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Research Article

Monotonic Limit Properties for Solutions of BSDEs with Continuous Coefficients

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This paper investigates the monotonic limit properties for the minimal and maximal solutions of certain one-dimensional backward stochastic differential equations with continuous coefficients.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$. Fix a terminal time $T > 0$, let $(\mathcal{F}_t)_{t\geq 0}$ be the natural σ-algebra generated by $(B_t)_{t\geq 0}$, and assume $\mathcal{F}_T = \mathcal{F}$. For every positive integer n, we use $|\cdot|$ to denote norm of Euclidean space $\mathbf{R}^n$. For $t \in [0,T]$, let $L^2(\Omega, \mathcal{F}_t, P)$ denote the set of all $\mathcal{F}_t$-measurable random variables ξ such that $E|\xi|^2 < +\infty$. Let $L^2_{\mathcal{F}}(0,T;\mathbf{R}^n)$ denote the set of $\mathcal{F}_t$-progressively measurable $\mathbf{R}^n$-valued processes $\{X_t, t \in [0,T]\}$ such that

$$\|X\|_2 \widehat{=} \left(E\int_0^T |X_t|^2 \mathrm{d}t \right)^{1/2} < +\infty. \tag{1.1}$$

This paper is concerned with the following one-dimensional BSDE:

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad t \in [0,T], \tag{1.2}$$

where the random function $g(\omega, t, y, z) : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ is progressively measurable for each (y,z) in $\mathbf{R} \times \mathbf{R}^d$, termed the generator of the BSDE(1), and ξ is an $\mathcal{F}_T$-measurable random variables termed the terminal condition. The triple (ξ, T, g) is called the parameters of the BSDE(1). In this paper, for each (ξ, T, g), by solution to the BSDE(1) we mean a pair of processes $(y_\cdot, z_\cdot)$ in $L^2_{\mathcal{F}}(0, T; \mathbf{R}^{1+d})$ which satisfies the BSDE(1) and y is a continuous process. Such equations, in nonlinear case, have been introduced by Pardoux and Peng [1]; they established an existence and uniqueness result of solutions of the BSDE(1) under the Lipschitz assumption of the generator g. Since then, these equations and their generalizations have been the subject of a great number of investigations, such as [2, 3]. Particularly, Lepeltier and San Martin [4] obtained the following result when the generator g is only continuous with a linear growth.

Proposition 1.1 (see [4, Theorem 1]). *Assume that the generator g satisfies*

(H1) *linear growth: there exists $K < +\infty$, for all $\omega, t, y, z, |g(\omega, t, y, z)| \leq K(1 + |y| + |z|)$;*

(H2) *for fixed ω, t, $g(\omega, t, \cdot, \cdot)$ is continuous.*

Then, if $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, the BSDE(1) has a unique minimal solution $(\underline{Y}_\cdot, \underline{Z}_\cdot)$ and a unique maximal solution $(\overline{Y}_\cdot, \overline{Z}_\cdot)$, which means that both $(\underline{Y}_\cdot, \underline{Z}_\cdot)$ and $(\overline{Y}_\cdot, \overline{Z}_\cdot)$ are the solution of (1.2)*, and for any other solution $(Y_\cdot, Z_\cdot)$ of* (1.2) *one has $\underline{Y}_\cdot \leq Y_\cdot \leq \overline{Y}_\cdot$. For convenience, for each $t \in [0, T]$, one denoted $\underline{Y}_t$ by $\underline{\mathcal{E}^g_{t,T}}[\xi]$, and $\overline{Y}_t$ by $\overline{\mathcal{E}^g_{t,T}}[\xi]$.*

This paper will work on the assumptions (H1) and (H2) and investigate the monotonic limit properties on the operators $\underline{\mathcal{E}^g_{t,T}}[\cdot]$ and $\overline{\mathcal{E}^g_{t,T}}[\cdot]$.

2. Main Results

In this section, we always assume that the generator g satisfies assumptions (H1) and (H2). The following Theorem 2.1 and Remark 2.2 are the main results of this paper.

Theorem 2.1. *Assume that the generator g satisfies assumptions (H1) and (H2). Let $t \in [0, T]$, $\xi^n \in L^2(\Omega, \mathcal{F}_t, P)$, $n \in \mathbf{N}$, and $E|\xi|^2 < +\infty$.*

If $\xi^n \uparrow \xi$, $P - a.s.$, then for all $s \in [0, t]$,

$$\lim_{n\to\infty} \uparrow \underline{\mathcal{E}^g_{s,t}}[\xi^n] = \underline{\mathcal{E}^g_{s,t}}\left[\lim_{n\to\infty} \xi^n\right] = \underline{\mathcal{E}^g_{s,t}}[\xi], \quad P - a.s. \tag{2.1}$$

If $\xi^n \downarrow \xi$, $P - a.s.$, then for all $s \in [0, t]$,

$$\lim_{n\to\infty} \downarrow \overline{\mathcal{E}^g_{s,t}}[\xi^n] = \overline{\mathcal{E}^g_{s,t}}\left[\lim_{n\to\infty} \xi^n\right] = \overline{\mathcal{E}^g_{s,t}}[\xi], \quad P - a.s. \tag{2.2}$$

Remark 2.2. If the condition "$\xi^n \uparrow \xi$" in Theorem 2.1 is replaced by "$\xi^n \downarrow \xi$", the conclusion of the first part of Theorem 2.1 does not hold in general. Similarly, the condition "$\xi^n \downarrow \xi$" in Theorem 2.1 cannot be replaced by "$\xi^n \uparrow \xi$" in general. For example, we consider the BSDE with

$$g(u, y, z) = 7y^{6/7}, \qquad \xi = 0. \tag{2.3}$$

It is easy to see that both

$$(y_r, z_r)_{r\in[0,t]} = (0,0)_{r\in[0,t]}, \qquad (Y_r, Z_r)_{r\in[0,t]} = \left((t-r)^7, 0\right)_{r\in[0,t]} \tag{2.4}$$

are solutions of BSDE

$$y_r = \int_r^t 7y_u^{6/7}\mathrm{d}u - \int_r^t z_u \mathrm{d}B_u, \quad r \in [0,t]. \tag{2.5}$$

For each $n \in \mathbf{N}$, set $\xi^n = 1/n$, then $\xi^n \in L^2(\Omega, \mathcal{F}_t, P)$, $E|\xi|^2 < +\infty$ and $\xi^n \downarrow \xi$, $P - a.s.$ However, one can verify that for each $s \in [0,t]$,

$$\underline{\mathcal{E}}^g_{s,t}[\xi^n] = \left(t - s + \frac{1}{\sqrt[7]{n}}\right)^7. \tag{2.6}$$

Consequently,

$$\lim_{n\to\infty} \underline{\mathcal{E}}^g_{s,t}[\xi^n] = Y_s \neq \underline{\mathcal{E}}^g_{s,t}[\xi] \leq 0. \tag{2.7}$$

So, the conclusion of the first part of Theorem 2.1 does not hold.

In order to prove Theorem 2.1, we need the following lemmas. Lemma 2.3 is actually a direct corollary of Theorem 1.1 in [5].

Lemma 2.3. *Assume that the generator g satisfies assumptions (H1) and (H2). Let $t \in [0,T]$ and $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, P)$. If $\xi \leq \xi'$, $P - a.s.$, then*

$$\begin{aligned} &\forall s \in [0,t], \quad \underline{\mathcal{E}}^g_{s,t}[\xi] \leq \underline{\mathcal{E}}^g_{s,t}[\xi'], \quad P - a.s., \\ &\forall s \in [0,t], \quad \overline{\mathcal{E}}^g_{s,t}[\xi] \leq \overline{\mathcal{E}}^g_{s,t}[\xi'], \quad P - a.s. \end{aligned} \tag{2.8}$$

From the procedure of the proof of Theorem 2.1 in [4], we can obtain the following Lemma 2.4.

Lemma 2.4. *If the function g satisfies (H1) and (H2), and one sets*

$$\begin{aligned} \underline{g}_m(t,y,z) &:= \inf_{(u,v)\in\mathbf{Q}^{1+d}} \left\{ g(t,u,v) + m\left(|y-u| + |z-v|\right)\right\}, \\ \overline{g}_m(t,y,z) &:= \sup_{(u,v)\in\mathbf{Q}^{1+d}} \left\{ g(t,u,v) - m\left(|y-u| + |z-v|\right)\right\}, \end{aligned} \tag{2.9}$$

then for any $m > K$, $\underline{g}_m$ and $\overline{g}_m$ are Lipschitz functions with constant m, that is, for any $y_1, y_2 \in \mathbf{R}$, $z_1, z_2 \in \mathbf{R}^d$ and $t \in [0,T]$,

$$\begin{aligned} \left|\underline{g}_m(t,y_1,z_1) - \underline{g}_m(t,y_2,z_2)\right| &\le m(|y_1 - y_2| + |z_1 - z_2|), \\ \left|\overline{g}_m(t,y_1,z_1) - \overline{g}_m(t,y_2,z_2)\right| &\le m(|y_1 - y_2| + |z_1 - z_2|). \end{aligned} \tag{2.10}$$

Moreover, let $t \in [0,T]$ and $\xi \in L^2(\Omega, \mathcal{F}_t, P)$, and let $(\underline{Y}^m_r, \underline{Z}^m_r)_{r\in[0,t]}$ and $(\overline{Y}^m_r, \overline{Z}^m_r)_{r\in[0,t]}$ be the unique solutions of the BSDEs with parameters $(\xi, t, \underline{g}_m)$ and $(\xi, t, \overline{g}_m)$, respectively. For convenience, from now on, we denoted $\underline{Y}^m_s$ by $\mathcal{E}^{\underline{g}_m}_{s,t}[\xi]$, and $\overline{Y}^m_s$ by $\mathcal{E}^{\overline{g}_m}_{s,t}[\xi]$ for each $s \in [0,t]$, then for each $s \in [0,t]$, we have

$$\begin{aligned} \lim_{m\to\infty} \uparrow \mathcal{E}^{\underline{g}_m}_{s,t}[\xi] &= \underline{\mathcal{E}^{g}_{s,t}}[\xi], \quad P-a.s., \\ \lim_{m\to\infty} \downarrow \mathcal{E}^{\overline{g}_m}_{s,t}[\xi] &= \overline{\mathcal{E}^{g}_{s,t}}[\xi], \quad P-a.s. \end{aligned} \tag{2.11}$$

Finally, the following Lemma 2.5 can be easily obtained by [6, Lemma 1].

Lemma 2.5. *Let $t \in [0,T]$, $\xi^n \in L^2(\Omega, \mathcal{F}_t, P)$, $n \in \mathbf{N}$, and $E|\xi|^2 < +\infty$, and let the generators g, $\underline{g}_m$ and $\overline{g}_m$ be defined as that in Lemma 2.4.*

If $\xi^n \uparrow \xi$, $P-a.s.$, then for all $s \in [0,t]$ and each $m > K$,

$$\lim_{n\to\infty} \uparrow \mathcal{E}^{\underline{g}_m}_{s,t}[\xi^n] = \mathcal{E}^{\underline{g}_m}_{s,t}\left[\lim_{n\to\infty} \xi^n\right] = \mathcal{E}^{\underline{g}_m}_{s,t}[\xi], \quad P-a.s. \tag{2.12}$$

If $\xi^n \downarrow \xi$, $P-a.s.$, then for all $s \in [0,t]$ and each $m > K$,

$$\lim_{n\to\infty} \downarrow \mathcal{E}^{\overline{g}_m}_{s,t}[\xi^n] = \mathcal{E}^{\overline{g}_m}_{s,t}\left[\lim_{n\to\infty} \xi^n\right] = \mathcal{E}^{\overline{g}_m}_{s,t}[\xi], \quad P-a.s. \tag{2.13}$$

Now, we are in the position to prove Theorem 2.1.

Proof of Theorem 2.1. We only prove the first part of this theorem, in the same way, one can complete the proof of the second part.

Since $\xi^n \in L^2(\Omega, \mathcal{F}_t, P)$ and $\lim_{n\to\infty} \uparrow \xi^n = \xi$, $P-a.s.$, one knows that $\xi \in \mathcal{F}_t$. Thus, by $E|\xi|^2 < +\infty$, we have $\xi \in L^2(\Omega, \mathcal{F}_t, P)$, then by Proposition 1.1, for each $s \in [0,t]$, both $\underline{\mathcal{E}^{g}_{s,t}}[\xi^n]$ and $\underline{\mathcal{E}^{g}_{s,t}}[\xi]$ are well defined. Moreover, according to Lemma 2.3, $\underline{\mathcal{E}^{g}_{s,t}}[\xi^n]$ is nondecreasing with respect to n and bounded by $\underline{\mathcal{E}^{g}_{s,t}}[\xi]$ from above, so in the sense of "almost surely," the limit of the sequence $\underline{\mathcal{E}^{g}_{s,t}}[\xi^n]$ must exist. Thus, in order to complete the proof of Theorem 2.1, we need only to prove that this limit is just $\underline{\mathcal{E}^{g}_{s,t}}[\xi]$.

Let functions $\underline{g}_m$ and $\overline{g}_m$ be defined for each $m > K$ as that in Lemma 2.4, then from Lemmas 2.3 and 2.4 one deduce that for each $n \in \mathbf{N}$, $m > K$ and $s \in [0,t]$,

$$\begin{aligned} 0 &\geq \underline{\mathcal{E}}^{g}_{s,t}[\xi^n] - \underline{\mathcal{E}}^{g}_{s,t}[\xi] = \underline{\mathcal{E}}^{g}_{s,t}[\xi^n] - \mathcal{E}^{\underline{g}_m}_{s,t}[\xi^n] + \mathcal{E}^{\underline{g}_m}_{s,t}[\xi^n] - \mathcal{E}^{\underline{g}_m}_{s,t}[\xi] + \mathcal{E}^{\underline{g}_m}_{s,t}[\xi] - \underline{\mathcal{E}}^{g}_{s,t}[\xi] \\ &\geq \mathcal{E}^{\underline{g}_m}_{s,t}[\xi^n] - \mathcal{E}^{\underline{g}_m}_{s,t}[\xi] + \mathcal{E}^{\underline{g}_m}_{s,t}[\xi] - \underline{\mathcal{E}}^{g}_{s,t}[\xi], \quad P - a.s. \end{aligned} \tag{2.14}$$

Letting $n \to \infty$ in (2.14), from Lemma 2.5 we get that for each $m > K$,

$$0 \geq \lim_{n \to \infty} \underline{\mathcal{E}}^{g}_{s,t}[\xi^n] - \underline{\mathcal{E}}^{g}_{s,t}[\xi] \geq \mathcal{E}^{\underline{g}_m}_{s,t}[\xi] - \underline{\mathcal{E}}^{g}_{s,t}[\xi], \quad P - a.s. \tag{2.15}$$

Furthermore, letting $m \to \infty$ in (2.15), from Lemma 2.4 we can easily deduce that for each $s \in [0,t]$,

$$\lim_{n \to \infty} \underline{\mathcal{E}}^{g}_{s,t}[\xi^n] = \underline{\mathcal{E}}^{g}_{s,t}[\xi], \quad P - a.s. \tag{2.16}$$

The proof of Theorem 2.1 is completed. □

According to Theorem 2.1, we can obtain the following theorem.

Theorem 2.6. *Assume that the generator g satisfies assumptions (H1) and (H2). Let $t \in [0,T]$ and $\xi^n \in L^2(\Omega, \mathcal{F}_t, P)$, $n \in \mathbf{N}$, Let $E|\eta|^2 < +\infty$ and $E|\zeta|^2 < +\infty$.*

If $\xi^n \geq \zeta$, $P - a.s.$ $(n \in \mathbf{N})$ with $E\left|\underline{\lim}_{n \to \infty} \xi^n\right|^2 < +\infty$, then for all $s \in [0,t]$,

$$\underline{\mathcal{E}}^{g}_{s,t}\left[\underline{\lim}_{n \to \infty} \xi^n\right] \leq \underline{\lim}_{n \to \infty} \underline{\mathcal{E}}^{g}_{s,t}[\xi^n], \quad P - a.s. \tag{2.17}$$

If $\xi^n \leq \eta$, $P - a.s.$ $(n \in \mathbf{N})$ with $E\left|\overline{\lim}_{n \to \infty} \xi^n\right|^2 < +\infty$, then for all $s \in [0,t]$,

$$\overline{\mathcal{E}}^{g}_{s,t}\left[\overline{\lim}_{n \to \infty} \xi^n\right] \geq \overline{\lim}_{n \to \infty} \overline{\mathcal{E}}^{g}_{s,t}[\xi^n], \quad P - a.s. \tag{2.18}$$

Proof. We only prove the first part of this theorem, the proof of the second part is similar.

Let us fix $s \in [0,t]$. Since $\xi^n \in L^2(\Omega, \mathcal{F}_t, P)$, then $\underline{\lim}_{n \to \infty} \xi^n \in \mathcal{F}_t$, and by the assumption of this theorem one knows that

$$\underline{\lim}_{n \to \infty} \xi^n \in L^2(\Omega, \mathcal{F}_t, P), \tag{2.19}$$

thus by Proposition 1.1, $\underline{\mathcal{E}}^{g}_{s,t}[\underline{\lim}_{n \to \infty} \xi^n]$ is well defined.

We set $Y_n = \inf_{k\geq n}\xi^k$, then $Y_n \in \mathcal{F}_t$, and $\zeta \leq Y_n \uparrow \underline{\lim}_{n\to\infty}\xi^n$, $P - a.s.$ So for each $n \in \mathbf{N}$,

$$E|Y_n|^2 \leq \max\left\{E|\zeta|^2, E\left|\underline{\lim}_{n\to\infty}\xi^n\right|^2\right\} < +\infty. \tag{2.20}$$

Thus,

$$Y_n \in L^2(\Omega, \mathcal{F}_t, P), \tag{2.21}$$

then $\underline{\mathcal{E}}^g_{s,t}[Y_n]$ is also well defined. Since $Y_n \leq \xi^n$, $P - a.s.$, by Lemma 2.3 we know that $\underline{\mathcal{E}}^g_{s,t}[Y_n] \leq \underline{\mathcal{E}}^g_{s,t}[\xi^n]$, $P - a.s.$, then applying Theorem 2.1 to the random variable sequence $\{Y_n\}$, we get

$$\underline{\mathcal{E}}^g_{s,t}\left[\underline{\lim}_{n\to\infty}\xi^n\right] = \underline{\mathcal{E}}^g_{s,t}\left[\lim_{n\to\infty}Y_n\right] = \lim_{n\to\infty}\underline{\mathcal{E}}^g_{s,t}[Y_n] \leq \underline{\lim}_{n\to\infty}\underline{\mathcal{E}}^g_{s,t}[\xi^n], \quad P - a.s. \tag{2.22}$$

The proof is completed. □

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