Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 573038, 9 pages
doi:10.1155/2009/573038

*Research Article*

# Characterizations of Strongly Compact Spaces

**Ahmad Al-Omari,[1] Takashi Noiri,[2] and Mohd. Salmi Md. Noorani[3]**

[1] *Department of Mathematics and Statistics, Faculty of Science, Mu'tah University, P.O. Box 7, Karak 61710, Jordan*
[2] *2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken 869-5142, Japan*
[3] *School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia*

Correspondence should be addressed to Ahmad Al-Omari, omarimutah1@yahoo.com

Received 4 July 2009; Accepted 30 September 2009

Recommended by Richard Wilson

A topological space $(X, \tau)$ is said to be strongly compact if every preopen cover of $(X, \tau)$ admits a finite subcover. In this paper, we introduce a new class of sets called $\mathcal{N}$-preopen sets which is weaker than both open sets and $\mathcal{N}$-open sets. Where a subset $A$ is said to be $\mathcal{N}$-preopen if for each $x \in A$ there exists a preopen set $U_x$ containing $x$ such that $U_x - A$ is a finite set. We investigate some properties of the sets. Moreover, we obtain new characterizations and preserving theorems of strongly compact spaces.



## 1. Introduction

It is well known that the effects of the investigation of properties of closed bounded intervals of real numbers, spaces of continuous functions, and solutions to differential equations are the possible motivations for the formation of the notion of compactness. Compactness is now one of the most important, useful, and fundamental notions of not only general topology but also other advanced branches of mathematics. Many researchers have pithily studied the fundamental properties of compactness and now the results can be found in any undergraduate textbook on analysis and general topology. The productivity and fruitfulness of the notion of compactness motivated mathematicians to generalize this notion. In 1982, Atia et al. [1] introduced a strong version of compactness defined in terms of preopen subsets of a topological space which they called strongly compact. A topological space $X$ is said to be strongly compact if every preopen cover of $X$ admits a finite subcover. Since then, many mathematicians have obtained several results concerning its properties. The notion of strongly compact relative to a topological space $X$ was introduced by Mashhour et al.

[2] in 1984. They established several characterizations of these spaces. In 1987, Ganster [3] obtained an interesting result that answered the question: what type of a space do we get when we take the one-point-compactification of a discrete space? He showed that this space is strongly compact. He proved that a topological space is strongly compact if and only if it is compact and every infinite subset of $X$ has nonempty interior. In 1988, Janković et al. [4] showed that a topological space $(X,\tau)$ is strongly compact if and only if it is compact and the family of dense sets in $(X,\tau)$ is finite. Quite recently Jafari and Noiri [5, 6], by introducing the class of firmly precontinuous functions, found some new characterizations of strongly compact spaces. They also obtained properties of strongly compact spaces by using nets, filterbases, precomplete accumulation points. The notion of preopen sets plays an important role in the study of strongly compact spaces. In this paper, first we introduce and study the notion of $\mathcal{N}$-preopen sets as a generalization of preopen sets. Then, by using $\mathcal{N}$-preopen sets, we obtain new characterizations and further preservation theorems of strongly compact spaces. We improve some of the results established by Mashhour et al. [2]. Throughout this paper, $(X,\tau)$ and $(Y,\sigma)$ stand for topological spaces on which no separation axiom is assumed unless otherwise stated. For a subset $A$ of $X$, the closure of $A$ and the interior of $A$ will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset $A$ of a topological space $X$ is said to be preopen [7] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. A subset $A$ is said to be $\mathcal{N}$-open [8] if for each $x \in A$ there exists an open set $U_x$ containing $x$ such that $U_x - A$ is a finite set. The complement of an $\mathcal{N}$-open subset is said to be $\mathcal{N}$-closed.

## 2. $\mathcal{N}$-Preopen Sets

In this section, we introduce and study the notion of $\mathcal{N}$-preopen sets.

*Definition 2.1.* A subset $A$ is said to be *$\mathcal{N}$-preopen* if for each $x \in A$ there exists a preopen set $U_x$ containing $x$ such that $U_x - A$ is a finite set. The complement of an $\mathcal{N}$-preopen subset is said to be $\mathcal{N}$-preclosed.

The family of all $\mathcal{N}$-preopen (resp., preopen, preclosed) subsets of a space $(X,\tau)$ is denoted by $\mathcal{N}\mathrm{PO}(X)$ (resp., $\mathrm{PO}(X)$, $\mathrm{PC}(X)$).

**Lemma 2.2.** *For a subset of a topological space $(X,\tau)$, both $\mathcal{N}$-openness and preopenness imply $\mathcal{N}$-preopenness.*

For a subset of a topological space, the following implications hold:

$$\begin{tikzcd} \text{open} \arrow[r] \arrow[d] & \text{preopen} \arrow[d] \\ \mathcal{N}\text{-open} \arrow[r] & \mathcal{N}\text{-preopen} \end{tikzcd} \tag{2.1}$$

*Example 2.3.* Let $X = [0,7]$ be the closed interval with a topology $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}, \{1,2,[3,5]\}\}$. Then $\{2,[3,5]\}$ is an $\mathcal{N}$-open set which is not preopen.

*Example 2.4.* Let $\mathbb{R}$ be the set of all real numbers with the usual topology. Then the set $\mathbb{Q}$ of all rational numbers is a preopen set which is not $\mathcal{N}$-open.

**Lemma 2.5.** *Let $(X, \tau)$ be a topological space. Then the union of any family of $\mathcal{N}$-preopen sets is $\mathcal{N}$-preopen.*

*Proof.* Let $\{U_i : i \in I\}$ be a family of $\mathcal{N}$-preopen subsets of $X$ and $x \in \bigcup_{i\in I} U_i$. Then $x \in U_j$ for some $j \in I$. This implies that there exists a preopen subset $V$ of $X$ containing $x$ such that $V \setminus U_j$ is finite. Since $V \setminus \bigcup_{i\in I} U_i \subseteq V \setminus U_j$, then $V \setminus \bigcup_{i\in I} U_i$ is finite. Thus $\bigcup_{i\in I} U_i \in \mathcal{N}\mathrm{PO}(X)$. □

Recall that a space $(X, \tau)$ is called submaximal if every dense subset of $X$ is open.

**Lemma 2.6** (see [9]). *For a topological space $(X, \tau)$, the followings are equivalent.*

(1) *$X$ is submaximal.*

(2) *Every preopen set is open.*

*Definition 2.7* (see [10]). A subset $A$ of a space $X$ is said to be $\alpha$-open if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$.

**Lemma 2.8** (see [11]). *Let $(X, \tau)$ be a topological space. Then the intersection of an $\alpha$-open set and a preopen set is preopen.*

**Theorem 2.9.** *Let $(X, \tau)$ be a submaximal topological space. Then $(X, \mathcal{N}PO(X))$ is a topological space.*

*Proof.* (1) We have $\phi, X \in \mathcal{N}\mathrm{PO}(X)$.

(2) Let $U, V \in \mathcal{N}\mathrm{PO}(X)$ and $x \in U \cap V$. Then there exist preopen sets $G, H \in X$ containing $x$ such that $G \setminus U$ and $H \setminus V$ are finite. And $(G \cap H) \setminus (U \cap V) = (G \cap H) \cap ((X \setminus U) \cup (X \setminus V)) \subseteq (G \cap (X \setminus U)) \cup (H \cap (X \setminus V))$. Thus $(G \cap H) \setminus (U \cap V)$ is finite, and since $X$ is submaximal by Lemma 2.6, the intersection of two preopen sets is preopen. Hence $U \cap V \in \mathcal{N}\mathrm{PO}(X)$.

(3) Let $\{U_i : i \in I\}$ be any family of $\mathcal{N}$-preopen sets of $X$. Then, by Lemma 2.5, $\bigcup_{i\in I} U_i$ is $\mathcal{N}$-preopen. □

The converse of above theorem is not true.

*Example 2.10.* Let $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. Then $(X, \mathcal{N}\mathrm{PO}(X))$ is a topological space and $(X, \tau)$ is not a submaximal topological space.

**Lemma 2.11.** *Let $(X, \tau)$ be a topological space. Then the intersection of an $\alpha$-open set and an $\mathcal{N}$-preopen set is $\mathcal{N}$-preopen.*

*Proof.* Let $U$ be $\alpha$-open and $A$ $\mathcal{N}$-preopen. Then for every $x \in A$, there exists a preopen set $V_x \subseteq X$ containing $x$ such that $V_x - A$ is finite, and also by Lemma 2.8, $U \cap V_x$ is preopen. Now for each $x \in U \cap A$, there exists a preopen set $U \cap V_x \subseteq X$ containing $x$ and

$$\begin{aligned}
(U \cap V_x) - (U \cap A) &= (U \cap V_x) \cap [(X - U) \cup (X - A)] \\
&= [(U \cap V_x) \cap (X - U)] \cup [(U \cap V_x) \cap (X - A)] \\
&= (U \cap V_x) - A \\
&\subseteq V_x - A.
\end{aligned} \tag{2.2}$$

Then $(U \cap V_x) - (U \cap A)$ is finite. Therefore $U \cap A$ is an $\mathcal{N}$-preopen set. □

The following lemma is well known and will be stated without the proof.

**Lemma 2.12.** *A topological space is a $T_1$-space if and only if every finite set is closed.*

**Proposition 2.13.** *If a topological space $X$ is a $T_1$-space, then every nonempty $\mathcal{N}$-preopen set contains a nonempty preopen set.*

*Proof.* Let $A$ be a nonempty $\mathcal{N}$-preopen set and $x \in A$, then there exists a preopen set $U_x$ containing $x$ such that $U_x - A$ is finite. Let $C = U_x - A = U_x \cap (X - A)$. Then $x \in U_x - C \subseteq A$ and by Lemmas 2.8 and 2.12, $U_x - C = U_x \cap (X - C)$ is preopen. □

The following example shows that if $X$ is not a $T_1$-space, then there exists a nonempty $\mathcal{N}$-preopen set which does not contain a nonempty preopen set.

*Example 2.14.* Let $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\{c\}$ is an $\mathcal{N}$-preopen set which does not contain any a nonempty preopen set.

**Lemma 2.15** (see [12]). *Let $A$ and $X_0$ be subsets of a topological space $X$.*

(1) *If $A \in PO(X)$ and $X_0 \in SO(X)$, then $A \cap X_0 \in PO(X_0)$.*

(2) *If $A \in PO(X_0)$ and $X_0 \in PO(X)$, then $A \in PO(X)$.*

**Lemma 2.16.** *Let $A$ and $X_0$ be subsets of a topological space $X$.*

(1) *If $A \in \mathcal{N}PO(X)$ and $X_0 \in \alpha O(X)$, then $A \cap X_0 \in \mathcal{N}PO(X_0)$.*

(2) *If $A \in \mathcal{N}PO(X_0)$ and $X_0 \in PO(X)$, then $A \in \mathcal{N}PO(X)$.*

*Proof.* (1) Let $x \in A \cap X_0$. Since $A$ is $\mathcal{N}$-preopen in $X$, there exists a preopen set $H$ of $X$ containing $x$ such that $H - A$ is finite. Since $X_0$ is $\alpha$-open, by Lemma 2.8 we have $H \cap X_0 \in \mathrm{PO}(X)$. Since $X_0 \in \alpha O(X) \subseteq \mathrm{SO}(X)$, by Lemma 2.15, $H \cap X_0 \in \mathrm{PO}(X_0)$, $x \in H \cap X_0$, and $(H \cap X_0) - (A \cap X_0)$ is finite. This shows that $A \cap X_0 \in \mathcal{N}\mathrm{PO}(X_0)$.

(2) If $A \in \mathcal{N}\mathrm{PO}(X_0)$, for each $x \in A$ there exists $V_x \in \mathrm{PO}(X_0)$ containing $x$ such that $V_x - A$ is finite. Since $X_0 \in \mathrm{PO}(X)$, by Lemma 2.15, $V_x \in \mathrm{PO}(X)$ and $V_x - A$ is finite and hence $A \in \mathcal{N}\mathrm{PO}(X)$. □

**Lemma 2.17.** *A subset $A$ of a space $X$ is $\mathcal{N}$-preopen if and only if for every $x \in A$, there exist a preopen subset $U_x$ containing $x$ and a finite subset $C$ such that $U_x - C \subseteq A$.*

*Proof.* Let $A$ be $\mathcal{N}$-preopen and $x \in A$, then there exists a preopen subset $U_x$ containing $x$ such that $U_x - A$ is finite. Let $C = U_x - A = U_x \cap (X - A)$. Then $U_x - C \subseteq A$. Conversely, let $x \in A$. Then there exist a preopen subset $U_x$ containing $x$ and a finite subset $C$ such that $U_x - C \subseteq A$. Thus $U_x - A \subseteq C$ and $U_x - A$ is a finite set. □

**Theorem 2.18.** *Let $X$ be a space and $F \subseteq X$. If $F$ is $\mathcal{N}$-preclosed, then $F \subseteq K \cup C$ for some preclosed subset $K$ and a finite subset $C$.*

*Proof.* If $F$ is $\mathcal{N}$-preclosed, then $X - F$ is $\mathcal{N}$-preopen, and hence for every $x \in X - F$, there exists a preopen set $U$ containing $x$ and a finite set $C$ such that $U - C \subseteq X - F$. Thus $F \subseteq X - (U - C) = X - (U \cap (X - C)) = (X - U) \cup C$. Let $K = X - U$. Then $K$ is a preclosed set such that $F \subseteq K \cup C$. □

## 3. Strongly Compact Spaces

*Definition 3.1.* (1) A topological space $X$ is said to be *strongly compact* [1] if every cover of $X$ by preopen sets admits a finite subcover.

(2) A subset $A$ of a space $X$ is said to be strongly compact relative to $X$ [2] if every cover of $A$ by preopen sets of $X$ admits a finite subcover.

**Theorem 3.2.** *If $X$ is a space such that every preopen subset of $X$ is strongly compact relative to $X$, then every subset of $X$ is strongly compact relative to $X$.*

*Proof.* Let $B$ be an arbitrary subset of $X$ and let $\{U_i : i \in I\}$ be a cover of $B$ by preopen sets of $X$. Then the family $\{U_i : i \in I\}$ is a preopen cover of the preopen set $\cup\{U_i : i \in I\}$. Hence by hypothesis there is a finite subfamily $\{U_{i_j} : j \in \mathbb{N}_0\}$ which covers $\cup\{U_i : i \in I\}$ where $\mathbb{N}_0$ is a finite subset of the naturals $\mathbb{N}$. This subfamily is also a cover of the set $B$. □

**Theorem 3.3.** *A subset $A$ of a space $X$ is strongly compact relative to $X$ if and only if for any cover $\{V_\alpha : \alpha \in \Lambda\}$ of $A$ by $\mathcal{N}$-preopen sets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $A \subseteq \cup\{V_\alpha : \alpha \in \Lambda_0\}$.*

*Proof.* Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of $A$ and $V_\alpha \in \mathcal{N}PO(X)$. For each $x \in A$, there exists $\alpha(x) \in \Lambda$ such that $x \in V_{\alpha(x)}$. Since $V_{\alpha(x)}$ is $\mathcal{N}$-preopen, there exists a preopen set $U_{\alpha(x)}$ such that $x \in U_{\alpha(x)}$ and $U_{\alpha(x)} \setminus V_{\alpha(x)}$ is finite. The family $\{U_{\alpha(x)} : x \in A\}$ is a preopen cover of $A$. Since $A$ is strongly compact relative to $X$, there exists a finite subset, says, $x_1, x_2, \dots, x_n$ such that $A \subseteq \cup\{U_{\alpha(x_i)} : i \in F\}$, where $F = \{1, 2, \dots, n\}$. Now, we have

$$\begin{aligned} A &\subseteq \bigcup_{i\in F}\left(\left(U_{\alpha(x_i)} \setminus V_{\alpha(x_i)}\right) \cup V_{\alpha(x_i)}\right) \\ &= \left(\bigcup_{i\in F}\left(U_{\alpha(x_i)} \setminus V_{\alpha(x_i)}\right)\right) \cup \left(\bigcup_{i\in F} V_{\alpha(x_i)}\right). \end{aligned} \tag{3.1}$$

For each $x_i$, $U_{\alpha(x_i)} \setminus V_{\alpha(x_i)}$ is a finite set and there exists a finite subset $\Lambda(x_i)$ of $\Lambda$ such that $(U_{\alpha(x_i)} \setminus V_{\alpha(x_i)}) \cap A \subseteq \cup\{V_\alpha : \alpha \in \Lambda(x_i)\}$. Therefore, we have $A \subseteq (\bigcup_{i\in F}(\cup\{V_\alpha : \alpha \in \Lambda(x_i)\})) \cup (\bigcup_{i\in F} V_{\alpha(x_i)})$. Hence $A$ is strongly compact relative to $X$.

Since every preopen set is $\mathcal{N}$-preopen, the proof of the converse is obvious. □

**Corollary 3.4.** *For any space $X$, the following properties are equivalent:*

(1) *$X$ is strongly compact;*

(2) *every $\mathcal{N}$-preopen cover of $X$ admits a finite subcover.*

**Theorem 3.5.** *For any space $X$, the following properties are equivalent:*

(1) *$X$ is strongly compact;*

(2) *every proper $\mathcal{N}$-preclosed set is strongly compact with respect to $X$.*

*Proof.* (1)⇒(2) Let $A$ be a proper $\mathcal{N}$-preclosed subset of $X$. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of $A$ by preopen sets of $X$. Now for each $x \in X - A$, there is a preopen set $V_x$ such that $V_x \cap A$ is finite. Then $\{U_\alpha : \alpha \in \Lambda\} \cup \{V_x : x \in X - A\}$ is a preopen cover of $X$. Since $X$ is strongly compact, there exist a finite subset $\Lambda_1$ of $\Lambda$ and a finite number of points, says, $x_1, x_2, \dots, x_n$ in $X - A$ such that $X = (\cup\{U_\alpha : \alpha \in \Lambda_1\}) \cup (\cup\{V_{x_i} : 1 \leq i \leq n\})$; hence $A \subset (\cup\{U_\alpha : \alpha \in \Lambda_1\}) \cup (\cup\{A \cap V_{x_i} : 1 \leq i \leq n\})$. Since $A \cap V_{x_i}$ is finite for each $i$, there exists a finite subset $\Lambda_2$ of $\Lambda$ such that $\cup\{A \cap V_{x_i} : 1 \leq i \leq n\} \subset \cup\{U_\alpha : \alpha \in \Lambda_2\}$. Therefore, we obtain $A \subset \cup\{U_\alpha : \alpha \in \Lambda_1 \cup \Lambda_2\}$. This shows that $A$ is strongly compact relative to $X$.

(2)⇒(1) Let $\{V_\alpha : \alpha \in \Lambda\}$ be any preopen cover of $X$. We choose and fix one $\alpha_0 \in \Lambda$. Then $\cup\{V_\alpha : \alpha \in \Lambda - \{\alpha_0\}\}$ is a preopen cover of a $\mathcal{N}$-preclosed set $X - V_{\alpha_0}$. There exists a finite subset $\Lambda_0$ of $\Lambda - \{\alpha_0\}$ such that $X - V_{\alpha_0} \subset \cup\{V_\alpha : \alpha \in \Lambda_0\}$. Therefore, $X = \cup\{V_\alpha : \alpha \in \Lambda_0 \cup \{\alpha_0\}\}$. This shows that $X$ is strongly compact. □

**Corollary 3.6** (see [2]). *If a space $X$ is strongly compact and $A$ is preclosed, then $A$ is strongly compact relative to $X$.*

**Theorem 3.7.** *Let $(X, \tau)$ be a submaximal topological space. Then $(X, \tau)$ is strongly compact if and only if $(X, \mathcal{N}PO(X))$ is compact.*

*Proof.* Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $(X, \mathcal{N}\mathrm{PO}(X))$. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in V_{\alpha(x)}$. Since $V_{\alpha(x)}$ is $\mathcal{N}$-preopen, there exists a preopen set $U_{\alpha(x)}$ of $X$ such that $x \in U_{\alpha(x)}$ and $U_{\alpha(x)} \setminus V_{\alpha(x)}$ is finite. The family $\{U_{\alpha(x)} : x \in X\}$ is a preopen cover of $(X, \tau)$. Since $(X, \tau)$ is strongly compact, there exists a finite subset, says, $x_1, x_2, \ldots, x_n$ such that $X = \cup\{U_{\alpha(x_i)} : i \in F\}$, where $F = \{1, 2, \ldots, n\}$. Now, we have

$$\begin{aligned} X &= \bigcup_{i\in F}\left(\left(U_{\alpha(x_i)} \setminus V_{\alpha(x_i)}\right) \cup V_{\alpha(x_i)}\right) \\ &= \left(\bigcup_{i\in F}\left(U_{\alpha(x_i)} \setminus V_{\alpha(x_i)}\right)\right) \cup \left(\bigcup_{i\in F} V_{\alpha(x_i)}\right). \end{aligned} \tag{3.2}$$

For each $x_i$, $U_{\alpha(x_i)} \setminus V_{\alpha(x_i)}$ is a finite set and there exists a finite subset $\Lambda(x_i)$ of $\Lambda$ such that $(U_{\alpha(x_i)} \setminus V_{\alpha(x_i)}) \subseteq \cup\{V_\alpha : \alpha \in \Lambda(x_i)\}$. Therefore, we have $X = (\bigcup_{i\in F}(\cup(V_\alpha : \alpha \in \Lambda(x_i)))) \cup (\bigcup_{i\in F} V_{\alpha(x_i)})$. Hence $\mathcal{N}\mathrm{PO}(X)$ is compact.

Conversely, let $\mathcal{U}$ be a preopen cover of $(X, \tau)$. Then $\mathcal{U} \subseteq \mathcal{N}\mathrm{PO}(X)$. Since $(X, \mathcal{N}\mathrm{PO}(X))$ is compact, there exists a finite subcover of $\mathcal{U}$ for $X$. Hence $(X, \tau)$ is strongly compact. □

## 4. Preservation Theorems

*Definition 4.1.* A function $f : X \to Y$ is said to be *$\mathcal{N}$-precontinuous* if $f^{-1}(V)$ is $\mathcal{N}$-preopen in $X$ for each open set $V$ in $Y$.

**Theorem 4.2.** *A function $f : X \to Y$ is $\mathcal{N}$-precontinuous if and only if for each point $x \in X$ and each open set $V$ in $Y$ with $f(x) \in V$, there is an $\mathcal{N}$-preopen set $U$ in $X$ such that $x \in U$ and $f(U) \subseteq V$.*

*Proof. Sufficiency*. Let $V$ be open in $Y$ and $x \in f^{-1}(V)$. Then $f(x) \in V$ and thus there exists a $U_x \in \mathcal{N}\mathrm{PO}(X)$ such that $x \in U_x$ and $f(U_x) \subseteq V$. Then $x \in U_x \subseteq f^{-1}(V)$ and $f^{-1}(V) = \cup\{U_x : x \in f^{-1}(V)\}$. Then by Lemma 2.5, $f^{-1}(V)$ is $\mathcal{N}$-preopen.

*Necessity*. Let $x \in X$ and let $V$ be an open set of $Y$ containing $f(x)$. Then $x \in f^{-1}(V) \in \mathcal{N}\mathrm{PO}(X)$ since $f$ is $\mathcal{N}$-precontinuous. Let $U = f^{-1}(V)$. Then $x \in U$ and $f(U) \subseteq V$. □

**Proposition 4.3.** *If $f : X \to Y$ is $\mathcal{N}$-precontinuous and $X_0$ is an $\alpha$-open set in $X$, then the restriction $f_{|X_0} : X_0 \to Y$ is $\mathcal{N}$-precontinuous.*

*Proof.* Since $f$ is $\mathcal{N}$-precontinuous, for any open set $V$ in $Y$, $f^{-1}(V)$ is $\mathcal{N}$-preopen in $X$. Hence by Lemma 2.11, $f^{-1}(V) \cap X_0$ is $\mathcal{N}$-preopen in $X$. Therefore, by Lemma 2.16, $(f_{|X_0})^{-1}(V) = f^{-1}(V) \cap X_0$ is $\mathcal{N}$-preopen in $X_0$. This implies that $f_{|X_0}$ is $\mathcal{N}$-precontinuous. □

**Proposition 4.4.** *Let $f : X \to Y$ be a function and let $\{A_\alpha : \alpha \in \Lambda\}$ be an $\alpha$-open cover of $X$. If the restriction $f_{|A_\alpha} : A_\alpha :\to Y$ is $\mathcal{N}$-precontinuous for each $\alpha \in \Lambda$, then $f$ is $\mathcal{N}$-precontinuous.*

*Proof.* Suppose that $V$ is an arbitrary open set in $Y$. Then for each $\alpha \in \Lambda$, we have $(f_{|A_\alpha})^{-1}(V) = f^{-1}(V) \cap A_\alpha \in \mathcal{N}PO(A_\alpha)$ because $f_{|A_\alpha}$ is $\mathcal{N}$-precontinuous. Hence by Lemma 2.16, $f^{-1}(V) \cap A_\alpha \in \mathcal{N}PO(X)$ for each $\alpha \in \Lambda$. By Lemma 2.5, we obtain $\cup\{f^{-1}(V) \cap A_\alpha : \alpha \in \Lambda\} = f^{-1}(V) \in \mathcal{N}PO(X)$. This implies that $f$ is $\mathcal{N}$-precontinuous. □

**Theorem 4.5.** *Let $f$ be an $\mathcal{N}$-precontinuous function from a space $X$ onto a space $Y$. If $X$ is strongly compact, then $Y$ is compact.*

*Proof.* Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of $Y$. Then $\{f^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an $\mathcal{N}$-preopen cover of $X$. Since $X$ is strongly compact, by Corollary 3.4, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \Lambda_0\}$; hence $Y = \cup\{V_\alpha : \alpha \in \Lambda_0\}$. Therefore $Y$ is compact. □

*Definition 4.6* (see [12]). A function $f : X \to Y$ is said to be *precontinuous* if $f^{-1}(V)$ is preopen in $X$ for each open set $V$ in $Y$.

It is clear that every precontinuous function is $\mathcal{N}$-precontinuous but not conversely.

*Example 4.7.* Let $X = \{a,b,c,d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$, and $\sigma = \{\phi, X, \{a,d\}, \{b,c\}\}$. Let $f : (X,\tau) \to (X,\sigma)$ be the identity function. Then $f$ is an $\mathcal{N}$-precontinuous function which is not precontinuous; because there exists $\{b,c\} \in \sigma$ such that $f^{-1}(\{b,c\}) \notin PO(X,\tau)$.

**Corollary 4.8** (see [2]). *Let $f$ be a precontinuous function from a space $X$ onto a space $Y$. If $X$ is strongly compact, then $Y$ is compact.*

*Definition 4.9* (see [13]). A function $f : X \to Y$ is said to be *M-preopen* if the image of each preopen set $U$ of $X$ is preopen in $Y$.

**Proposition 4.10.** *If $f : X \to Y$ is M-preopen, then the image of an $\mathcal{N}$-preopen set of $X$ is $\mathcal{N}$-preopen in $Y$.*

*Proof.* Let $f : X \to Y$ be $M$-preopen and $W$ an $\mathcal{N}$-preopen subset of $X$. For any $y \in f(W)$, there exists $x \in W$ such that $f(x) = y$. Since $W$ is $\mathcal{N}$-preopen, there exists a preopen set $U$ such that $x \in U$ and $U - W = C$ is finite. Since $f$ is $M$-preopen, $f(U)$ is preopen in $Y$ such that $y = f(x) \in f(U)$ and $f(U) - f(W) \subseteq f(U - W) = f(C)$ is finite. Therefore, $f(W)$ is $\mathcal{N}$-preopen in $Y$. □

*Definition 4.11* (see [14]). A function $f : X \to Y$ is said to be *preirresolute* if $f^{-1}(V)$ is preopen in $X$ for each preopen set $V$ in $Y$.

**Proposition 4.12.** *If $f : X \to Y$ is a preirresolute injection and $A$ is $\mathcal{N}$-preopen in $Y$, then $f^{-1}(A)$ is $\mathcal{N}$-preopen in $X$.*

*Proof.* Assume that $A$ is an $\mathcal{N}$-preopen subset of $Y$. Let $x \in f^{-1}(A)$. Then $f(x) \in A$ and there exists a preopen set $V$ containing $f(x)$ such that $V - A$ is finite. Since $f$ is preirresolute, $f^{-1}(V)$ is a preopen set containing $x$. Thus $f^{-1}(V) - f^{-1}(A) = f^{-1}(V - A)$ and it is finite. It follows that $f^{-1}(A)$ is $\mathcal{N}$-preopen in X. □

*Definition 4.13.* A function $f : X \to Y$ is said to be $\mathcal{N}$-*preclosed* if $f(A)$ is $\mathcal{N}$-preclosed in $Y$ for each preclosed set $A$ of X.

**Theorem 4.14.** *If $f : X \to Y$ is an $\mathcal{N}$-preclosed surjection such that $f^{-1}(y)$ is strongly compact relative to X for each $y \in Y$, and $Y$ is strongly compact, then X is strongly compact.*

*Proof.* Let $\{U_\alpha : \alpha \in \Lambda\}$ be any preopen cover of X. For each $y \in Y$, $f^{-1}(y)$ is strongly compact relative to X and there exists a finite subset $\Lambda(y)$ of $\Lambda$ such that $f^{-1}(y) \subset \cup\{U_\alpha : \alpha \in \Lambda(y)\}$. Now we put $U(y) = \cup\{U_\alpha : \alpha \in \Lambda(y)\}$ and $V(y) = Y - f(X - U(y))$. Then, since $f$ is $\mathcal{N}$-preclosed, $V(y)$ is an $\mathcal{N}$-preopen set in $Y$ containing $y$ such that $f^{-1}(V(y)) \subset U(y)$. Since $\{V(y) : y \in Y\}$ is an $\mathcal{N}$-preopen cover of $Y$, by Corollary 3.4 there exists a finite subset $\{y_k : 1 \le k \le n\} \subseteq Y$ such that $Y = \bigcup_{k=1}^{n} V(y_k)$. Therefore, $X = f^{-1}(Y) = \bigcup_{k=1}^{n} f^{-1}(V(y_k)) \subseteq \bigcup_{k=1}^{n} U(y_k) = \bigcup_{k=1}^{n}\{U_\alpha : \alpha \in \Lambda(y_k)\}$. This shows that X is strongly compact. □

*Definition 4.15* (see [2]). A function $f : X \to Y$ is said to be $M$-*preclosed* if $f(A)$ is preclosed in $Y$ for each preclosed set $A$ of X.

It is clear that every $M$-preclosed function is $\mathcal{N}$-preclosed but not conversely.

*Example 4.16.* Let $X = \{a, b, c, d\}$ with topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be the function defined by setting $f(a) = c$, $f(b) = d$, $f(c) = a$, and $f(d) = b$. Then $f$ is an $\mathcal{N}$-preclosed function which is not $M$-preclosed; because there exists $\{c\} \in \mathrm{PC}(X)$ such that $f(\{c\}) = \{a\} \notin \mathrm{PC}(X)$.

**Corollary 4.17** (see [2]). *If $f : X \to Y$ is an $M$-preclosed surjection such that $f^{-1}(y)$ is strongly compact relative to X for each $y \in Y$, and $Y$ is strongly compact, then X is strongly compact.*

*Definition 4.18.* A function $f : X \to Y$ is said to be $\delta_{\mathcal{N}}$-*continuous* if for each $x \in X$ and each preopen set $V$ of $Y$ containing $f(x)$, there exists an $\mathcal{N}$-preopen set $U$ of $X$ containing $x$ such that $f(U) \subseteq V$.

It is clear that every preirresolute function is $\delta_{\mathcal{N}}$-continuous but the converse is not true.

*Example 4.19.* Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then the function $f : (X, \tau) \to (X, \tau)$, defined as $f(a) = c$, $f(b) = d$, $f(c) = a$, and $f(d) = b$, is $\delta_N$-continuous but it is not preirresolute.

**Theorem 4.20.** *Let $f : X \to Y$ be a $\delta_{\mathcal{N}}$-continuous surjection from X onto $Y$. If X is strongly compact, then $Y$ is strongly compact.*

*Proof.* Let $\{V_\alpha : \alpha \in \Lambda\}$ be a preopen cover of $Y$. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since $f$ is $\delta_{\mathcal{N}}$-continuous, there exists an $\mathcal{N}$-preopen set $U_{\alpha(x)}$ of X containing $x$ such that $f(U_{\alpha(x)}) \subseteq V_{\alpha(x)}$. So $\{U_{\alpha(x)} : x \in X\}$ is an $\mathcal{N}$-preopen cover of the strongly compact space X, and by Corollary 3.4 there exists a finite subset $\{x_k : 1 \le k \le n\} \subseteq X$ such that $X = \bigcup_{k=1}^{n} U_{\alpha(x_k)}$. Therefore $Y = f(X) = f(\bigcup_{k=1}^{n} U_{\alpha(x_k)}) = \bigcup_{k=1}^{n} f(U_{\alpha(x_k)}) \subseteq \bigcup_{k=1}^{n} V_{\alpha(x_k)}$. This shows that $Y$ is strongly compact. □

**Corollary 4.21** (see [2]). *Let $f : X \to Y$ be a preirresolute surjection from $X$ onto $Y$. If $X$ is strongly compact, then $Y$ is strongly compact.*

## Acknowledgments

This work is financially supported by the Ministry of Higher Education, Malaysia under FRGS Grant no: UKM-ST-06-FRGS0008-2008. The authors also would like to thank the referees for useful comments and suggestions.

## References

[1] R. H. Atia, S. N. El-Deeb, and I. A. Hasanein, "A note on strong compactness and S-closedness," *Matematički Vesnik*, vol. 6, no. 1, pp. 23–28, 1982.
[2] A. S. Mashhour, M. E. Abd El-Monsef, I. A. Hasanein, and T. Noiri, "Strongly compact spaces," *Delta Journal of Science*, vol. 8, no. 1, pp. 30–46, 1984.
[3] M. Ganster, "Some remarks on strongly compact spaces and semi-compact spaces," *Bulletin Malaysian Mathematical Society*, vol. 10, no. 2, pp. 67–70, 1987.
[4] D. S. Janković, I. L. Reilly, and M. K. Vamanamurthy, "On strongly compact topological spaces," *Questions and Answers in General Topology*, vol. 6, no. 1, pp. 29–40, 1988.
[5] S. Jafari and T. Noiri, "Strongly compact spaces and firmly precontinuous functions," *Research Reports. Yatsushiro National College of Technology*, vol. 24, pp. 97–100, 2002.
[6] S. Jafari and T. Noiri, "More on strongly compact spaces," *Missouri Journal of Mathematical Sciences*, vol. 19, no. 1, pp. 52–61, 2007.
[7] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deeb, "On precontinuous and weak precontinuous mappings," *Proceedings of the Mathematical and Physical Society of Egypt*, vol. 53, pp. 47–53, 1982.
[8] A. Al-Omari and M. S. M. Noorani, "New characterizations of compact spaces," submited.
[9] I. L. Reilly and M. K. Vamanamurthy, "On some questions concerning preopen sets," *Kyungpook Mathematical Journal*, vol. 30, no. 1, pp. 87–93, 1990.
[10] O. Njåstad,, "On some classes of nearly open sets," *Pacific Journal of Mathematics*, vol. 15, pp. 961–970, 1965.
[11] V. Popa and T. Noiri, "Almost weakly continuous functions," *Demonstratio Mathematica*, vol. 25, no. 1-2, pp. 241–251, 1992.
[12] A. S. Mashhour, I. A. Hasanein, and S. N. El-Deeb, "A note on semicontinuity and precontinuity," *Indian Journal of Pure and Applied Mathematics*, vol. 13, no. 10, pp. 1119–1123, 1982.
[13] A. S. Mashhour, M. E. Abd El-Monsef, and I. A. Hasanein, "On pretopological spaces," *Bulletin Mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie*, vol. 28, no. 1, pp. 39–45, 1984.
[14] I. L. Reilly and M. K. Vamanamurthy, "On $\alpha$-continuity in topological spaces," *Acta Mathematica Hungarica*, vol. 45, no. 1-2, pp. 27–32, 1985.