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Research Article

Fixed Points Approximation and Solutions of Some Equilibrium and Variational Inequalities Problems

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We prove a new strong convergence theorem for an element in the intersection of the set of common fixed points of a countable family of nonexpansive mappings, the set of solutions of some variational inequality problems, and the set of solutions of some equilibrium problems using a new iterative scheme. Our theorem generalizes and improves some recent results.

1. Introduction

Let H be a real Hilbert space; a mapping $B : D(B) \to H$ is said to be *monotone* if for all $x, y \in D(B)$

$$\langle Bx - By, x - y \rangle \geq 0. \tag{1.1}$$

For some $\lambda > 0$, the mapping B is said to be *λ-inverse strongly monotone* if

$$\langle Bx - By, x - y \rangle \geq \lambda \|Bx - By\|^2. \tag{1.2}$$

A λ-inverse strongly monotone map is some time called *λ-cocoercive.* A map B is said to be *relaxed λ-cocoercive* if there exists a constant $\lambda > 0$ such that

$$\langle Bx - By, x - y \rangle \geq -\lambda \|Bx - By\|^2 \quad \forall x, y \in K. \tag{1.3}$$

B is said to be relaxed (λ, γ)-*cocoercive*, if there exist $\lambda, \gamma > 0$ such that

$$\langle Bx - By, x - y \rangle \geq -\lambda \|Bx - By\|^2 + \gamma \|x - y\|^2. \tag{1.4}$$

A map $B : H \to H$ is said to be λ-*Lipschitzian* if there exists a real number $\lambda \geq 0$ such that

$$\|Bx - By\| \leq \lambda \|x - y\| \quad \forall x, y \in H. \tag{1.5}$$

B is a *contraction* map, if in the above inequality $\lambda \in [0, 1)$ and *nonexpansive* if $\lambda = 1$.

Let K be a nonempty, closed, and convex subset of a real Hilbert space H. A variational inequality problem is searched for $x^* \in K$ such that

$$\langle Bx^*, y - x^* \rangle \geq 0 \quad \forall y \in K, \tag{1.6}$$

where B is some nonlinear mapping of K into H. Inequality (1.6) is called the variational inequality.

Recall that for each $x \in H$ there exists a unique nearest point in K to x denoted by $P_K x$. That is, $\|x - P_K x\| \leq \|x - y\|$ for all $y \in K$. P_K is called a *metric projection* of H onto K. The mapping P_K is nonexpansive in this setting, that is, $\|P_K x - P_K y\| \leq \|x - y\|$ for all $x, y \in H$. It is also known that P_K satisfies the following inequality $\|P_K x - P_K y\|^2 \leq \langle x - y, P_K x - P_K y \rangle$.

The solution set of the problem (1.6) is denoted by $\mathrm{VI}(K, B)$. It is well known (see [1]) that $x^* \in \mathrm{VI}(K, B)$ if and only if

$$x^* = P_K(x^* - \lambda B x^*), \quad \forall \lambda > 0. \tag{1.7}$$

A monotone map B is said to be *maximal* if the graph $\Gamma(B)$ of B is not properly contained in the graph of any other monotone map, where $\Gamma(B) = \{(x, y) \in H \times H : y \in Bx\}$ for a multivalued map B. It is also known that B is maximal monotone if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in \Gamma(B)$ implies $f \in Bx$. Let B be a monotone mapping defined from K into H and let $N_K q$ be a normal cone to K at $q \in K$, that is, $N_K q = \{p \in H : \langle q - u, p \rangle \geq,$ for all $u \in K\}$. Define a map M by

$$Mq = \begin{cases} Bq + N_K q, & q \in K, \\ \emptyset, & q \notin K. \end{cases} \tag{1.8}$$

Then, M is maximal monotone and $x^* \in M^{-1}(0) \Leftrightarrow x^* \in \mathrm{VI}(K, B)$, see, for example, [2].

Let $G : K \times K \to \mathbb{R}$ be a bifunction on a closed convex nonempty subset K of a real Hilbert space H. An equilibrium problem is searched for $x^* \in K$ such that

$$G(x^*, y) \geq 0 \quad \forall y \in K. \tag{1.9}$$

The set of solutions of the equilibrium problem above is denoted by $\mathrm{EP}(G)$.

Several physical problems (such as the theories of lubrications, filterations and flows, moving boundary problems, see, e.g., [1, 3]) can be reduced to variational inequality or equilibrium problems. Consequently, these problems have solutions as the solutions of these resultant variational inequality or equilibrium problems.

Maingé [4] introduced a Halpern-type scheme and proved a strong convergence theorem for family of nonexpansive mappings in Hilbert space.

Recently, S. Takahashi and W. Takahashi [5] introduced an iterative scheme which they used to study the problem of approximating a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping.

More recently, Kumam and Katchang [6], W. Kumam and P. Kumam [7], Li and Su [8] and many others (see, e.g. [9–12] and the references contained in them) studied the problem of fixed point approximations and solutions of some equilibrium and, or solutions of some variational inequalities problems.

In this paper we introduce a new iterative scheme for approximation of a common element in the intersection of the set of fixed points of some countable family of nonexpansive mappings, the set of solutions of some equilibrium problem, and the set of solutions of some variational inequality problem and prove a new theorem. Our theorem generalizes and improves some recent results.

2. Preliminaries

For a sequence $\{x_n\}$ the notation $x_n \to x^*$ and $x_n \rightharpoonup x^*$ means that the sequence $\{x_n\}$ converges strongly and weakly to x^*, repectively. A Banach space E is said to satisfy an Opial's condition (or in other words is an Opial's space) if for a sequence $\{x_n\}$ in E with $x_n \rightharpoonup x^*$, then

$$\liminf_{n\to\infty}\|x_n - x^*\| < \liminf_{n\to\infty}\|x_n - y\| \quad \text{for any } y \in E\ y \neq x^*. \tag{2.1}$$

It is well known that Hilbert spaces are Opial's spaces (see [13]).

In the sequel we shall make use of the following results.

Lemma 2.1 (see [14]). *Let K be a nonempty closed convex subset of H and let G be a bifunction of $K \times K$ into $\mathbb{R}$ satisfying*

($A1$) $G(x,x) = 0$ *forall* $x \in K$;

($A2$) *G is monotone, i.e.* $G(x,y) + G(y,x) \leq 0$ *forall* $x, y \in K$;

($A3$) *forall* $x, y, z \in K$, $\limsup_{t\to 0+} G(t_z + (1+t)x, y) \leq G(x,y)$;

($A4$) *forall* $x \in K$, *$G(x,.)$ is convex and lower semicontinuous.*

Let $r > 0$ and $x \in H$. Then there exists $z \in K$ such that

$$G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in K. \tag{2.2}$$

Lemma 2.2 (see [15]). *Let K be a nonempty closed convex subset of H and let G be a bifunction of $K \times K$ into $\mathbb{R}$ satisfying $(A1)$–$(A4)$. For $r > 0$ and $x \in H$ define a map $T_r : H \to K$ by*

$$T_r x = \left\{ z \in K : G(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0,\ \forall y \in K \right\}. \tag{2.3}$$

Then, the following holds:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \tag{2.4}$$

(3) $\mathrm{Fix}(T_r) = \mathrm{EP}(G)$;

(4) $\mathrm{EP}(G)$ *is closed and convex.*

Lemma 2.3 (see [16]). *Let H be a real inner product space. Then, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \quad \forall x, y \in H. \tag{2.5}$$

Lemma 2.4 (see [17]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all integers $n \geq 0$ and $\limsup(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim \|y_n - x_n\| = 0$.*

Lemma 2.5 (see [18]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0, \tag{2.6}$$

Where (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0; (n \geq 0), \sum \gamma_n < \infty$. *Then, $a_n \to 0$ as $n \to \infty$.*

3. Main Results

In the sequel we assume that the sequences $\{\alpha_n\}, \{\sigma_{i,n}\}_i \subset (0,1)$ satisfy $\sum_{i \geq 1} \sigma_{i,n} = 1 - \alpha_n$.

Theorem 3.1. *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Let G be a bifunction from $K \times K$ to $\mathbb{R}$ which satisfies conditions $(A1)$–$(A4)$. Let $\{T_i, i = 1, 2, 3, \ldots\}$ be family of nonexpansive mappings of K into H and let B be a μ-Lipschitzian, relaxed (λ, γ)-cocoercive map of*

K into H such that $F := \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{EP}(G) \cap \mathrm{VI}(K,B) \neq \emptyset$. For an arbitrary but fixed $\delta \in (0,1)$, let $\{x_n\}$ and $\{y_n\}$ be sequences generated by

$$\begin{gathered} x_1, u \in H, \\ G(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n \rangle \geq 0, \quad \forall \eta \in K, \\ x_{n+1} = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta \sum_{i \geq 1} \sigma_{in} T_i P_K (I - s_n B) y_n, \quad n \geq 1, \end{gathered} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\sigma_{in}\}$ are sequences in $[0,1]$ and $\{r_n\}$ and $\{s_n\}$ are sequences in $[0,\infty)$ satisfying

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $\{s_n\} \subset [a,b]$ *for some a, b satisfying* $0 \leq a \leq b \leq 2(\gamma - \lambda\mu^2)/\mu^2$,

(C4) $\lim_{n\to\infty}|s_{n+1} - s_n| = 0$, $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$, $\lim_{n\to\infty} \sum_{i\geq 1} |\sigma_{i,n+1} - \sigma_{i,n}| = 0$,

(C5) $\liminf_{n\to\infty} r_n > 0$.

Then, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_F u$.

Proof. First, we show that $(I - s_n B)$ is nonexpansive, actually, using the property of B we have for $x, y \in H$,

$$\begin{aligned} \|(I - s_n B)x - (I - s_n B)y\|^2 &= \|x - y - s_n(Bx - By)\|^2 \\ &= \|x-y\|^2 - 2s_n\langle x - y, Bx - By\rangle + s_n^2\|Bx - By\|^2 \\ &\leq \|x-y\|^2 - 2s_n\left[-\lambda\|Bx - By\|^2 + \gamma\|x-y\|^2\right] + s_n^2\|Bx - By\|^2 \\ &\leq \|x-y\|^2 + 2s_n\mu^2\lambda\|x-y\|^2 - 2s_n\gamma\|x-y\|^2 + \mu^2 s_n^2\|x-y\|^2 \\ &= \left(1 + 2s_n\mu^2\lambda - 2s_n\gamma + \mu^2 s_n^2\right)\|x-y\|^2 \\ &\leq \|x-y\|^2, \end{aligned} \tag{3.2}$$

and thus $(I - s_n B)$ is nonexpansive.

Let $x^* \in F$; since $y_n = T_{r_n} x_n$, $n \in \mathbb{N}$ and the fact that T_{r_n} is firmly nonexpansive (and hence nonexpansive) we have the following:

$$\|y_n - x^*\| = \|T_{r_n} x_n - T_{r_n} x^*\| \leq \|x_n - x^*\|. \tag{3.3}$$

We claim that $\{x_n\}$ satisfies

$$\|x_n - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\} \quad \forall n \geq 1. \tag{3.4}$$

We prove this by induction. Cearly the result is true for $n = 1$. Assume that the result holds for $n = k$ for some $k \in \mathbb{N}$. Then, for $n = k + 1$ we have

$$\begin{aligned}\|x_{k+1} - x^*\| &= \|\alpha_k(u - x^*) + (1-\delta)(1-\alpha_k)(x_k - x^*) \\ &\quad + \delta\sum_{i\geq 1}\sigma_{ik}T_iP_K(I - s_kB)(y_k - x^*)\| \\ &\leq \alpha_k\|u - x^*\| + (1-\delta)(1-\alpha_k)\|x_k - x^*\| \\ &\quad + \delta\sum_{i\geq 1}\sigma_{ik}\|T_iP_K(I - s_kB)y_k - T_iP_K(I - s_kB)x^*\| \\ &\leq \alpha_k\|u - x^*\| + (1-\alpha_k)\|x_k - x^*\| \\ &\leq \max\{\|u - x^*\|, \|x_k - x^*\|\}.\end{aligned} \tag{3.5}$$

Hence the result, and so $\{x_n\}$ is bounded. Furthermore, $\{y_n\}$, $\{T_iP_K(y_n - s_nBy_n)\}$ and $\{By_n\}$ are each bounded.

We now show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Note that $y_n = T_{r_n}x_n$, $y_{n+1} = T_{r_{n+1}}x_{n+1}$, so that

$$\begin{aligned} G(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n\rangle \geq 0 \quad \forall \eta \in K, \\ G(y_{n+1}, \eta) + \frac{1}{r_{n+1}}\langle \eta - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0 \quad \forall \eta \in K. \end{aligned} \tag{3.6}$$

Using (3.6) and $(A2)$, we have

$$\left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0. \tag{3.7}$$

which implies that

$$\left\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - x_n - \frac{r_n}{r_{n+1}}(y_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.8}$$

from which we get

$$\|y_{n+1} - y_n\|^2 \leq \|y_{n+1} - y_n\|\left[\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|y_{n+1} - x_{n+1}\|\right]. \tag{3.9}$$

If, without loss of generality M, m are real numbers such that $r_n > m > 0$ for all n and $M := \sup_{n,i}\{\|y_n - x_n\|, \|T_i P_K(I - s_n B)y_n\|, \|By_n\|\}$ we then have

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|y_{n+1} - x_{n+1}\| \\ &\le \|x_{n+1} - x_n\| + \frac{M}{m}|r_{n+1} - r_n|.\end{aligned} \tag{3.10}$$

Now, define two sequences $\{\beta_n\}$ and $\{z_n\}$ by $\beta_n := (1-\delta)\alpha_n + \delta$ and $z_n := (x_{n+1} - x_n + \beta_n x_n)/\beta_n$. Then,

$$z_n = \frac{\alpha_n u + \delta \sum_{i\ge1} \sigma_{i,n} T_i P_K(I - s_n B)y_n}{\beta_n}. \tag{3.11}$$

Observe that $\{z_n\}$ is bounded and that

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le{}& \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right| \|u\| \\ &+ \frac{\delta(1 - \alpha_{n+1})}{\beta_{n+1}} \|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \\ &+ \frac{\delta}{\beta_{n+1}} \sum_{i\ge1} |\sigma_{i,n+1} - \sigma_{i,n}| \|T_i P_K(I - s_{n+1}B)y_n\| \\ &+ \frac{\delta}{\beta_{n+1}\beta_n} |\beta_n - \beta_{n+1}| \sum_{i\ge1} \sigma_{i,n+1} \|T_i P_K(I - s_{n+1}B)y_n\| \\ &+ \frac{\delta(1 - \alpha_n)\|By_n\|}{\beta_n} |s_n - s_{n+1}|.\end{aligned} \tag{3.12}$$

Using (3.10) we have that

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le{}& \left|\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n}\right| \|u\| \\ &+ \left|\frac{\delta(1 - \alpha_{n+1})}{\beta_{n+1}} - 1\right| \|x_{n+1} - x_n\| + \frac{\delta(1 - \alpha_{n+1})M}{\beta_{n+1} m} |r_{n+1} - r_n| \\ &+ \frac{\delta M}{\beta_{n+1}} \sum_{i\ge1} |\sigma_{i,n+1} - \sigma_{i,n}| + \frac{\delta M}{\beta_{n+1}\beta_n} \left|\beta_n - \beta_{n+1}\right| + \frac{\delta(1 - \alpha_n)M}{\beta_n} |s_n - s_{n+1}|.\end{aligned} \tag{3.13}$$

This implies

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0, \tag{3.14}$$

and by Lemma 2.4, $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Hence,

$$\|x_{n+1} - x_n\| = \beta_n \|z_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

Using (3.10) we have

$$\|y_{n+1} - y_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{3.16}$$

We now have

$$\begin{aligned} \|y_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \\ &\le \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^* \rangle \\ &= \langle y_n - x^*, x_n - x^* \rangle \\ &= \frac{1}{2}\left(\|y_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - y_n\|^2\right), \end{aligned} \tag{3.17}$$

so that

$$\|y_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - y_n\|^2. \tag{3.18}$$

But,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 = &\Bigg\| \alpha_n(u - x^*) + (1-\delta)(1-\alpha_n)(x_n - x^*) \\ &+\delta\left[\sum_{i\ge1}\sigma_{in}T_iP_K(I - s_nB)y_n - T_iP_K(I - s_nB)x^*\right]\Bigg\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + (1-\delta)(1-\alpha_n)\|x_n - x^*\|^2 \\ &\quad + \delta\sum_{i\ge1}\sigma_{in}\|T_iP_K(I - s_nB)y_n - T_iP_K(I - s_nB)x^*\|^2 \\ &\le \alpha_n\|u - x^*\|^2 + (1-\delta)(1-\alpha_n)\|x_n - x^*\|^2 + \delta(1-\alpha_n)\|y_n - x^*\|^2. \end{aligned} \tag{3.19}$$

Putting (3.18) in (3.19), we have

$$\begin{aligned} \delta(1-\alpha_n)\|x_n - y_n\|^2 &\le \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &= \alpha_n\|u - x^*\|^2 + (\|x_n - x^*\| - \|x_{n+1} - x^*\|)(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\ &\le \alpha_n\|u - x^*\|^2 + (\|x_n - x_{n+1}\|)(\|x_n - x^*\| + \|x_{n+1} - x^*\|), \end{aligned} \tag{3.20}$$

and hence,

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.21}$$

Observe also that if $w_n := P_K(I - s_nB)y_n$, then

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|P_K(I - s_nB)y_n - P_K(I - s_nB)x^*\|^2 \\
&\le \|y_n - x^* - s_n(By_n - Bx^*)\|^2 \\
&= \|y_n - x^*\|^2 - 2s_n\langle y_n - x^*, By_n - Bx^*\rangle + s_n^2\|By_n - Bx^*\|^2 \\
&\le \|y_n - x^*\|^2 - 2s_n\left[-\lambda\|By_n - Bx^*\|^2 + \gamma\|y_n - x^*\|^2\right] + s_n^2\|By_n - Bx^*\|^2 \\
&= \|y_n - x^*\|^2 + 2s_n\lambda\|By_n - Bx^*\|^2 - 2s_n\gamma\|y_n - x^*\|^2 + s_n^2\|By_n - Bx^*\|^2 \\
&\le \|y_n - x^*\|^2 + \left[2s_n\lambda + s_n^2 - \frac{2s_n\gamma}{\mu^2}\right]\|By_n - Bx^*\|^2.
\end{aligned} \tag{3.22}$$

Also,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \Bigg\|\alpha_n(u - x^*) + (1-\delta)(1-\alpha_n)(x_n - x^*) \\
&\quad +\delta\left[\sum_{i\ge1}\sigma_{in}T_iP_K(I - s_nB)y_n - T_iP_K(I - s_nB)x^*\right]\Bigg\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + (1-\delta)(1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad + \delta\sum_{i\ge1}\sigma_{in}\|T_iP_K(I - s_nB)y_n - T_iP_K(I - s_nB)x^*\|^2 \\
&= \alpha_n\|u - x^*\|^2 + (1-\delta)(1-\alpha_n)\|x_n - x^*\|^2 + \delta\sum_{i\ge1}\sigma_{in}\|T_iw_n - x^*\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + (1-\delta)(1-\alpha_n)\|x_n - x^*\|^2 + \delta(1-\alpha_n)\|w_n - x^*\|^2 \\
&\le \alpha_n\|u - x^*\|^2 + (1-\delta)(1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad + \delta(1-\alpha_n)\left[\|y_n - x^*\|^2 + \left[2s_n\lambda + s_n^2 - \frac{2s_n\gamma}{\mu^2}\right]\|By_n - Bx^*\|^2\right] \\
&\le \alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 + \delta(1-\alpha_n)\left[2s_n\lambda + s_n^2 - \frac{2s_n\gamma}{\mu^2}\right]\|By_n - Bx^*\|^2,
\end{aligned} \tag{3.23}$$

which implies

$$-\delta(1-\alpha_n)\left[2s_n\lambda + s_n^2 - \frac{2s_n\gamma}{\mu^2}\right]\|By_n - Bx^*\|^2 \le \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \tag{3.24}$$

so that

$$\lim_{n\to\infty}\|By_n - Bx^*\| = 0. \tag{3.25}$$

We go further to prove that for each $i \in \mathbb{N}$,

$$\lim_{n\to\infty}\|T_iP_K(I - s_n)By_n - y_n\| = 0. \tag{3.26}$$

Consider the following estimates:

$$\begin{aligned}
\|(T_iP_K(I - s_nB))y_n - y_n\|^2 &= \|(T_iP_K(I - s_nB))y_n - x^* + x^* - y_n\|^2 \\
&= \|x^* - y_n\|^2 + 2\langle (T_iP_K(I - s_nB))y_n - x^*, x^* - y_n\rangle \\
&\quad + \|(T_iP_K(I - s_nB))y_n - x^*\|^2 \\
&\le 2\|x^* - y_n\|^2 + 2\langle (T_iP_K(I - s_nB))y_n - y_n + y_n - x^*, x^* - y_n\rangle \\
&= 2\|y_n - x^*\|^2 + 2\langle (T_iP_K(I - s_nB))y_n - y_n, x^* - y_n\rangle - 2\|y_n - x^*\|^2 \\
&= 2\langle (T_iP_K(I - s_nB))y_n - y_n, x^* - y_n\rangle.
\end{aligned} \tag{3.27}$$

Using (3.1), we have

$$\begin{aligned}
\langle x_{n+1} - x^*, y_n - x^*\rangle &= \alpha_n\langle u - x^*, y_n - x^*\rangle + (1-\alpha_n)(1-\delta)\langle x_n - x^*, y_n - x^*\rangle \\
&\quad + \delta\sum_{i\ge1}\sigma_{i,n}\langle (T_iP_K(I - s_nB))y_n - y_n + y_n - x^*, y_n - x^*\rangle \\
&= \alpha_n\langle u - x^*, y_n - x^*\rangle + (1-\alpha_n)(1-\delta)\langle x_n - y_n + y_n - x^*, y_n - x^*\rangle \\
&\quad + \delta\sum_{i\ge1}\sigma_{i,n}\langle (T_iP_K(I - s_nB))y_n - y_n, y_n - x^*\rangle + \delta(1-\alpha_n)\|y_n - x^*\|^2 \\
&= \alpha_n\langle u - x^*, y_n - x^*\rangle + (1-\alpha_n)(1-\delta)\langle x_n - y_n, y_n - x^*\rangle \\
&\quad + \delta\sum_{i\ge1}\sigma_{i,n}\langle (T_iP_K(I - s_nB))y_n - y_n, y_n - x^*\rangle + (1-\alpha_n)(1-\delta)\|y_n - x^*\|^2 \\
&\quad + \delta(1-\alpha_n)\|y_n - x^*\|^2,
\end{aligned} \tag{3.28}$$

which implies

$$\begin{aligned}
\delta\sum_{i\geq 1}\sigma_{i,n}\langle (T_iP_K(I-s_nB))y_n-y_n,x^*-y_n\rangle &= \alpha_n\langle u-x^*,y_n-x^*\rangle \\
&\quad+(1-\alpha_n)(1-\delta)\langle x_n-y_n,y_n-x^*\rangle \\
&\quad+(1-\alpha_n)\langle y_n-x^*,y_n-x^*\rangle \\
&\quad-\langle x_{n+1}-x^*,y_n-x^*\rangle \\
&= \alpha_n\langle u-x_{n+1},y_n-x^*\rangle \\
&\quad+(1-\alpha_n)(1-\delta)\langle x_n-y_n,y_n-x^*\rangle \\
&\quad+(1-\alpha_n)\langle y_n-x_{n+1},y_n-x^*\rangle.
\end{aligned} \tag{3.29}$$

Using this and (3.27), we get

$$\begin{aligned}
\frac{\delta}{2}\sum_{i\geq 1}\sigma_{i,n}\|(T_iP_K(I-s_nB))y_n-y_n\|^2 &\leq \alpha_n\langle u-x_{n+1},y_n-x^*\rangle \\
&\quad+(1-\alpha_n)(1-\delta)\langle x_n-y_n,y_n-x^*\rangle \\
&\quad+(1-\alpha_n)\langle y_n-x_{n+1},y_n-x^*\rangle.
\end{aligned} \tag{3.30}$$

Since $\{x_n\}$, $\{y_n\}$ are each bounded and using (3.21), we have that

$$\lim_{n\to\infty}\|(T_iP_K(I-s_nB))y_n-y_n\|=0\quad \forall i\in\mathbb{N}. \tag{3.31}$$

Using this and (3.21), we also have

$$\lim_{n\to\infty}\|(T_iP_K(I-s_nB))y_n-x_n\|=0\quad \forall i\in\mathbb{N}. \tag{3.32}$$

Next we show that $\limsup_{n\to\infty}\langle u-w,x_n-w\rangle \leq 0$, where $w=P_{\bigcap_{i=1}^{\infty}\mathrm{Fix}(T_i)\cap \mathrm{EP}(G)\cap \mathrm{VI}(K,B)}u$. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\limsup_{n\to\infty}\langle u-w,x_n-w\rangle=\lim_{j\to\infty}\langle u-w,x_{n_j}-w\rangle$.

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $y_{n_j}\rightharpoonup z$. Then, $z\in\bigcap_{i=1}^{\infty}\mathrm{Fix}(T_iP_K(I-s_nB))$,otherwise, for $i\in\mathbb{N}$, we have

$$\begin{aligned}
\liminf_{j\to\infty}\|y_{n_j}-z\| &< \liminf_{j\to\infty}\left\|y_{n_j}-T_iP_K\left(I-s_{n_j}B\right)z\right\| \\
&\leq \liminf_{j\to\infty}\left[\left\|y_{n_j}-T_iP_K\left(I-s_{n_j}B\right)y_{n_j}\right\|+\left\|T_iP_K\left(I-s_{n_j}B\right)y_{n_j}-T_iP_K\left(I-s_{n_j}B\right)z\right\|\right] \\
&\leq \liminf_{j\to\infty}\|y_{n_j}-z\|,
\end{aligned} \tag{3.33}$$

which is a contradiction, so $z\in\bigcap_{i=1}^{\infty}\mathrm{Fix}(T_iP_K(I-s_nB))$.

Next we show that $z \in \mathrm{EP}(G)$. Since $y_n = T_{r_n} x_n$, we have

$$G(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n \rangle \geq 0, \quad \forall \eta \in K. \tag{3.34}$$

It follows from $(A2)$ that

$$\left\langle \eta - y_n, \frac{y_n - x_n}{r_n} \right\rangle \geq G(\eta, y_n), \tag{3.35}$$

and so

$$\left\langle \eta - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq G(\eta, y_{n_i}). \tag{3.36}$$

Since $(y_{n_i} - x_{n_i})/r_{n_i} \to 0$, $y_{n_i} \rightharpoonup z$ and using $(A4)$, we have $G(\eta, z) \leq 0$ for all $\eta \in K$. For a real number t, $0 < t \leq 1$ and $\eta \in K$, let $\eta_t = t\eta + (1-t)z$. Clearly $\eta_t \in K$, so that using $(A1)$ and $(A4)$, we have

$$0 = G(\eta_t, \eta_t) \leq tG(\eta_t, \eta) + (1-t)G(\eta_t, z) \leq tG(\eta_t, \eta). \tag{3.37}$$

This implies $G(\eta_t, \eta) \geq 0$, and using this and $(A3)$ we have that $G(z, \eta) \geq 0$ for all $\eta \in K$ and hence $z \in \mathrm{EP}(G)$.

Next we show that $z \in \mathrm{VI}(K, B)$.

Let $x^* \in F$ then, we have the following:

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|P_K(I - s_n B)y_n - P_K(I - s_n B)x^*\|^2 \\
&\leq \langle y_n - s_n B y_n - (x^* - s_n B x^*), w_n - x^* \rangle \\
&= \langle y_n - x^*, w_n - x^* \rangle - s_n \langle B y_n - B x^*, w_n - x^* \rangle \\
&\leq \frac{1}{2}\left[\|w_n - x^*\|^2 + \|y_n - x^*\|^2 - \|w_n - y_n\|^2\right] - s_n \langle B y_n - B x^*, w_n - x^* \rangle,
\end{aligned} \tag{3.38}$$

so that

$$\|w_n - x^*\|^2 \leq \|y_n - x^*\|^2 - \|w_n - y_n\|^2 - 2s_n \langle B y_n - B x^*, w_n - x^* \rangle. \tag{3.39}$$

We then have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n(u-x^*)+(1-\alpha_n)(1-\delta)(x_n-x^*)+\delta\sum_{i\geq 1}\sigma_{i,n}[T_iw_n-x^*]\|^2 \\
&\leq \alpha_n\|u-x^*\|^2+(1-\alpha_n)(1-\delta)\|x_n-x^*\|^2+(1-\alpha_n)\delta\|w_n-x^*\|^2 \\
&\leq \alpha_n\|u-x^*\|^2+(1-\alpha_n)(1-\delta)\|x_n-x^*\|^2 \\
&\quad +(1-\alpha_n)\delta\Big[\|y_n-x^*\|^2-\|w_n-y_n\|^2-2s_n\langle By_n-Bx^*,w_n-x^*\rangle\Big] \\
&\leq \alpha_n\|u-x^*\|^2+(1-\alpha_n)(1-\delta)\|x_n-x^*\|^2 \\
&\quad +(1-\alpha_n)\delta\Big[\|x_n-x^*\|^2-\|w_n-y_n\|^2-2s_n\langle By_n-Bx^*,w_n-x^*\rangle\Big] \\
&= \alpha_n\|u-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-\delta(1-\alpha_n)\|w_n-y_n\|^2 \\
&\quad -2s_n\delta(1-\alpha_n)\langle By_n-Bx^*,w_n-x^*\rangle,
\end{aligned} \tag{3.40}$$

so that

$$\begin{aligned}
\delta(1-\alpha_n)\|w_n-y_n\|^2 &\leq \alpha_n\|u-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\quad -2\delta s_n(1-\alpha_n)\langle By_n-Bx^*,w_n-x^*\rangle,
\end{aligned} \tag{3.41}$$

and as $\alpha_n \to 0$, $\|x_n-x_{n+1}\| \to 0$ and $\|By_n-Bx^*\| \to 0$ as $n\to\infty$, we get

$$\lim_{n\to\infty}\|y_n-w_n\|=0. \tag{3.42}$$

Using this and (3.21) we also have

$$\lim_{n\to\infty}\|x_n-w_n\|=0. \tag{3.43}$$

As B is a relaxed (λ,γ)-cocoercive and using condition $(C3)$ we have for $x,y\in H$

$$\langle Bx-By,x-y\rangle \geq (-\lambda)\|Bx-By\|^2+\gamma\|x-y\|^2 \geq \left(\gamma-\lambda\mu^2\right)\|x-y\|^2 \geq 0, \tag{3.44}$$

and so B is monotone. If

$$Mq = \begin{cases} Bq + N_K q, & q \in K, \\ \emptyset, & q \notin K, \end{cases} \tag{3.45}$$

then M is maximal monotone. Let $\Gamma(M)$ denote the graph of M.

Let $(q,p) \in \Gamma(M)$. Since $p - Bq \in N_K q$ and $w_n \in K$, we have $\langle p - Bq, q - w_n\rangle \geq 0$ by definition of $N_K q$. Also, as $w_n = P_K(y_n - s_n B y_n)$ (using property of the projection P_K), we have

$$\langle w_n - (y_n - s_n B y_n), q - w_n\rangle \geq 0, \tag{3.46}$$

and hence

$$\left\langle \frac{w_n - y_n}{s_n} + By_n, q - w_n \right\rangle \geq 0. \tag{3.47}$$

Using this, we obtain the following estimates:

$$\begin{aligned} \langle p, q - w_{n_i}\rangle &\geq \langle Bq, q - w_{n_i}\rangle \\ &\geq \langle Bq, q - w_{n_i}\rangle - \left\langle \frac{w_{n_i} - y_{n_i}}{s_{n_i}} + By_{n_i}, q - w_{n_i}\right\rangle \\ &= \left\langle Bq - \frac{w_{n_i} - y_{n_i}}{s_n} - By_{n_i}, q - w_{n_i}\right\rangle \\ &= \langle Bq - Bw_{n_i}, q - w_{n_i}\rangle + \langle Bw_{n_i} - By_{n_i}, q - w_{n_i}\rangle - \left\langle \frac{w_{n_i} - y_{n_i}}{s_n}, q - w_{n_i}\right\rangle \\ &\geq \langle Bw_{n_i} - By_{n_i}, q - w_{n_i}\rangle - \left\langle \frac{w_{n_i} - y_{n_i}}{s_n}, q - w_{n_i}\right\rangle, \end{aligned} \tag{3.48}$$

which implies $\langle p, q - z\rangle \geq 0$ (letting $i \to \infty$).

Since M is maximal monotone, we obtained that $z \in M^{-1}(0)$ and hence $z \in \mathrm{VI}(K,B)$.

We mention here that since we proved that $z \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i P_K(I - s_n B))$ and $z \in \mathrm{VI}(K,B)$, then $z \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$. So, clearly $z \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{EP}(G) \cap \mathrm{VI}(K,B)$.

Since $w = P_{\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{EP}(G) \cap \mathrm{VI}(K,B)} u$, we have

$$\limsup_{n\to\infty} \langle u - w, x_n - w\rangle = \lim_{i\to\infty} \langle u - w, x_{n_i} - w\rangle = \langle u - w, z - w\rangle \leq 0. \tag{3.49}$$

Hence, $\limsup_{n\to\infty} \langle u - w, x_n - w\rangle \leq 0$.

From the recursion formula (3.1) and Lemma 2.3, we have

$$\begin{aligned}
\|x_{n+1}-w\|^2 &= \Big\|\alpha_n(u-w)+(1-\alpha_n)(1-\delta)(x_n-w) \\
&\qquad +\delta\sum_{i\geq 1}\sigma_{i,n}[T_iP_K(I-s_nB)y_n-w]\Big\|^2 \\
&\leq \Big\|(1-\alpha_n)(1-\delta)(x_n-w)+\delta\sum_{i\geq 1}\sigma_{i,n}[T_iP_K(I-s_nB)y_n-w]\Big\|^2 \\
&\quad + 2\alpha_n\langle u-w, x_{n+1}-w\rangle \\
&\leq \Big[(1-\alpha_n)(1-\delta)\|x_n-w\|+\delta\sum_{i\geq 1}\sigma_{i,n}\|T_iP_K(I-s_nB)y_n-w\|\Big]^2 \\
&\quad + 2\alpha_n\langle u-w, x_{n+1}-w\rangle \\
&\leq (1-\alpha_n)\|x_n-w\|^2+2\alpha_n\langle u-w, x_{n+1}-w\rangle.
\end{aligned} \tag{3.50}$$

Using Lemma 2.5, we have that $\{x_n\}$ and consequently $\{y_n\}$ converge to w and the proof is complete. □

The following corollaries follow from Theorem 3.1.

Corollary 3.2. *Let K, G and B be as in Theorem 3.1 and let $\{T_i,\ i=1,2,3,\ldots,N\}$ be finite family of nonexpansive mappings of K into H. Let $F := \bigcap_{i=1}^{N}\operatorname{Fix}(T_i)\cap \operatorname{EP}(G)\cap \operatorname{VI}(K,B)\neq\emptyset$. For an arbitrary but fixed $\delta\in(0,1)$, let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{gathered}
x_1, u\in H, \\
G(y_n,\eta)+\frac{1}{r_n}\langle \eta-y_n, y_n-x_n\rangle \geq 0, \quad \forall \eta\in K, \\
x_{n+1}=\alpha_n u+(1-\delta)(1-\alpha_n)x_n+\delta\sum_{i=1}^{N}\sigma_{in}T_iP_K(I-s_nB)y_n, \quad n\geq 1,
\end{gathered} \tag{3.51}$$

where $\{\alpha_n\}$ and $\{\sigma_{in}\}$ are sequences in $[0,1]$ and $\{r_n\}$ and $\{s_n\}$ are sequences in $[0,\infty)$ satisfying

(C1) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{i=1}^{N}\sigma_{in}=(1-\alpha_n)$,

(C2) $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(C3) $\{s_n\}\subset[a,b]$ *for some* a,b *satisfying* $0\leq a\leq b\leq 2(\gamma-\lambda\mu^2)/\mu^2$,

(C4) $\lim_{n\to\infty}|s_{n+1}-s_n|=0$, $\lim_{n\to\infty}|r_{n+1}-r_n|=0$, $\lim_{n\to\infty}\sum_{i=1}^{N}|\sigma_{i,n+1}-\sigma_{i,n}|=0$,

(C5) $\liminf_{n\to\infty}r_n>0$.

Then, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_F u$.

Corollary 3.3. *Let K and G be as in Theorem 3.1. Let T be a nonexpansive map of K into H. Let $F := \mathrm{Fix}(T) \cap \mathrm{EP}(G) \neq \emptyset$. For an arbitrary but fixed $\delta \in (0,1)$, let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{gathered} x_1, u \in H, \\ G(y_n, \eta) + \frac{1}{r_n}\langle \eta - y_n, y_n - x_n \rangle \geq 0, \quad \forall \eta \in K, \\ x_{n+1} = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta(1-\alpha_n)Ty_n, \quad n \geq 1, \end{gathered} \tag{3.52}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{r_n\}$ is a sequences in $[0,\infty)$ satisfying

(C1) $\lim_{n\to\infty}\alpha_n = 0$

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

(C4) $\liminf_{n\to\infty} r_n > 0$.

Then, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_F u$.

Remark 3.4. Prototypes of the sequences $\{\alpha_n\}$ and $\{\sigma_{i,n}\}$ in our theorem are the following:

$$\alpha_n := \frac{1}{n+1}, \qquad \sigma_{i,n} := \frac{n}{2^i(n+1)} \quad \forall i \in \mathbb{N}. \tag{3.53}$$

References

[1] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, vol. 88 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1980.

[2] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.

[3] M. A. Noor, "General variational inequalities and nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 810–822, 2007.

[4] P. Maingé, "Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 469–479, 2007.

[5] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[6] P. Kumam and P. Katchang, "A viscosity of extragradient approximation method for finding equilibrium problems, variational inequalities and fixed point problems for nonexpansive mappings," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 475–486, 2009.

[7] W. Kumam and P. Kumam, "Hybrid iterative scheme by a relaxed extragradient method for solutions of equilibrium problems and a general system of variational inequalities with application to optimization," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 640–656, 2009.

[8] H. Li and Y. Su, "Strong convergence theorem by a new hybrid method for equilibrium problems and variational inequality problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 2, pp. 847–855, 2009.

[9] P. Kumam and C. Jaiboon, "A new hybrid iterative method for mixed equilibrium problems and variational inequality problem for relaxed cocoercive mappings with application to optimization problems," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 510–530, 2009.

[10] X. Qin, Y. J. Cho, and S. M. Kang, "Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 99–112, 2010.

[11] R. Wangkeeree and U. Kamraksa, "An iterative approximation method for solving a general system of variational inequality problems and mixed equilibrium problems," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 615–630, 2009.

[12] R. Wangkeeree and R. Wangkeeree, "Strong convergence of the iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems of an infinite family of nonexpansive mappings," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 719–733, 2009.

[13] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.

[14] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[15] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.

[16] W. V. Petryshyn, "A characterization of strict convexity of Banach spaces and other uses of duality mappings," *Journal of Functional Analysis*, vol. 6, pp. 282–291, 1970.

[17] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[18] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.