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Research Article

Some Monotonicity Results for the Ratio of Two-Parameter Symmetric Homogeneous Functions

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Applying well properties of homogeneous functions, some monotonicity results for the ratio of two-parameter symmetric homogeneous functions are presented, which give an easier access to find two-parameter symmetric homogeneous means having ratio simple monotonicity properties proposed by L. Losonczi. As an application, a chain of inequalities of ratio of bivariate means is established.

1. Introduction

Let $\Phi(a,b)$ and $\Psi(c,d)$ be bivariate means. For what means Φ and Ψ does the following inequality

$$\frac{\Phi(a,b)}{\Phi(c,d)} \leq \frac{\Psi(a,b)}{\Psi(c,d)} \tag{1.1}$$

hold true? where

$$a,b,c,d > 0, \qquad \frac{b}{a} \geq \frac{d}{c} \geq 1. \tag{1.2}$$

Define that

$$M_p = M_p(a,b) := M^{1/p}(a^p,b^p), \quad M = A,H,L,I, \tag{1.3}$$

where A, H, L, and I stand for arithmetic mean, Heronian mean, logarithmic mean, and exponential mean (identric mean) of two positive numbers a and b, respectively.

In 1988 Wang et al. [1] proved that for $a, b, c, d > 0$ with $b/a \geq d/c \geq 1$ the following inequalities of ratio of bivariate means

$$\frac{G(a,b)}{G(c,d)} \leq \frac{L(a,b)}{L(c,d)} \leq \frac{A_{1/3}(a,b)}{A_{1/3}(c,d)} \tag{1.4}$$

hold, with equalities if and only if $b/a = d/c$. That same year, Chen et al. [2] presented second inequalities of ratio of bivariate means:

$$\frac{A_{1/2}(a,b)}{A_{1/2}(c,d)} \leq \frac{H(a,b)}{H(c,d)} \leq \frac{A_{2/3}(a,b)}{A_{2/3}(c,d)}, \tag{1.5}$$

where the constant $1/2$ and $2/3$ both are best possible.

In 1994, Pearce et al. [3] proved that the function

$$p \longrightarrow \frac{L_p(a,b)}{L_p(c,d)} \quad (p \in \mathbb{R}) \tag{1.6}$$

is nondecreasing, provided that $a, b, c, d > 0$ with $b/a \geq d/c$. Here $L_p(a,b) := S_{p+1,1}(a,b)$ is the generalized logarithmic mean and $S_{p,q}(a,b)$ is the Stolarsky mean of $a, b > 0$ with parameters $p, q \in \mathbb{R}$ defined by

$$S_{p,q}(a,b) = \begin{cases} \left(\dfrac{q(a^p - b^p)}{p(a^q - b^q)}\right)^{1/(p-q)}, & p \neq q,\ pq \neq 0, \\ \left(\dfrac{a^p - b^p}{p(\ln a - \ln b)}\right)^{1/p}, & p \neq 0,\ q = 0, \\ \left(\dfrac{a^q - b^q}{q(\ln a - \ln b)}\right)^{1/q}, & p = 0,\ q \neq 0, \\ \exp\left(\dfrac{a^p \ln a - b^p \ln b}{a^p - b^p} - \dfrac{1}{p}\right), & p = q \neq 0, \\ \sqrt{ab}, & p = q = 0. \end{cases} \tag{1.7}$$

Also, $S_{p,q}(a,a) = a$. In a few years, Chen and Qi [4–7] also proved equivalent results.

In [8] the author has proven that inequality (1.1) is valid for power means of certain order, logarithmic, identric, and the Heronian mean of order ω. Neuman et al. [9] obtained inequalities of the form (1.1) for the Stolarsky, Gini, Schwab-Borchardt, and the lemniscatic means.

Recently Chen [10, 11] established a more general result than Pearce and Pečarić's: let a, b, c, d be fixed positive numbers with $a \neq b$, $c \neq d$ and let p, q be real numbers. Then the function

$$R_{p,q}(a,b;c,d) := \frac{S_{p,q}(a,b)}{S_{p,q}(c,d)} \tag{1.8}$$

is increasing with both p and q according to (1.2). Soon after, Losonczi studied four monotonicity properties of the ratio

$$R_{p,q}(a,b,c) := \frac{S_{p,q}(a,b)}{S_{p,q}(a,c)} \left(p,q \in \mathbb{R}, 0 < a < b < c\right) \tag{1.9}$$

in the parameters p, q and completely solve the comparison problem

$$R_{p,q}(a,b,c) \leq R_{r,s}(a,b,c) \left(p,q,r,s \in \mathbb{R},\ 0 < a < b < c\right) \tag{1.10}$$

for this ratio [12]. This generalizes Chen's result. Also, an open problem was proposed by the author.

Let $M_{p,q}(p,q \in \mathbb{R})$ be a two-parameter, symmetric, and homogeneous mean defined for positive variables and let us form the ratio

$$R_{p,q}(a,b,c) := \frac{M_{p,q}(a,b)}{M_{p,q}(a,c)} \quad \left(p,q \in \mathbb{R},\ 0 < a < b < c\right). \tag{1.11}$$

For what means $M_{p,q}$ has this ratio simple monotonicity properties?

The more general form of two-parameter, symmetric, and homogeneous means is the so-called two-parameter homogenous functions first introduced by Yang [13]. For conveniences, we record it as follows.

Definition 1.1. Assume that $f \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \cup \{0\}$ is n-order homogeneous, and continuous and exists first partial derivatives and $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$, $(p,q) \in \mathbb{R} \times \mathbb{R}$.

If $f(x,y) > 0$ for $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $x \neq y$ and $f(x,x) = 0$ for all $x \in \mathbb{R}_+$, then define that

$$\begin{aligned}
\mathscr{H}_f(p,q;a,b) &:= \left(\frac{f(a^p,b^p)}{f(a^q,b^q)}\right)^{1(p-q)} \quad (p \neq q, pq \neq 0),\\
\mathscr{H}_f(p,p;a,b) &:= \lim_{q \to p} \mathscr{H}_f(a,b;p,q) = G_{f,p}(a,b) \quad (p = q \neq 0),
\end{aligned} \tag{1.12}$$

where

$$G_{f,p}(a,b) = G_f^{1/p}(a^p,b^p), \qquad G_f(x,y) = \exp\left(\frac{x f_x(x,y) \ln x + y f_y(x,y) \ln y}{f(x,y)}\right), \tag{1.13}$$

and $f_x(x,y)$ and $f_y(x,y)$ denote first-order partial derivative for first and second variables of $f(x,y)$, respectively.

If $f(x,y) > 0$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$, then define further

$$\begin{aligned}
\mathcal{H}_f(p,0;a,b) &:= \left(\frac{f(a^p,b^p)}{f(1,1)}\right)^{1/p} \quad (p \neq 0, q = 0),\\
\mathcal{H}_f(0,q;a,b) &:= \left(\frac{f(a^q,b^q)}{f(1,1)}\right)^{1/q} \quad (p = 0, q \neq 0),\\
\mathcal{H}_f(0,0;a,b) &:= \lim_{p\to 0}\mathcal{H}_f(a,b;p,0) = a^{f_x(1,1)/f(1,1)} b^{f_y(1,1)/f(1,1)} \quad (p = q = 0).
\end{aligned} \tag{1.14}$$

Since $f(x,y)$ is a homogeneous function, $\mathcal{H}_f(a,b;p,q)$ is also one and called a homogeneous function with parameters p and q, and simply denoted by $\mathcal{H}_f(p,q)$ sometimes.

The aim of this paper is to investigate the monotonicity of the ratio defined by

$$R_f(p,q) := \frac{\mathcal{H}_f(p,q;a,b)}{\mathcal{H}_f(p,q;c,d)} \quad \left(p,q \in \mathbb{R}, a,b,c,d > 0 \quad \text{with } \frac{b}{a} > \frac{d}{c} \geq 1\right) \tag{1.15}$$

and presents four types of monotonicity of $R_f(p,q)$ in the parameters p and q, which give an easier access to find two-parameter symmetric homogeneous means having ratio simple monotonicity properties mentioned by Losonczi [12].

2. Properties and Lemmas

Before formulating our main results, let us recall the properties and lemmas of two-parameter homogeneous functions.

Property 2.1. $\mathcal{H}_f(p,q)$ is symmetric with respect to p,q, that is,

$$\mathcal{H}_f(p,q) = \mathcal{H}_f(q,p). \tag{2.1}$$

Property 2.2. If $f(x,y)$ is symmetric with respect to x and y, then

$$\begin{aligned}
\mathcal{H}_f(-p,-q;a,b) &= \frac{G^{2n}}{\mathcal{H}_f(p,q;a,b)},\\
\mathcal{H}_f(p,-p;a,b) &= G^n,
\end{aligned} \tag{2.2}$$

where $G = \sqrt{ab}$.

Property 2.3 (see [14, (1.13)]). If $G_{f,t}$ is continuous on $[q,p]$ or $[p,q]$, then

$$\ln \mathcal{H}_f(p,q) = \frac{1}{p-q}\int_q^p \ln G_{f,t}\,dt, \tag{2.3}$$

where $G_{f,t}$ is defined by (1.13).

It is worth mentioning that the following function

$$T(t) = T(t; a, b) := \ln f(a^t, b^t), \quad t \neq 0 \tag{2.4}$$

is well behaved, whose properties as useful lemmas read as follows.

Lemma 2.4 (see [14, (1.14), (1.15), (2.10), (2.11)]). *Suppose that $f : \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, x) : x \in \mathbb{R}_+\} \to \mathbb{R}_+)$ is a symmetric, n-order homogenous and three-time differentiable function. Then*

$$T(t) - T(-t) = 2nt \ln G, \tag{2.5}$$

$$T'(t) + T'(-t) = 2n \ln G, \tag{2.6}$$

$$T''(-t) = T''(t), \tag{2.7}$$

$$T'''(-t) = -T'''(t), \tag{2.8}$$

where $G = \sqrt{ab}$.

Remark 2.5. If $f(1,1) := \lim_{x\to 1} f(x, 1) > 0$, then $T(t)$ can be extended continuously by defining $T(0) := \lim_{t\to 0} T(t) = \ln f(1,1)$, with the result that $T(t)$ is also three times derivable at $t = 0$. Particularly, $T'(0) := n \ln G$. Thus (2.6) can be written as

$$T'(t) + T'(-t) = 2T'(0). \tag{2.9}$$

Lemma 2.6 (see [14, Lemma 3, 4]). *Suppose that $f : \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, x) : x \in \mathbb{R}_+\} \to \mathbb{R}_+$ is an n-order homogenous and three times differentiable function. Then*

$$T'(t) = \frac{a^t f_x(a^t, b^t) \ln a + b^t f_b(a^t, b^t) \ln b}{f(a^t, b^t)}, \tag{2.10}$$

$$T''(t) = -xy\mathcal{I}\ln^2\left(\frac{b}{a}\right), \quad \mathcal{I} = (\ln f)_{xy}, \tag{2.11}$$

$$T'''(t) = -Ct^{-3}\mathcal{J}, \quad \mathcal{J} = (x - y)(x\mathcal{I})_x, \quad C = \frac{xy\ln^3(x/y)}{x - y} > 0, \tag{2.12}$$

where $x = a^t$, $y = b^t$, $G_f(x, y)$ is defined by (1.13).

Remark 2.7. Comparing (1.13) with (2.10), we see that $T'(t) = \ln G_{f,t}(a, b)$. Thus (2.3) can be written as

$$\begin{aligned} \ln \mathcal{H}_f(p, q) &= \begin{cases} \dfrac{1}{p - q}\displaystyle\int_q^p T'(t)dt & \text{if } p \neq q \\ T'(q) & \text{if } p = q \end{cases} \\ &= \int_0^1 T'(tp + (1 - t)q)dt. \end{aligned} \tag{2.13}$$

Based on properties and lemmas above, the author has investigated the monotonicity and log-convexity of two-parameter homogeneous functions and obtained a series of valuable results in [13, 14], which yield some new and interesting inequalities for means. Recently, two results on monotonicity and log-convexity of a four-parameter homogeneous containing Stolarsky mean and Gini mean have been presented in [15].

In the processes of proofs on [13–15], two decision functions play an important role, which

$$\begin{aligned} \mathcal{I} = \mathcal{I}(x,y) &= \frac{\partial^2 \ln f(x,y)}{\partial x \partial y} = (\ln f(x,y))_{xy} = (\ln f)_{xy}, \\ \mathcal{J} = \mathcal{J}(x,y) &= (x-y)\frac{\partial(x\mathcal{I})}{\partial x} = (x-y)(x\mathcal{I})_x. \end{aligned} \tag{2.14}$$

In next section we will encounter other two key decision functions defined by

$$\mathcal{T}_2(x,y) := -xy\mathcal{I}\ln^2\left(\frac{x}{y}\right), \tag{2.15}$$

$$\mathcal{T}_3(x,y) := -xy(x\mathcal{I})_x\ln^3\left(\frac{x}{y}\right), \tag{2.16}$$

where $\mathcal{I} = (\ln f)_{xy}$, $x = a^t$, $y = b^t$. Combining (2.11), (2.12) with (2.15), (2.16) we have the following relations:

$$T''(t) = t^{-2}\mathcal{T}_2(x,y), \tag{2.17}$$

$$T'''(t) = t^{-3}\mathcal{T}_3(x,y), \tag{2.18}$$

where $x = a^t$, $y = b^t$.

Moreover, it is easy to verify that $\mathcal{T}_2(x,y)$ and $\mathcal{T}_3(x,y)$ both are zero-order homogeneous functions due to homogeneity of $f(x,y)$, and thus,

$$\begin{aligned} \mathcal{T}_2(x,y) &= \mathcal{T}_2\left(\frac{x}{y},1\right) = \mathcal{T}_2\left(1,\frac{y}{x}\right), \\ \mathcal{T}_3(x,y) &= \mathcal{T}_3\left(\frac{x}{y},1\right) = \mathcal{T}_3\left(1,\frac{y}{x}\right). \end{aligned} \tag{2.19}$$

3. Main Results and Proofs

Next let us consider the monotonicities of ratio of two-parameter homogeneous functions defined by (1.15). In what follows, we always assume $b/a \neq d/c$.

Theorem 3.1 (first monotonicity property). *Suppose that $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a symmetric, homogenous, and two time-differentiable function; $\mathcal{T}_2(1,u)$ is strictly increasing (decreasing) with $u > 1$; (1.2) is satisfied. Then $R_f(p,q)$ is strictly increasing (decreasing) in either p or q unless $b/a = d/c$.*

Proof. Since $\mathcal{H}_f(p,q)$ is symmetric with respect to p and q, it only needs to prove the log-convexity of $R_f(p,q)$ in parameter p.

Direct partial derivative calculation for (2.13) leads to

$$\frac{\partial \ln \mathcal{H}_f(p,q)}{\partial p} = \int_0^1 tT''(tp+(1-t)q)\,dt. \tag{3.1}$$

From (1.15), we have

$$\frac{\partial \ln R_f(p,q)}{\partial p} = \int_0^1 t\left(T''(tp+(1-t)q;a,b) - T''(tp+(1-t)q;c,d)\right)dt. \tag{3.2}$$

Since $\mathcal{T}_2(1,u)$ is strictly increasing (decreasing) with $u > 1$ and by (2.7), (2.17), and assumption (1.2), we have always

$$\begin{aligned} T''(t;a,b) - T''(t;c,d) &= T''(|t|;a,b) - T''(|t|;c,d) \\ &= t^{-2}\mathcal{T}_2\left(a^{|t|},b^{|t|}\right) - t^{-2}\mathcal{T}_2\left(c^{|t|},d^{|t|}\right) \\ &= t^{-2}\left(\mathcal{T}_2\left(1,\left(\frac{b}{a}\right)^{|t|}\right) - \mathcal{T}_2\left(1,\left(\frac{d}{c}\right)^{|t|}\right)\right) > (<)0. \end{aligned} \tag{3.3}$$

It follows that

$$\frac{\partial \ln R_f(p,q)}{\partial p} > (<)0. \tag{3.4}$$

This proof is completed. □

The next monotonicity result is a direct corollary of Theorem 3.1 actually.

Theorem 3.2 (second monotonicity property). *The conditions are the same as those of Theorem 3.1. Then for fixed $m \in \mathbb{R}$, the function $R_f(p,p+m)$ is strictly increasing (decreasing) with p unless $b/a = d/c$.*

Proof. Under the same conditions as Theorem 3.1, the function $R_f(p,q)$ is strictly increasing (decreasing) in either p or q. Hence for $p_1, p_2 \in \mathbb{R}$ with $p_1 < p_2$, we have

$$R_f(p_1,p_1+m) < R_f(p_2,p_1+m) < R_f(p_2,p_2+m), \tag{3.5}$$

which indicates that the function $R_f(p,p+m)$ is strictly increasing (decreasing) with p.

The proof ends. □

To investigate the third and fourth monotonicity properties, we need a useful lemma.

Lemma 3.3. *Let $f(x)$ be odd and continuous on $(-m, m)$ $(m > 0)$. Then*

$$\int_s^r f(x)dx = \int_{|s|}^{|r|} f(x)dx \tag{3.6}$$

is always true for arbitrary $r, s \in (-m, m)$.

Proof. By the additivity of definite integral we have

$$\int_s^r f(x)dx = \int_s^{|s|} f(x)dx + \int_{|s|}^{|r|} f(x)dx + \int_{|r|}^{r} f(x)dx. \tag{3.7}$$

According to the property of definite integral of odd functions, our required result is obtain immediately.

This lemma is proved. □

Theorem 3.4 (third monotonicity property). *Suppose that $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a symmetric, homogenous, and three-time differentiable function; $\mathcal{T}_3(1, u)$ is strictly increasing (decreasing) with $u > 1$; (1.2) is satisfied. Then for fixed $m \neq 0$, the function $R_f(p, 2m - p)$ is*

(1) *strictly decreasing (increasing) with p on $(-\infty, m)$ and increasing (decreasing) with p on (m, ∞) if $m > 0$ unless $b/a = d/c$;*

(2) *strictly increasing (decreasing) with p on $(-\infty, m)$ and decreasing (increasing) with p on (m, ∞) if $m < 0$ unless $b/a = d/c$.*

Proof. By (2.13), $\ln \mathcal{H}_f(p, 2m - p)$ can be expressed in integral form as

$$\ln \mathcal{H}_f(p, 2m - p) = \int_0^1 T'(t_1(t))dt, \tag{3.8}$$

where $t_1(t) = tp + (1 - t)(2m - p)$. Direct partial derivative calculation leads to

$$\frac{\partial \ln \mathcal{H}_f(p, 2m - p)}{\partial p} = \int_0^1 (2t - 1)T''(t_1(t))dt, \tag{3.9}$$

which can be spiltted into a sum of two integrals:

$$\int_0^{1/2} (2t - 1)T''(t_1(t))dt + \int_{1/2}^1 (2t - 1)T''(t_1(t))dt. \tag{3.10}$$

Substituting $t = 1 - v$ in the first integral above yields

$$\int_0^{1/2} (2t - 1)T''(t_1(t))dt = -\int_{1/2}^1 (2v - 1)T''(t_2(v))dv, \tag{3.11}$$

where $t_2(t) = (1-t)p + t(2m-p)$. Hence

$$\begin{aligned}
\frac{\partial \ln \mathcal{H}_f(p,2m-p)}{\partial p} &= -\int_{1/2}^{1}(2v-1)T''(t_2(v))dv + \int_{1/2}^{1}(2t-1)T''(t_1(t))dt \\
&= \int_{1/2}^{1}(2t-1)\left(T''(t_1(t)) - T''(t_2(t))\right)dt \\
&= \int_{1/2}^{1}(2t-1)\left(\int_{t_2(t)}^{t_1(t)} T'''(s)ds\right)dt \\
&= \int_{1/2}^{1}(2t-1)\left(\int_{|t_2(t)|}^{|t_1(t)|} T'''(s)ds\right)dt \ \left(\text{by Lemma (3.3)}\right).
\end{aligned} \tag{3.12}$$

From (1.15), we have

$$\begin{aligned}
\frac{\partial \ln R_f(p,2m-p)}{\partial p} &= \frac{\partial \ln \mathcal{H}_f(p,2m-p;a,b)}{\partial p} - \frac{\partial \ln \mathcal{H}_f(p,2m-p;c,d)}{\partial p} \\
&= \int_{1/2}^{1}(2t-1)\left(\int_{|t_2(t)|}^{|t_1(t)|}\left(T'''(s;a,b) - T'''(s;c,d)\right)ds\right)dt.
\end{aligned} \tag{3.13}$$

Since $\mathcal{T}_3(1,u)$ is strictly increasing (decreasing) with $u > 1$, by (2.18) and (1.2), we have always

$$\begin{aligned}
T'''(t;a,b) - T'''(t;c,d) &= t^{-3}\left(\mathcal{T}_3(a^t,b^t) - \mathcal{T}_3(c^t,d^t)\right) \\
&= t^{-3}\left(\mathcal{T}_3\left(1,\left(\frac{b}{a}\right)^t\right) - \mathcal{T}_3\left(1,\left(\frac{d}{c}\right)^t\right)\right) > (<)0 \quad \text{for } t > 0.
\end{aligned} \tag{3.14}$$

It follows from $2t-1 \geq 0$ $(t \in 1/2, 1]$ that $\partial \ln R_f(p,2m-p)/\partial p$ is positive (negative) if $|t_1(t)| > |t_2(t)|$, zero if $|t_2(t)| = |t_2(t)|$, and negative (positive) if $|t_1(t)| < |t_2(t)|$. However,

$$|t_1(t)|^2 - |t_2(t)|^2 = 8m\left(t - \frac{1}{2}\right)(p-m), \tag{3.15}$$

and hence

$$\frac{\partial \ln R_f(p,2m-p)}{\partial p}\begin{cases} > (<)0 & \text{if } m > 0, \quad p > m \text{ or } m < 0, \quad p < m, \\ = 0 & \text{if } m = 0, \\ < (>)0 & \text{if } m > 0, \quad p < m \text{ or } m < 0, \quad p > m. \end{cases} \tag{3.16}$$

This completes the proof. □

Theorem 3.5 (fourth monotonicity property). *The conditions are the same as those of Theorem 3.1. Then for fixed $r, s \in \mathbb{R}$, the function $R_f(pr, ps)$ is strictly increasing (decreasing) with p if $r + s > 0$ and decreasing (increasing) if $r + s < 0$.*

Proof. By (2.13), $\ln \mathscr{H}_f(pr, ps)$ can be expressed in integral form as

$$\ln \mathscr{H}_f(pr, ps) = \begin{cases} \dfrac{1}{r-s}\displaystyle\int_s^r T'(pt)\,dt & \text{if } r \neq s, \\ T'(pr) & \text{if } r = s. \end{cases} \tag{3.17}$$

A partial derivative calculation yields

$$\frac{\partial \ln \mathscr{H}_f(pr, ps)}{\partial p} = \begin{cases} \dfrac{1}{r-s}\displaystyle\int_s^r tT''(pt)\,dt & \text{if } r \neq s, \\ rT''(pr) & \text{if } r = s. \end{cases} \tag{3.18}$$

(1) In the case of $r \neq s$ (2.7) implies that $tT''(pt) = tT''(|pt|)$ is odd and makes use of Lemma 3.3 and (3.18) can be written as

$$\frac{\partial \ln \mathscr{H}_f(pr, ps)}{\partial p} = \frac{1}{r-s}\int_{|s|}^{|r|} tT''(|pt|)\,dt = \frac{r+s}{|r|-|s|}\int_{|s|}^{|r|} tT''(|pt|)\,dt, \tag{3.19}$$

and then

$$\frac{\partial \ln R_f(pr, ps)}{\partial p} = \frac{r+s}{|r|-|s|}\int_{|s|}^{|r|} t\left(T''(|pt|; a, b) - T''(|pt|; c, d)\right)dt. \tag{3.20}$$

Since $\mathcal{T}_2(1, u)$ strictly increasing (decreasing) with $u > 1$ and by assumption (1.2), so (3.3) is true, which indicates that $T''(|t|; a, b) - T''(|t|; c, d) > (<)0$. It follows that

$$\frac{\partial \ln R_f(pr, ps)}{\partial p} \begin{cases} > (<)0 & \text{if } r + s > 0, \\ < (>)0 & \text{if } r + s < 0. \end{cases} \tag{3.21}$$

This shows that $R_f(pr, ps)$ is strictly increasing (decreasing) with p if $r + s > 0$ and decreasing (increasing) if $r + s < 0$.

(2) In the case of $r = s$. Similarly, by (3.18), (2.7), (1.2), and (3.3) we have

$$\frac{\partial \ln R_f(pr, ps)}{\partial p} = r\left(T''(|pr|; a, b) - T''(|pr|; c, d)\right) \begin{cases} > (<)0 & \text{if } r > 0, \\ < (>)0 & \text{if } r < 0. \end{cases} \tag{3.22}$$

Combining two cases above, the proof is accomplished. □

4. Applications

As applications of main results in this paper, next let us prove the monotonicity of ratio of Stolarsky means. We will see that the methods provided by this paper are simple and effective.

It is easy to verify that the two-parameter logarithmic mean is just Stolarsky mean, that is, $\mathscr{L}_L(p,q;a,b) = S_{p,q}(a,b)$. Consequently, the monotonicities of ratio of Stolarsky means depend on the monotonicities of $\mathcal{T}_2(1,u)$ and $\mathcal{T}_3(1,u)$ defined by (2.15) and (2.16).

Some simple calculations yield

$$\begin{aligned}
\mathcal{I} &= (\ln L)_{xy} = \frac{1}{(x-y)^2} - \frac{1}{xy(\ln x - \ln y)^2},\\
\mathcal{T}_2(x,y) &= -xy\mathcal{I}\ln^2\left(\frac{x}{y}\right) = -xy\frac{\ln^2(x/y)}{(x-y)^2} + 1,\\
\frac{d\mathcal{T}_2(1,u)}{du} &= -2(u-1)^{-3}\ln^2 u\left(\frac{u-1}{\ln u} - \frac{u+1}{2}\right),\\
(x\mathcal{I})_x &= -\frac{x+y}{(x-y)^3} + \frac{2}{xy(\ln x - \ln y)^3},\\
\mathcal{T}_3(x,y) &= -xy(x\mathcal{I})_x\ln^3\left(\frac{x}{y}\right) = -2 + \frac{xy(x+y)}{(x-y)^3}\ln^3\left(\frac{x}{y}\right),\\
\frac{d\mathcal{T}_3(1,u)}{du} &= 6u(u-1)^4\ln^3 u\left(\frac{u^2-1}{\ln u^2} - \frac{((u^2+1)/2) + 2\sqrt{u^2}}{3}\right).
\end{aligned} \tag{4.1}$$

Making use of the well-known inequalities $L(x,y) < (x+y)/2$ $(x,y>0)$ and $L(x,y) < ((x+y)/2 + 2\sqrt{xy})/3(x,y>0)$ [16], we see that $d\mathcal{T}_2(1,u)/du > 0$ if $u>1$ and $d\mathcal{T}_3(1,u)/du < 0$ if $u > 1$.

Applying our main results, we can obtain all theorems involving monotonicity of ratio of Stolarsky means in Section 2 of [12]. Here we have no longer list.

Lastly, as concrete applications of the monotonicity of ratio of Stolarsky means, we now show a refined chain of inequalities of ratio of means involving logarithmic mean, exponential mean (identric mean), arithmetic mean, geometric mean, and Heronian mean, which is a generalization of inequalities in [14, (5.5)] and contains (1.4).

For convenience of statement in the following theorem, corresponding to (1.3) let us define further that

$$\widetilde{M_p} := M^{1/p}(c^p, d^p), \quad M = A, H, L, I, \tag{4.2}$$

where A,H,L, and I stand for arithmetic mean, Heronian mean, logarithmic mean, and exponential mean (identric mean) of two positive numbers c and d, respectively.

Theorem 4.1. *Suppose that* a, b, c, d *satisfy assumption* (1.2). *Then the following inequalities*

$$\begin{aligned}\frac{A^{1/3}G^{2/3}}{\widetilde{A}^{1/3}\widetilde{G}^{2/3}} \le \frac{\sqrt{GH}}{\sqrt{\widetilde{G}\widetilde{H}}} \le \frac{G^{2/5}A_{1/3}^{1/5}A_{2/3}^{2/5}}{\widetilde{G}^{2/5}\widetilde{A}_{1/3}^{1/5}\widetilde{A}_{2/3}^{2/5}} \le \frac{L}{\widetilde{L}} \\ \le \frac{A_{1/5}^{1/3}A_{2/5}^{2/3}}{\widetilde{A}_{1/5}^{1/3}\widetilde{A}_{2/5}^{2/3}} \le \frac{H_{1/2}}{\widetilde{H}_{1/2}} \le \frac{A_{1/3}}{\widetilde{A}_{1/3}} \le \frac{H_{2/5}^{2}A_{1/5}^{-1}}{\widetilde{H}_{2/5}^{2}\widetilde{A}_{1/5}^{-1}} \le \frac{I_{1/2}}{\widetilde{I}_{1/2}}\end{aligned} \tag{4.3}$$

hold, with equalities if and only if $b/a = d/c$.

Proof. By the third monotonicity property, we see that $R_{p,1-p}(a, b; c, d)$ is strictly decreasing in p on $(1/2, \infty)$. Put $p = 2, 3/2, 4/3, 1, 4/5, 3/4, 2/3, 3/5, 1/2$ in $R_{p,1-p}(a, b; c, d)$ and by some calculations, the chain of inequalities (4.3) is derived immediately, with equalities if and only if $b/a = d/c$ because the monotonicity of $R_{p,1-p}(a, b; c, d)$ is strict.

The proof is finished. □

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