an **I-lattice** [17].

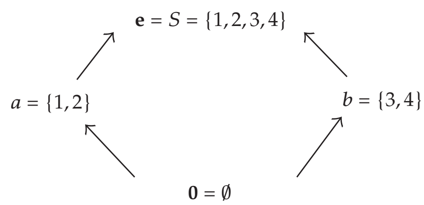

Figure 1: $\mathcal{L}$ in Example 3.4.

Proposition 3.6. *As an immediate consequence of Definition 3.5, one has the following.*

(i) *$\mathcal{L}$ is compact $\Rightarrow$ $\mathcal{L}$ is $\aleph_0$-compact $\Rightarrow$ $\mathcal{L}$ is an I-lattice $\Rightarrow$ $\mathcal{L}$ is a P-lattice.*

(ii) *The following are equivalent:*

(a) *$\mathcal{L}$ is compact ($\aleph_0$-compact),*

(b) *every prime filter is fixed (has cmp),*

(c) *every ultrafilter is fixed (has cmp).*

Definition 3.7. (i) $\mathcal{L}$ is **$\aleph_0$-paracompact** if and only if whenever there exists $\{a_n\} \subseteq \mathcal{L}$ with $a_n \downarrow \mathbf{0}$, there exists $\{b_n\} \subseteq \mathcal{L}$ such that $a_n \leq b_n^*$ and $b_n^* \downarrow \mathbf{0}$.

(ii) When $\mathcal{L} \subseteq \tau(\mathcal{L}^*)$, we say that $\mathcal{L}$ is **perfect**.

Proposition 3.8. *Every perfect lattice is $\aleph_0$-paracompact.*

Proof. Assume $\mathcal{L}$ is perfect and $a_n \downarrow \mathbf{0}$. For each a_n, there exists $\{b_{n_\alpha}\} \subseteq \mathcal{L}$ such that $a_n = \bigwedge_\alpha b_{n_\alpha}^*$. Let $c_n^* = \wedge\{b_{k_\alpha}^* : k \leq n\}$. Then $c_n^* \geq a_n$ and $c_n^* \downarrow \mathbf{0}$. □

Example 3.9. The zero sets in a topological space are perfect (i.e., *complement generated* in the sense of [11]) and thus $\aleph_0$-paracompact.

Definition 3.10. Let $a, a_i, b \in \mathcal{B}$, and $f, f_i, g, g_i \in \mathcal{L}$. Then the following are given.

(i) $\mathcal{L}$ is $\mathbf{T}_1$ if and only if for all $a_1 \nleq a_2$ there exists $g^* \in \mathcal{L}^*$ such that $a_2 \leq g^*$ but $a_1 \nleq g^*$.

(ii) $\mathcal{L}$ is **Hausdorff** if and only if for all $a_1, a_2 \neq \mathbf{0}$ with $a_1 \wedge a_2 = \mathbf{0}$ there exist $g_1^*, g_2^* \in \mathcal{L}^*$ such that $a_1 \wedge g_1^* \neq \mathbf{0}, a_2 \wedge g_2^* \neq \mathbf{0}$, and $g_1^* \wedge g_2^* = \mathbf{0}$.

(iii) $\mathcal{L}$ is **regular** if and only if for all $b \in \mathcal{B}$, $b \neq \mathbf{0}$, $f \in \mathcal{L}$, $b \wedge f = \mathbf{0}$ there exist $g_1^*, g_2^* \in \mathcal{L}^*$ such that $b \wedge g_1^* \neq \mathbf{0}, f \leq g_2^*$, and $g_1^* \wedge g_2^* = \mathbf{0}$.

The following proposition provides an example.

Proposition 3.11. *Let $\mathcal{B}$ be the power set of a topological space X, let $\mathcal{L}$ be the lattice of closed sets in X, and let T be complement. X is a T_1 (Hausdorff, regular) space if and only if $\mathcal{L}$ is T_1 (Hausdorff, regular).*

Proof. Let $\mathcal{L}$ be the lattice of closed sets in X. Suppose $\mathcal{L}$ is a T_1 lattice. Let $a_1 \neq a_2$ be atoms in $\mathcal{B}$. Since $a_1 \nleq a_2$, there exists $g_2' \in \mathcal{L}'$ such that $a_2 < g_2'$ but $a_1 \nleq g_2'$. By symmetry, there exists $g_1' \in \mathcal{L}'$ such that $a_1 \leq g_1'$ but $a_2 \nleq g_1'$. Thus X is a T_1 space.

Now suppose X is a T_1 topological space. Let $b, c \in \mathcal{B}$, $b \nleq c$. Then there exists an atom $a \leq b$ but $a \nleq c$. Thus for all atoms $a_\alpha \leq c, a \neq a_\alpha$, and there exists $g_\alpha' \in \mathcal{L}'$ such that $a_\alpha \leq g_\alpha'$ but $a \nleq g_\alpha'$. Let $g' = \bigvee_\alpha g_\alpha'$. Then $c \leq g', b \nleq g'$, and thus $\mathcal{L}$ is T_1.

The proofs for Hausdorff and regular are similar. □

Proposition 3.12. *Suppose $T(a) \wedge a = \mathbf{0}$ and $\mathcal{L}$ is regular. If F_1 is prime and $F_1 \subseteq F_2$, then $\bigwedge F_1 = \bigwedge F_2$.*

Proof. Suppose there exists $F_1 \subseteq F_2$ such that $\bigwedge F_1 \neq \bigwedge F_2$. Let $a = \bigwedge F_1$. $a \nleq \bigwedge F_2$ implies there exists $f \in F_2$ with $a \nleq f$. But then $b = a \wedge f' \neq \mathbf{0}$ and $b \wedge f = \mathbf{0}$. By regularity, there exist $c_1^*, c_2^* \in \mathcal{L}^*$ such that $b \bigwedge c_1^* \neq \mathbf{0}$, $f \leq c_2^*$, and $c_1^* \wedge c_2^* = \mathbf{0}$. Now $f \leq c_2^*$ implies that $c_2 \notin F_2 \supseteq F_1$. From $b \wedge c_1^* \neq \mathbf{0}$, it follows that $a \nleq c_1$ and thus $c_1 \notin F_1$. But $c_1^* \wedge c_2^* = \mathbf{0}$ implies that $c_1 \vee c_2 = \mathbf{e}$, and thus F_1 is not prime. □

Example 3.13. The closed sets in a regular topological space form an R-lattice.

Definition 3.14. $\mathcal{L}$ is **normal** if and only if for all $f_1, f_2 \in \mathcal{L}$, $f_1 \wedge f_2 = \mathbf{0}$ there exist $g_1^*, g_2^* \in \mathcal{L}^*$ such that $f_1 \leq g_1^*$, $f_2 \leq g_2^*$, and $g_1^* \wedge g_2^* = \mathbf{0}$.

The following proposition demonstrates an example of an application.

Proposition 3.15. *Let X be a topological space. Let T be complement and let $\mathcal{L}'$ be the lattice of open sets. Then the following are equivalent.*

(a) *$\mathcal{L}'$ is normal.*

(b) *$a', b' \in \mathcal{L}'$ and $a' \wedge b' = \mathbf{0} \Rightarrow \overline{a'} \wedge \overline{b'} = \mathbf{0}$.*

(c) *$a' \in \mathcal{L}' \Rightarrow \overline{a'} \in \mathcal{L}'$ (i.e., X is extremally disconnected).*

Proof. Consider the following. (i) (a) implies (b): let $a', b' \in \mathcal{L}'$, $a' \wedge b' = \mathbf{0}$. Then there exist $c, d \in \mathcal{L}$ such that $a' \leq c$, $b' \leq d$, and $c \wedge d = \mathbf{0}$ (by normality). But $\overline{a'} \leq c$ and $\overline{b'} \leq d$ implies that $a' \wedge b' = \mathbf{0}$.

(ii) (b) implies (c): let $a' \in \mathcal{L}'$. Let $b = \overline{a'}$ so that $a' \wedge b' = \mathbf{0}$. Then $\overline{a'} \wedge \overline{b'} = \mathbf{0}$ (by hypothesis), so that $b \wedge \overline{b'} = \mathbf{0}$. Then $b \leq (\overline{b'})' = b^0$, so that $b = b^0$ and $b \in \mathcal{L}'$. Therefore $\overline{a'} \in \mathcal{L}'$ by definition of b.

(iii) (c) implies (b): let $a', b' \in \mathcal{L}', a' \wedge b' = \mathbf{0}$, so that $\overline{a'} \wedge b' = \mathbf{0}$. Since $\overline{a'} \in \mathcal{L}'$, then $\overline{a'} \wedge \overline{b'} = \mathbf{0}$.

(iv) (b) implies (a): $\overline{a'}, \overline{b'}$ are elements of $\mathcal{L}$. □

Proposition 3.16. *Let $T(a) \wedge a = \mathbf{0}$. If $\mathcal{L}$ is normal and F is a prime $\mathcal{L}$-filter contained in two $\mathcal{L}$-ultrafilters G and H, then $G = H$.*

Proof. Let G and H be two distinct ultrafilters and let $F \subseteq G \cap H$. Then there exist $g \in G$, $h \in H$ with $g \wedge h = \mathbf{0}$. By normality, there exist $a^*, b^* \in \mathcal{L}^*$ such that $g \leq a^*, h \leq b^*$, and $a^* \wedge b^* = \mathbf{0}$. But $g \wedge a = \mathbf{0}$ implies that $a \notin G$, and $h \wedge b = \mathbf{0}$ implies that $b \notin H$, so that $a, b \notin G \cap H \supseteq F$, that is, $a, b \notin F$. But $a \vee b = \mathbf{e}$ implies that F is not prime. □

Definition 3.17. $\mathcal{L}$ is $\aleph_0$**-normal** if and only if it is normal and $\aleph_0$-paracompact.

Example 3.18. A normal topological space is countably paracompact if and only if the lattice of closed sets is $\aleph_0$-paracompact [18]. Willard [16] calls such a space *binormal*.

Proposition 3.19. *Let T be complement in $\mathcal{B}$. If $\mathcal{L}$ is $\aleph_0$-normal and F_1 is a prime filter with cmp, then $F_1 \subseteq F_2$ implies that F_2 has cmp.*

Proof. Let $F_1 \subseteq F_2$ where F_1 is a prime $\mathcal{L}$-filter and F_2 is an $\mathcal{L}$-filter without cmp. Then there exists $\{f_k\} \subseteq F_2$ such that $\bigwedge_k \{f_k\} = \mathbf{0}$. Let $g_n = \bigwedge_1^n f_k$. $\{g_n\} \subseteq F_2$ and $g_n \downarrow \mathbf{0}$. Thus there exist $\{b_n\} \subseteq \mathcal{L}$ such that $g_n \leq b_n'$ and $b_n' \downarrow \mathbf{0}$. Now $g_n \wedge b_n = \mathbf{0}$, so there exist two sequences $\{c_n\}, \{d_n\}$ such that $g_n \leq c_n'$, $b_n \leq d_n'$, $c_n \wedge d_n = \mathbf{0}$, for all n. Since $c_n \vee d_n = \mathbf{e}$ and F_1 is prime, we have $c_n \in F_1$ or $d_n \in F_1$, for all n. But $g_n \leq c_n'$ implies that $c_n \notin F_1$, so $d_n \in F_1$, for all n. And since $b_n' \geq d_n$ implies that $d_n \downarrow \mathbf{0}$, we have that F_1 does not have cmp. $\square$

Remark 3.20. If $\mathcal{L}$ is a normal lattice that has the stronger property that whenever $\{a_n\} \subseteq \mathcal{L}$ such that $\bigvee_n a_n^* = \mathbf{e}$ there exists $\{b_n\} \subseteq \mathcal{L}$ such that $\bigvee_n b_n = \mathbf{e}$ and for all n, $a_n^* \geq b_n$, then we need only to assume that $T(a) \wedge a = \mathbf{0}$.

Corollary 3.21. *Let T be complement in $\mathcal{B}$. If $\mathcal{L}$ is $\aleph_0$-normal, then $\mathcal{L}$ is a P-lattice.*

4. Behavior of Filters and Measures Under T

In this section we look at the behavior of filters and measures with respect to T. It is interesting to see an example where T does not "behave as nicely" as complement.

Definition 4.1. Let $D \subseteq \mathcal{L}$. $\widetilde{\mathbf{D}} = \{a^* \in \mathcal{L}^* : a \notin D\}$.

Proposition 4.2. *Let F be a prime $\mathcal{L}$-filter. $\widetilde{F}$ is a prime $\mathcal{L}^*$-filter.*

Proof. (a) $\mathbf{0} \notin \widetilde{F}$ since $\mathbf{e} \in F$.

(b) $a^* \leq b^*, a^* \in \widetilde{F} \Rightarrow a \geq b$ and $a \notin F \Rightarrow b \notin F \Rightarrow b^* \in \widetilde{F}$.

(c) Let $a^*, b^* \in \widetilde{F}$, so that $a, b \notin F$; equivalently, $a \vee b \notin F$. But then $(a \vee b)^* \in \widetilde{F}$, and so $a^* \wedge b^* \in \widetilde{F}$. Thus (by (a), (b), and (c)), $\widetilde{F}$ is a filter.

(d) Let $a^*, b^* \in \mathcal{L}^*, a^* \vee b^* \geq c^* \in \widetilde{F}$. $(a \wedge b)^* \geq a^* \vee b^* \geq c^* \in \widetilde{F} \Rightarrow a \wedge b \leq c \notin F \Rightarrow a \wedge b \notin F \Rightarrow a \notin F$ or $b \notin F \Rightarrow a^* \in \widetilde{F}$ or $b^* \in \widetilde{F}$. Thus $\widetilde{F}$ is prime. $\square$

Definition 4.3. A filter F is **coultra** if and only if $\widetilde{F}$ is $\mathcal{L}^*$-ultra. μ_F is **coregular** if and only if F is coultra.

Proposition 4.4. *$\widetilde{F}$ is $\mathcal{L}^*$-ultra if and only if $a \in F$ is equivalent to $a^* \leq (b^*)'$ for some $b^* \in \widetilde{F}$ (i.e., $a^* \notin \widetilde{F} \Leftrightarrow$ there exists $b^* \in \widetilde{F}$ such that $a^* \leq (b^*)'$).*

Remark 4.5. Proposition 4.4 generalizes a theorem of [12] which we get by taking T to be complement.

As demonstrated by the following example, we can associate measures with the prime filters on $\mathcal{L}$ as usual, but they may lack some of the properties to which we are accustomed.

Example 4.6. Let $S = \{1, 2, 3, 4\}$. Let $\mathcal{B} = \mathcal{P}(S)$, with set union and intersection as the join and meet operations. Figure 2 defines the sublattice $\mathcal{L}$.

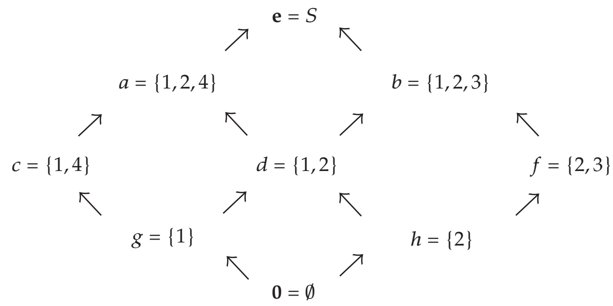

Figure 2: $\mathcal{L}$ in Example 4.6.

Define T on $\mathcal{L}$ as follows:

$$
\begin{gathered}
T(\mathbf{e}) = \mathbf{0}, \quad T(\mathbf{0}) = \mathbf{e}, \\
T(a) = g, \quad T(g) = a, \\
T(b) = h, \quad T(h) = b, \\
T(c) = c, \quad T(d) = d, \quad T(f) = f.
\end{gathered}
\tag{4.1}
$$

Incidentally, $\mathcal{L} = \mathcal{L}^*$ and for all $a \in \mathcal{L}$, $T(T(a)) = a$. However $a \wedge T(a)$ does not necessarily $= \mathbf{0}$.

Now T can be extended to all of $\mathcal{A}(\mathcal{L}) = \mathcal{B}$ by defining $T(a \vee b) = T(a) \wedge T(b)$, $T(a \wedge b) = T(a) \vee T(b)$, and $T(a') = T(a)'$.

Let $F = F_f = \{x \in \mathcal{L} : x \geq f\} = \{\mathbf{e}, b, f\}$. $\widetilde{F} = \{a^*, d^*, h^*, c^*, g^*, \mathbf{0}^*\} = \{g, d, b, c, a, \mathbf{e}\} = F_g$. Since $\widetilde{F}$ is an $\mathcal{L}^*$-ultrafilter, F is a coultra filter. F is a prime filter so we have the associated measure $\mu = \mu_F = \mu_f$, where

$$
\mu(x) = \begin{cases} 1 & \text{if } x \geq f, \\ 0 & \text{otherwise.} \end{cases}
\tag{4.2}
$$

Now consider that $\{3\} \in \mathcal{A}(\mathcal{L})$. $\mu(\{2,3\}) = 1$, $\mu(\{2\}) = 0$, so that $\mu(\{3\}) = \mu(\{2,3\}) - \mu(\{2\}) = 1$. But $\{3\} \ngeq$ any element of $\mathcal{L}^*$ whose measure is one, so μ is not $\mathcal{L}^*$-regular even though it is $\mathcal{L}$-coregular.

Note that

$$
\begin{aligned}
&\mu(a) = 0 \text{ and } \mu(a^*) = \mu(g) = 0, \\
&\mu(f) = 1 \text{ and } \mu(f^*) = \mu(f) = 1, \\
&\mu(h) = 0 \text{ but } \mu(h^*) = \mu(b) = 1, \\
&\mu(b) = 1 \text{ but } \mu(b^*) = \mu(h) = 0.
\end{aligned}
$$

Thus, T is not necessarily measure inverting $(\mu(T(a)) = 1 - \mu(a) \forall \mu)$, preserving, increasing, or decreasing.

$$
\begin{tikzcd}
\text{Compact} \arrow[r] \arrow[d] \arrow[dr] & \text{Complete} \arrow[r] & \text{Prime complete} \arrow[r] \arrow[d] & \text{Max complete} \\
\aleph_0\text{-compact} \arrow[d] \arrow[dr] & \text{Comax compact} \arrow[r] \arrow[d] & \text{Comax complete} & \\
I\text{-lattice} \arrow[d] & \text{Comax-}\aleph_0\text{-compact} & & \\
P\text{-lattice} & & &
\end{tikzcd}
$$

Figure 3: Lattice Implications.

Remark 4.7. If $T(a) \wedge a = \mathbf{0}$ and T is measure inverting, then every $\mathcal{L}$-coregular measure is $\mathcal{L}^*$-regular. These concepts are equivalent when T is complement.

From now on we assume that $T(a \wedge b) = T(a) \vee T(b)$. Now $\mathcal{L}^*$ is a lattice.

Example 4.8. See Examples 4.6 and 3.4.

Proposition 4.9. *F is a prime $\mathcal{L}$-filter if $\widetilde{F}$ is a prime $\mathcal{L}^*$-filter. (Please see Definition 4.1 and Proposition 4.2 for the converse.)*

Proof. (a) $\mathbf{0} \notin F$ since $\mathbf{e} \in \widetilde{F}$.

(b) $a, b \in F \Rightarrow a^*, b^* \notin \widetilde{F} \Rightarrow a^* \vee b^* \notin \widetilde{F} \Rightarrow (a \wedge b)^* \notin \widetilde{F} \Rightarrow a \wedge b \in F$.

(c) $a \leq b$ and $a \in F \Rightarrow b^* \leq a^*$ and $a^* \notin \widetilde{F} \Rightarrow b^* \notin \widetilde{F} \Rightarrow b \in F$.

(d) $a \vee b \in F \Rightarrow (a \vee b)^* \notin \widetilde{F} \Rightarrow a^* \wedge b^* \notin \widetilde{F} \Rightarrow a^* \notin \widetilde{F}$ or $b^* \notin \widetilde{F} \Rightarrow a \in F$ or $b \in F$. □

Corollary 4.10. (a) *F is a prime $\mathcal{L}$-filter if and only if $\widetilde{F}$ is a prime $\mathcal{L}^*$-filter.*

(b) *F is a prime filter if F is a coultra filter.*

5. Covering Properties

In this section we define some covering properties for $\mathcal{L}$ and show that they are analogous to the topological ones. In particular, when T is taken to be complement, we get topological results as corollaries.

Definition 5.1. $\mathcal{L}$ is **comax compact** if and only if every coultra filter is fixed. $\mathcal{L}$ is **comax $\aleph_0$-compact** if and only if every coultra filter has cmp.

Proposition 5.2. *Every comax compact R-lattice is compact.*

Proof. Let F be a prime $\mathcal{L}$-filter. Form $\widetilde{F}$ and extend it to $\widetilde{G}$, an $\mathcal{L}^*$-ultrafilter. G is a prime $\mathcal{L}$-filter and fixed. $G \subseteq F$ implies that F is fixed since $\mathcal{L}$ is an R-lattice. Thus $\mathcal{L}$ is compact by Proposition 3.6. □

Corollary 5.3. *If $T(a) \wedge a = \mathbf{0}$ and $\mathcal{L}$ is a comax compact regular lattice, then $\mathcal{L}$ is compact.*

Definition 5.4. $\mathcal{L}$ is **(prime, max, comax) complete** if and only if every (prime, ultra-, coultra) filter with cmp is fixed.

We have the implications in Figure 3.

Proposition 5.5. *If $\mathcal{L}$ is (comax-)$\aleph_0$-compact, then it is (comax) complete if and only if it is (comax) compact.*

Proposition 5.6. *If $\mathcal{L}$ is an R-lattice, then $\mathcal{L}$ is prime complete if and only if it is comax complete.*

Proof. In the proof of Proposition 5.2, let F have cmp. □

Proposition 5.7. *If $\mathcal{L}$ is a max complete I-lattice, then $\mathcal{L}$ is complete.*

Proof. Let F be a filter with cmp. Since $\mathcal{L}$ is an I-lattice, F can be extended to G, an ultrafilter with cmp. G is fixed because $\mathcal{L}$ is max complete. Hence F is fixed and $\mathcal{L}$ is complete. □

Corollary 5.8. *If L is a max complete P-lattice, then $\mathcal{L}$ is prime complete.*

Proof. In the proof of Proposition 5.7, take F to be a prime filter. □

Remark 5.9. When T is defined on $\tau(\mathcal{L})$ (as when $\mathcal{L} = \tau(\mathcal{L})$) and $T(\bigwedge_\alpha a_\alpha) = \bigvee_\alpha T(a_\alpha) \forall a_\alpha \in \mathcal{L}$, our definitions coincide with the conventional topological ones. (See Examples 3.4 and 4.6.) In particular T may be taken to be complement.

Proposition 5.10. *Let T be defined on $\tau(\mathcal{L})$ with $T(\bigwedge_\alpha a_\alpha) = \bigvee_\alpha T(a_\alpha)$ for all $a_\alpha \in \mathcal{L}$.*

(a) *$\mathcal{L}$ is compact if and only if $\mathbf{e} = \bigvee_\alpha a_\alpha^* \Rightarrow \mathbf{e} = \bigvee_1^n a_{\alpha_i}^*$.*

(b) *$\mathcal{L}$ is complete if and only if $\mathbf{e} = \bigvee_\alpha a_\alpha^* \Rightarrow \mathbf{e} = \bigvee_1^\infty a_{\alpha_i}^*$.*

(c) *$\mathcal{L}$ is $\aleph_0$-compact if and only if $\mathbf{e} = \bigvee_1^\infty a_i^* \Rightarrow \mathbf{e} = \bigvee_{j=1}^n a_{i_j}^*$.*

Proof. We will prove only part (b). Parts (a) and (c) have similar proofs.

Suppose the condition holds. Let $F = \{f_\alpha\}$ be a free $\mathcal{L}$-filter. Then $\bigwedge_\alpha f_\alpha = \mathbf{0}$ implies that $\bigvee_\alpha f_\alpha^* = \mathbf{e}$, so that there exists $\{f_{\alpha_i}\} \subseteq \{f_\alpha\}$ such that $\bigvee_i f_{\alpha_i}^* = \mathbf{e}$. But then $\bigwedge_i f_\alpha^* = \mathbf{0}$ and F does not have cmp, so $\mathcal{L}$ is complete.

Conversely, suppose that the condition does not hold. Then there exists $\{a_\alpha\}$ such that $\bigvee_\alpha a_\alpha^* = \mathbf{e}$ but $\bigvee_i a_{\alpha_i}^* \neq \mathbf{e}$ for any countable subset. Then $\mathbf{0} \neq \bigwedge_i a_{\alpha_i}$ and $\{a_\alpha\}$ is a subbase for a filter F. F has cmp since $\{f_j\} \subseteq F$ implies that $\bigwedge_j f_j \geq \bigwedge_j a_{\alpha_j} > \mathbf{0}$. F is free since $\mathbf{e} = \bigvee_\alpha a_\alpha^*$ implies that $\bigwedge_\alpha a_\alpha = \mathbf{0}$, and therefore, $\mathcal{L}$ is not complete. □

Corollary 5.11. *Let X be a topological space. Take $\mathcal{L}$ to be the closed sets and take T to be complement. Then X is compact (Lindelöf, countably compact) if and only if $\mathcal{L}$ is compact (complete, $\aleph_0$-compact).*

Corollary 5.12. *X is a realcompact I-space if and only if it is a Lindelöf space [14].*

Corollary 5.13. *If X is regular, countably paracompact, and almost realcompact, then X is realcompact [6].*

6. Lattice Interrelations

In this section we investigate the implications between the properties of two lattices when one is a sublattice of the other.

Proposition 6.1. *Let $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \tau(\mathcal{L}_1)$*

(a) *If $\mathcal{L}_1$ is complete, then $\mathcal{L}_2$ is complete.*

(b) *If $\mathcal{L}_1$ is prime complete, then $\mathcal{L}_2$ is prime complete.*

(c) *If $\mathcal{L}_1$ is a P-lattice, then that $\mathcal{L}_1$ is max complete implies that $\mathcal{L}_2$ is prime complete.*

Proof. (a) Let F be an $\mathcal{L}_2$-filter with cmp, $G = F \cap \mathcal{L}_1$, and $b \in F$. There exists $\{a_\alpha\} \subseteq \mathcal{L}_1$ such that $b = \bigwedge_\alpha a_\alpha$. As $b \leq a_\alpha$, for all α, we have $a_\alpha \in F \cap \mathcal{L}_1$ (which $= G$), for all α. Now let $\{b_\beta\} \subseteq F$. $\bigwedge_\beta b_\beta = \bigwedge_{\alpha,\beta} a_{\alpha,\beta} \neq \mathbf{0}$, since G is fixed. Thus F is fixed and $\mathcal{L}_2$ is complete.

(b) Let F in (a) be prime. Then G is prime.

(c) Since $\mathcal{L}_1$ is a P-lattice, G may be extended to an $\mathcal{L}_1$-ultrafilter H with cmp. Since $\mathcal{L}_1$ is max complete, H is fixed, and hence G and F are fixed. □

Corollary 6.2. *Let $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \tau(\mathcal{L}_1)$ and let $\mathcal{L}_1$ be a P-lattice. If $\mathcal{L}_1$ is max complete, then $\mathcal{L}_2$ is max and comax complete.*

Corollary 6.3. *Let $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \tau(\mathcal{L}_1)$ and let $\mathcal{L}_1$ be $\aleph_0$-normal. If $\mathcal{L}_1$ is max complete, then $\mathcal{L}_2$ is max and comax complete.*

Corollary 6.4. *If X is a normal, countably paracompact space, then Z-replete implies F-replete and realcompact implies α-complete [13].*

The following proposition generalizes two results of Alexandroff [5].

Proposition 6.5. *Let $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \tau(\mathcal{L}_1)$.*

(a) *If $\mathcal{L}_1$ is compact, then $\mathcal{L}_2$ is compact.*

(b) *If $\mathcal{L}_1$ is compact and normal, then $\mathcal{L}_2$ is normal.*

Proof. (a) Let F be an $\mathcal{L}_2$-filter. Let $G = F \cap \mathcal{L}_1$. The proof follows as in Proposition 6.1.

(b) Let $a_1, a_2 \in \mathcal{L}_2$, $a_1 \wedge a_2 = \mathbf{0}$, where $a_1 = \bigwedge_\beta b_\beta$, $a_2 = \bigwedge_\gamma c_\gamma$, and of course $b_\beta, c_\gamma \in \mathcal{L}_1$. $a_1 \leq b_\beta$ and $a_2 \leq c_\gamma$, for all β, γ. Now there must exist $b_{\beta_0}, c_{\gamma_0}$ such that $b_{\beta_0} \wedge c_{\gamma_0} = \mathbf{0}$. (If not, $\{b_\beta, c_\gamma\}$ would form a subbase for a free filter in a compact space.) By normality of $\mathcal{L}_1$, there exist $d_1^*, d_2^* \in \mathcal{L}_1^*$ such that $d_1^* \geq b_{\beta_0}$, $d_2^* \geq c_{\gamma_0}$, and $d_1^* \wedge d_2^* = \mathbf{0}$. But then $d_1^* \geq a_1$ and $d_2^* \geq a_2$ and so $\mathcal{L}_2$ is normal. □

Proposition 6.6. *Let $\delta(\mathcal{L})$ denote the smallest set containing $\mathcal{L}$ and closed under countable meets. Let $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \delta(\mathcal{L}_1)$. If $\mathcal{L}_1$ is $\aleph_0$-compact, then $\mathcal{L}_2$ is $\aleph_0$-compact.*

Proposition 6.7. *Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$. If $\mathcal{L}_2$ is complete, then $\mathcal{L}_1$ is complete.*

Proof. Let F be an $\mathcal{L}_1$-filter with cmp. Let $G = \{a \in \mathcal{L}_2 : a \geq f \text{ for some } f \in F\}$. G has cmp since $\bigwedge g_i \geq \bigwedge f_i > \mathbf{0}$, $g_i \in G$, and $f_i \in F$. $\mathcal{L}_2$ is complete so G is fixed. Thus F is fixed and $\mathcal{L}_1$ is complete. □

Corollary 6.8. *Let$\mathcal{L}_1 \subseteq \mathcal{L}_2$ where $\mathcal{L}_2$ is an I-lattice.*

(a) *If $\mathcal{L}_2$ is max complete, then $\mathcal{L}_1$ is complete.*

(b) *If $\mathcal{L}_2$ is a comax complete R-lattice, then $\mathcal{L}_1$ is complete.*

Proof. (a) Since $\mathcal{L}_2$ is an I-lattice, $\mathcal{L}_2$ that is max complete implies that $\mathcal{L}_2$ is complete, which implies by Proposition 6.7 that $\mathcal{L}_1$ is complete.

(b) Since $\mathcal{L}_2$ is an R-lattice, its being comax complete implies that it is prime complete. Since $\mathcal{L}_2$ is an I-lattice, $\mathcal{L}_2$ is complete and thus $\mathcal{L}_1$ is complete by Proposition 6.7. □

Definition 6.9. $\mathcal{L}_2$ is an $\mathcal{L}_1$-**P-lattice** if and only if every $\mathcal{L}_1$ prime filter with cmp is contained in an $\mathcal{L}_2$ ultrafilter with cmp.

Definition 6.10. (i) $\mathcal{L}_2$ is $\mathcal{L}_1$**-normal** if and only if for all $f_1, f_2 \in \mathcal{L}_2$ with $f_1 \wedge f_2 = \mathbf{0}$ there exist $g_1^*, g_2^* \in {\mathcal{L}_1}^*$ such that $f_1 \leq g_1^*$, $f_2 \leq g_2^*$, and $g_1^* \wedge g_2^* = \mathbf{0}$.

(ii) $\mathcal{L}_2$ is $\mathcal{L}_1$-$\aleph_0$**-paracompact** if and only if for each $\{a_n\} \subseteq \mathcal{L}_2$ such that $a_n \downarrow \mathbf{0}$ there exists a sequence $\{b_n\} \subseteq \mathcal{L}_1$ such that $a_n \leq b_n^*$ and $b_n^* \downarrow \mathbf{0}$.

(iii) $\mathcal{L}_2$ is $\mathcal{L}_1$-$\aleph_0$**-normal** if and only if $\mathcal{L}_2$ is $\mathcal{L}_1$-normal and $\mathcal{L}_1$-$\aleph_0$-paracompact.

Proposition 6.11. *Let T be complement. If $\mathcal{L}_2$ is $\mathcal{L}_1$-$\aleph_0$-normal and F_1 is a prime $\mathcal{L}_1$-filter with cmp contained in an $\mathcal{L}_2$-filter F_2, then F_2 has cmp.*

Proof. Similar to the proof of Proposition 3.19. □

Corollary 6.12. *Let T be complement.*

(a) *That $\mathcal{L}_2$ is $\mathcal{L}_1$-$\aleph_0$-normal implies that $\mathcal{L}_2$ is an $\mathcal{L}_1$-P-lattice.*

(b) *If $\mathcal{L}_2$ is $\aleph_0$-paracompact and $\mathcal{L}_1$ separates $\mathcal{L}_2$, then $\mathcal{L}_2$ is $\mathcal{L}_1$-$\aleph_0$-paracompact.*

Corollary 6.13. (a) *Let $\mathcal{L}_2$ be an $\mathcal{L}_1$-P-lattice. That $\mathcal{L}_2$ is max complete implies that $\mathcal{L}_1$ is prime complete.*

(b) *If in addition $\mathcal{L}_2$ is an R-lattice, then that $\mathcal{L}_2$ is comax complete implies that $\mathcal{L}_1$ is prime complete.*

We get Frolík's [8] theorems as our final corollary.

Corollary 6.14. (a) *If $\mathcal{L}_1 \subseteq \mathcal{L}_2$ and $\mathcal{L}_2$ is $\mathcal{L}_1$-$\aleph_0$-normal and prime complete, then $\mathcal{L}_1$ is prime complete.*

(b) *Let X be a normal T_2 space. If X is almost realcompact and countably paracompact, then X is realcompact.*

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