Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 736746, 15 pages
doi:10.1155/2009/736746

Research Article

General Approach to Regions of Variability via Subordination of Harmonic Mappings

Sh. Chen,[1] S. Ponnusamy,[2] and X. Wang[1]

[1] *Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China*
[2] *Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India*

Correspondence should be addressed to X. Wang, xtwang@hunnu.edu.cn

Received 17 October 2009; Accepted 20 November 2009

Recommended by Narendra Kumar Govil

Using subordination, we determine the regions of variability of several subclasses of harmonic mappings. We also graphically illustrate the regions of variability for several sets of parameters for certain special cases.

Introduction

A planar harmonic mapping in a simply connected domain $D \subset \mathbb{C}$ is a complex-valued function $f = u + iv$ defined in D for which both u and v are real harmonic in D, that is, $\Delta f = 4 f_{z\overline{z}} = 0$, where Δ represents the Laplacian operator. The mapping f can be written as a sum of an analytic and antianalytic functions, that is, $f = h + \overline{g}$. We refer to [1] and the book of Duren [2] for many interesting results on planar harmonic mappings.

We note that the composition $f \circ \phi$ of a harmonic function f with an analytic function ϕ is harmonic, but this is not true for the function $\phi \circ f$, that is, an analytic function of a harmonic function need not be harmonic. It is known that [2, Theorem 2.4] the only univalent harmonic mappings of $\mathbb{C}$ onto $\mathbb{C}$ are the affine mappings $g(z) = \beta z + \gamma \overline{z} + \eta (|\beta| \neq |\gamma|)$. Motivated by the work of [3], we say that F is an affine harmonic mapping of a harmonic mapping of f if and only if F has the form

$$F := F_{\alpha}(f) = f + \alpha \overline{f} \tag{1}$$

for some $\alpha \in \mathbb{C}$ with $|\alpha| < 1$. Obviously, an affine transformation applied to a harmonic mapping is again harmonic. The affine harmonic mappings $F_\alpha(f)$ and f share many properties in common (see [4]).

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{A}_0 = \{h \in \mathcal{H} : h(0) = 0\}$. Also, let $\mathcal{S}_0$ be the subclass of $\mathcal{A}_0$ consisting of functions that are univalent in $\mathbb{D}$. For a given $\phi \in \mathcal{S}_0$, we will denote by $\mathcal{A}_0(\phi)$ and $\mathcal{S}_0(\phi)$ the subsets defined by $\{h \in \mathcal{A}_0 : h \prec \phi\}$ and $\{h \in \mathcal{S}_0 : h \prec \phi\} \cup \{0\}$, respectively. From now onwards, we use the notation $f \prec g$, or, $f(z) \prec g(z)$ in $\mathbb{D}$ for analytic functions f and g on $\mathbb{D}$ to mean the subordination, namely there exists $\omega \in \mathcal{B}_0$ such that $f(z) = g(\omega(z))$. Here $\mathcal{B}_0$ denotes the class of analytic maps φ of the unit disk $\mathbb{D}$ into itself with the normalization $\varphi(0) = 0$. We remark that if g is univalent in $\mathbb{D}$, then the subordination $f \prec g$ is equivalent to the condition that $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. This fact will be used in our investigation. Moreover, the special choices of ϕ have been the subjects of extensive studies; we suggest that the reader to consult the books of Pommerenke [5], Duren [6] and of Miller and Mocanu [7] for general back ground material.

We denote by $\mathcal{A}_{a,b}$ the class of functions $f \in \mathcal{H}$ with $f(0) = (b-a)/2$, and $-a < \operatorname{Re} f(z) < b$ for $z \in \mathbb{D}$. We note that if $a > 0$, then each function $f \in \mathcal{A}_{a,a}$ obviously satisfy the normalization condition $f(0) = 0$. A function $f \in \mathcal{H}$ is called a *Bloch function* if

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) \left|f'(z)\right| < \infty. \tag{2}$$

Then the set of all Bloch functions forms a complex Banach space $\mathcal{B}$ with the norm $\|\cdot\|$ given by

$$\|f\| = \left|f(0)\right| + \|f\|_{\mathcal{B}}, \tag{3}$$

see [8]. Every bounded function in $\mathcal{H}$ is Bloch, but there are unbounded Bloch functions, as can be seen also from the following result which shows that $\mathcal{A}_{a,b} \subset \mathcal{B}$.

Proposition 1. *If $f \in \mathcal{A}_{a,b}$, then $\|f\|_{\mathcal{B}} \le 2(b+a)/\pi$. The constant $2(b+a)/\pi$ is sharp. In particular, if $f \in \mathcal{A}_{a,a}$ then $\|f\|_{\mathcal{B}} \le 4a/\pi$ and the constant $4a/\pi$ is sharp.*

Proof. Let

$$P(z) = \frac{b+a}{i\pi} \log\left(\frac{1+z}{1-z}\right) + \frac{b-a}{2}, \quad z \in \mathbb{D}. \tag{4}$$

Then $P(0) = (b-a)/2$,

$$P'(z) = \frac{2(b+a)}{i\pi(1-z^2)} \tag{5}$$

and P maps $\mathbb{D}$ univalently onto the vertical strip $\{w : -a < \operatorname{Re}\ w < b\}$, and $\|P\|_{\mathcal{B}} = 2(b+a)/\pi$. Consequently, if $f \in \mathcal{A}_{a,b}$, then we have $f \prec P$ and so, there exists a Schwarz function $\omega \in \mathcal{B}_0$

such that $f(z) = P(\omega(z))$. Thus, as $\omega(0) = 0$, the Schwarz-Pick lemma gives that

$$\left(1-|z|^2\right)\left|f'(z)\right| = \left(1-|z|^2\right)\left|\omega'(z)\right|\left|P'(\omega(z))\right| \le \left(1-|\omega|^2\right)\left|P'(\omega)\right| \le \|P\|_{\mathcal{B}} \tag{6}$$

so that $\|f\|_{\mathcal{B}} \le 2(b+a)/\pi$, with equality for $f(z) = P_\alpha(z)$, where

$$P_\alpha(z) = \frac{b+a}{i\pi}\log\left(\frac{1+ze^{i\alpha}}{1-ze^{i\alpha}}\right) + \frac{b-a}{2}, \quad \alpha \in \mathbb{R}. \tag{7}$$

□

It may be interesting to remark that the function $f(z) = \sum_{n=1}^{\infty} z^{2^n}$ belongs to $\mathcal{B}$ [9, Theorem 1] is a good example of a Bloch function which is not in H^p-space for any p. Bloch functions are intimately close with univalent functions (see [5]).

In order to state our main results, we introduce some basics. For given $a, b > 0$, let $\mathcal{S}_{a,b}$ be the class of functions $f \in \mathcal{A}_0$ and $-a < \operatorname{Re} f(z) < b$ for $z \in \mathbb{D}$. Now, we define

$$\mathcal{S}_{a,b,u} = \{f : f \in \mathcal{S}_{a,b} \text{ and } f \text{ is univalent}\} \cup \{0\}. \tag{8}$$

We note that each function in $\mathcal{S}_{a,b}$ has the normalization $f(0) = 0$. For any fixed $z_0 \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$, we consider the following sets:

$$\begin{aligned}
V_{\phi,\mathcal{H}}(z_0) &= \{F_\alpha(f)(z_0) : f \in \mathcal{S}_0(\phi)\}, \\
V_{\phi,\mathcal{H}}(z_0,\lambda) &= \{F_\alpha(f)(z_0) : f \in \mathcal{A}_0(\phi),\ f'(0) = \lambda\phi'(0)\}, \\
V_{\mathcal{H},\mathcal{S}_{a,b,u}}(z_0) &= \{F_\alpha(f)(z_0) : f \in \mathcal{S}_{a,b,u}\}, \\
V_{\mathcal{H},\mathcal{S}_{a,b}}(z_0,\lambda) &= \left\{F_\alpha(f)(z_0) : f'(0) = \lambda\frac{b+a}{i\pi}\left(1-e^{-2\pi a i/(b+a)}\right),\ f \in \mathcal{S}_{a,b}\right\}.
\end{aligned} \tag{9}$$

We now recall the definition of subordination for the harmonic case from [10, page 162]. Let f and F be two harmonic functions defined on $\mathbb{D}$. We say f is subordinate to F, denoted by $f \prec F$, if $f(z) = F(\omega(z))$, where $\omega \in \mathcal{B}_0$. Obviously, if f_1 and f_2 are two harmonic functions in $\mathbb{D}$, then

$$f_1 \prec f_2 \Longleftrightarrow F_\alpha(f_1) \prec F_\alpha(f_2). \tag{10}$$

Here we see that $\overline{\alpha}$ is the analytic dilatation for both $F_\alpha(f_1)$ and $F_\alpha(f_2)$.

For each fixed $z_0 \in \mathbb{D}$, using extreme function theory, it has been shown by Grunsky (see, e. g., Duren [6, Theorem 10.6]) that the region of variability of

$$V_{\mathcal{S}}(z_0) = \left\{\log\frac{f(z_0)}{z_0} : f \in \mathcal{S}\right\} \tag{11}$$

is precisely a closed disk, where $\mathcal{S} = \{f \in \mathcal{S}_0 : f'(0) = 1\}$. Recently, by using the Herglotz representation formula for analytic functions, many authors have discussed region of variability problems for a number of classical subclasses of univalent and analytic functions in the unit disk $\mathbb{D}$ (see [11, 12] and the references therein). Because the class of harmonic univalent mappings includes the class of conformal mappings, it is natural to study the class of harmonic mappings. In the following, we will use the method of subordination and determine the regions of variability for $V_{\phi,\mathcal{H}}(z_0)$, $V_{\phi,\mathcal{H}}(z_0,\lambda)$, $V_{\mathcal{H},\mathcal{S}_{a,b,u}}(z_0)$ and $V_{\mathcal{H},\mathcal{S}_{a,b}}(z_0,\lambda)$, respectively.

Theorem 1. *The boundary $\partial V_{\phi,\mathcal{H}}(z_0)$ of $V_{\phi,\mathcal{H}}(z_0)$ is the Jordan curve given by*

$$(-\pi,\pi] \ni \theta \longmapsto \phi\left(e^{i\theta}z_0\right) + \alpha\overline{\phi\left(e^{i\theta}z_0\right)}. \tag{12}$$

Proof. We define $V_\phi(z_0) = \{f(z_0):f \in \mathcal{S}_0(\phi)\}$. In order to determine the set $V_\phi(z_0)$, we first recall that each $f \in \mathcal{S}_0(\phi) \setminus \{0\}$ can be written as $f(z) = \phi(\omega(z))$ for some $\omega \in \mathcal{B}_0 \setminus \{0\}$. By the Riemann mapping theorem, $\omega = \phi^{-1} \circ f$ is univalent and analytic in $\mathbb{D}$ with $\omega(0) = 0$. It follows from the classical Schwarz lemma that for any $\omega \in \mathcal{B}_0$, we have $|\omega(z)| \le |z|$ in $\mathbb{D}$. Because, in our situation ω is also univalent in $\mathbb{D}$, we easily show that the region of variability

$$V^{\mathcal{B}}(z_0) = \{\omega(z_0) : \omega \in (\mathcal{B}_0 \cap \mathcal{S}_0) \cup \{0\}\} \tag{13}$$

coincides with the set $\{z:|z| \le |z_0|\}$. Hence the region of variability $V_\phi(z_0)$ is precisely the set $\{\phi(z):|z| \le |z_0|\}$. We remark that $V_\phi(z_0)$ depends only on $|z_0|$, because $\mathcal{S}_0$ is preserved under rotation and therefore, we may assume that $0 < z_0 < 1$. Finally, the region of variability $V_{\phi,\mathcal{H}}(z_0)$ follows from $V_\phi(z_0)$. The proof of this theorem is complete. □

There are many choices for ϕ for which Theorem 1 is applicable. For example, if we choose ϕ to be

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\beta - 1, \tag{14}$$

for some $0 < \beta \le 2$, then we have following result from Theorem 1.

Corollary 1. *The boundary $\partial V_{\phi_0,\mathcal{H}}(z_0)$ of $V_{\phi_0,\mathcal{H}}(z_0)$ is the Jordan curve given by*

$$(-\pi,\pi] \ni \theta \longmapsto \left(\frac{1+e^{i\theta}z_0}{1-e^{i\theta}z_0}\right)^\beta + \alpha\,\overline{\left(\frac{1+e^{i\theta}z_0}{1-e^{i\theta}z_0}\right)^\beta} - 1 - \alpha. \tag{15}$$

Theorem 2. *The boundary $\partial V_{\phi,\mathcal{H}}(z_0,\lambda)$ of $V_{\phi,\mathcal{H}}(z_0,\lambda)$ is the Jordan curve given by*

$$(-\pi,\pi] \ni \theta \longmapsto \phi\left(z_0\delta\left(e^{i\theta}z_0,\lambda\right)\right) + \alpha\,\overline{\phi(z_0\delta(e^{i\theta}z_0,\lambda))}, \tag{16}$$

where

$$\delta(cz,\lambda) = \frac{cz+\lambda}{1+cz\overline{\lambda}} \quad \left(c \in \overline{\mathbb{D}}\right). \tag{17}$$

Proof. Let $f \in \mathcal{A}_0$ such that $f \prec \phi$ for some $\phi \in \mathcal{S}_0$. Because $f \prec \phi$, there exists a Schwarz function $\omega = \phi^{-1} \circ f \in \mathcal{B}_0$ with $\omega'(0) = f'(0)/\phi'(0) = \lambda$, where $|\lambda| \le 1$. Therefore, for any fixed $z_0 \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \mathbb{C}$ with $0 < |\lambda| \le 1$, it is natural to consider the set

$$V_\phi(z_0, \lambda) = \left\{ f(z_0) : f \in \mathcal{A}_0(\phi),\ f'(0) = \lambda\phi'(0) \right\}. \tag{18}$$

First, we determine $V_\phi(z_0, \lambda)$. Then the determination of the set $V_{\phi,\mathcal{L}}(z_0, \lambda)$ follows from $V_\phi(z_0, \lambda)$. Now, we define

$$F_\omega(z) = \frac{\omega(z)/z - \lambda}{1 - \left(\overline{\lambda}\omega(z)/z\right)}, \quad \text{i.e.,} \ \omega(z) = \frac{z(F_\omega(z) + \lambda)}{1 + F_\omega(z)\overline{\lambda}}. \tag{19}$$

We observe that $F_\omega \in \mathcal{B}_0$. By the Schwarz lemma, we have $|F_\omega(z)| \le |z|$. If we set

$$\mathcal{B}_0^\lambda = \left\{ F_\omega : \omega \in \mathcal{B}_0,\ \omega'(0) = \lambda \right\} \tag{20}$$

then the region of variability $\{\omega(z_0){:}\omega \in \mathcal{B}_0^\lambda\}$ coincides with the set $\{z{:}|z| \le |z_0|\}$. It follows from the two expressions in (19) that $V_\phi(z_0, \lambda)$ coincides with the set

$$\left\{ \phi(z_0\delta(z, \lambda)) : |z| \le |z_0|,\ \text{where } \delta(z, \lambda) = \frac{z + \lambda}{1 + z\overline{\lambda}} \right\}. \tag{21}$$

The proof of this theorem is complete. □

The case $\lambda = 0$ of Theorem 2 gives the following result.

Corollary 2. *The boundary $\partial V_{\phi,\mathcal{L}}(z_0, 0)$ of $V_{\phi,\mathcal{L}}(z_0, 0)$ is the Jordan curve given by*

$$(-\pi, \pi] \ni \theta \longmapsto \phi\left(z_0^2 e^{i\theta}\right) + \alpha\ \overline{\phi(z_0^2 e^{i\theta} z_0)}. \tag{22}$$

If $\phi_0(z)$ is given by (14) for some $0 < \beta \le 2$, then $\phi_0'(0) = 2\beta$ and $V_{\phi_0,\mathcal{L}}(z_0, \lambda)$ reduces to

$$V_{\phi_0,\mathcal{L}}(z_0, \lambda) = \left\{ F_\alpha(f)(z_0) : f \in \mathcal{A}_0(\phi_0),\ f'(0) = 2\beta\lambda \right\} \tag{23}$$

and the corresponding $\omega(z)$ in the proof of the theorem will be precisely of the form

$$\omega(z) = \frac{(1 + f(z))^{1/\beta} - 1}{(1 + f(z))^{1/\beta} + 1}. \tag{24}$$

This observation gives the following corollary.

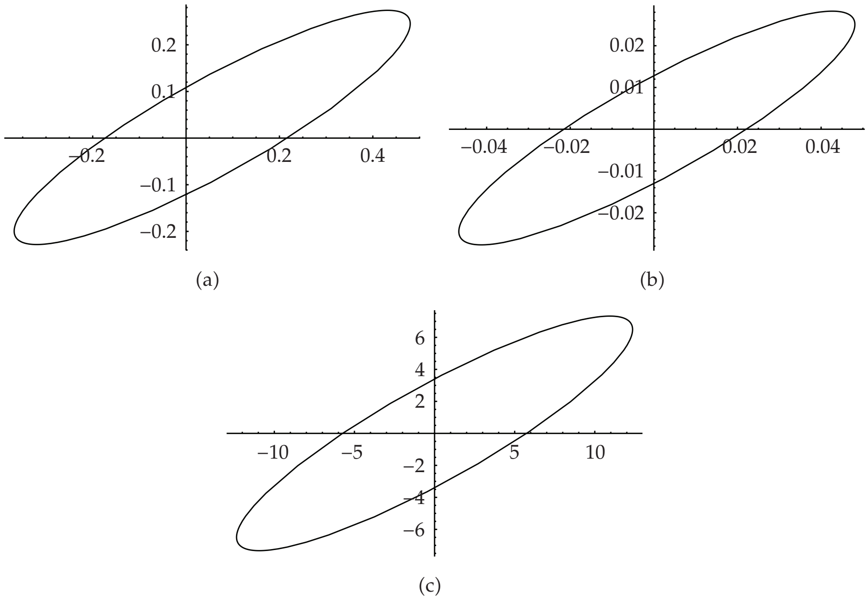


Figure 1

Corollary 3. *The boundary* $\partial V_{\phi_0,\mathcal{L}}(z_0,\lambda)$ *of* $V_{\phi_0,\mathcal{L}}(z_0,\lambda)$ *is the Jordan curve given by*

$$(-\pi,\pi] \ni \theta \longmapsto \left(\frac{1+z_0\delta(e^{i\theta}z_0,\lambda)}{1-z_0\delta(e^{i\theta}z_0,\lambda)}\right)^{\beta} + \alpha\,\overline{\left(\frac{1+z_0\delta(e^{i\theta}z_0,\lambda)}{1-z_0\delta(e^{i\theta}z_0,\lambda)}\right)^{\beta}} - 1 - \alpha, \tag{25}$$

where $\phi_0(z)$ *and* $\delta(cz,\lambda)$ *are given by* (14) *and* (17), *respectively.*

The boundary $\partial V_{\phi_0,\mathcal{L}}(z_0,0)$ of $V_{\phi_0,\mathcal{L}}(z_0,0)$ is the Jordan curve given by

$$(-\pi,\pi] \ni \theta \longmapsto \left(\frac{1+z_0^2e^{i\theta}}{1-z_0^2e^{i\theta}}\right)^{\beta} + \alpha\,\overline{\left(\frac{1+z_0^2e^{i\theta}}{1-z_0^2e^{i\theta}}\right)^{\beta}} - 1 - \alpha. \tag{26}$$

Theorem 3. *The boundary* $\partial V_{\mathcal{L},\mathcal{S}_{a,b,u}}(z_0)$ *of* $V_{\mathcal{L},\mathcal{S}_{a,b,u}}(z_0)$ *is the Jordan curve given by*

$$(-\pi,\pi] \ni \theta \longmapsto \frac{a+b}{i\pi}\left[\log\left(\frac{1-z_0e^{i\theta}e^{-2\pi ai/(a+b)}}{1-z_0e^{i\theta}}\right) - \alpha\,\overline{\log\left(\frac{1-z_0e^{i\theta}e^{-2\pi ai/(a+b)}}{1-z_0e^{i\theta}}\right)}\right]. \tag{27}$$

Proof. We define $V_{\mathcal{S}_{a,b,u}}(z_0) = \{f(z_0) : f \in \mathcal{S}_{a,b,u}\}$. It suffices to determine $V_{\mathcal{S}_{a,b,u}}(z_0)$ as the region of variability $V_{\mathcal{L},\mathcal{S}_{a,b,u}}(z_0)$ follows from $V_{\mathcal{S}_{a,b,u}}(z_0)$. In order to do this, first we consider

$$T(z) = \frac{a+b}{i\pi}\log w(z), \quad w(z) = \frac{1-ze^{-2\pi ai/(a+b)}}{1-z}. \tag{28}$$

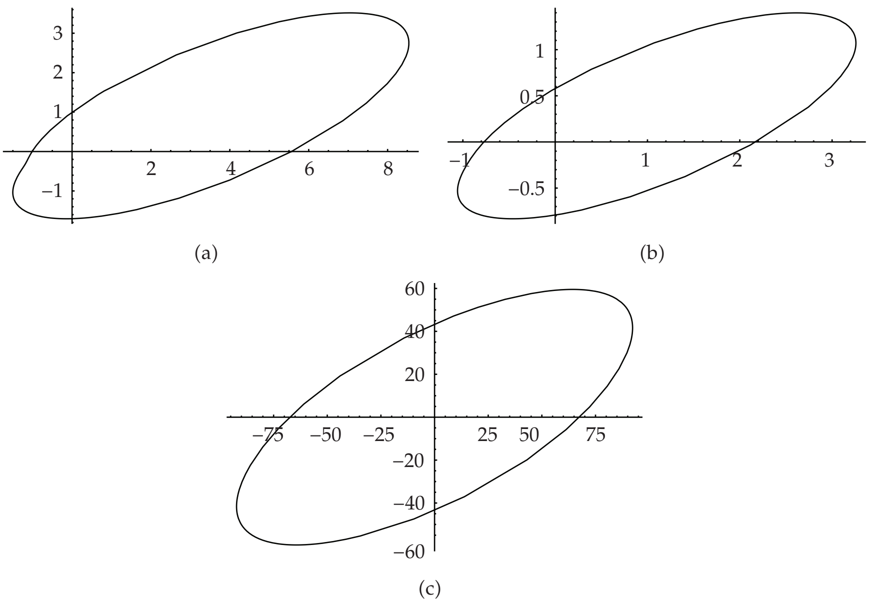


Figure 2

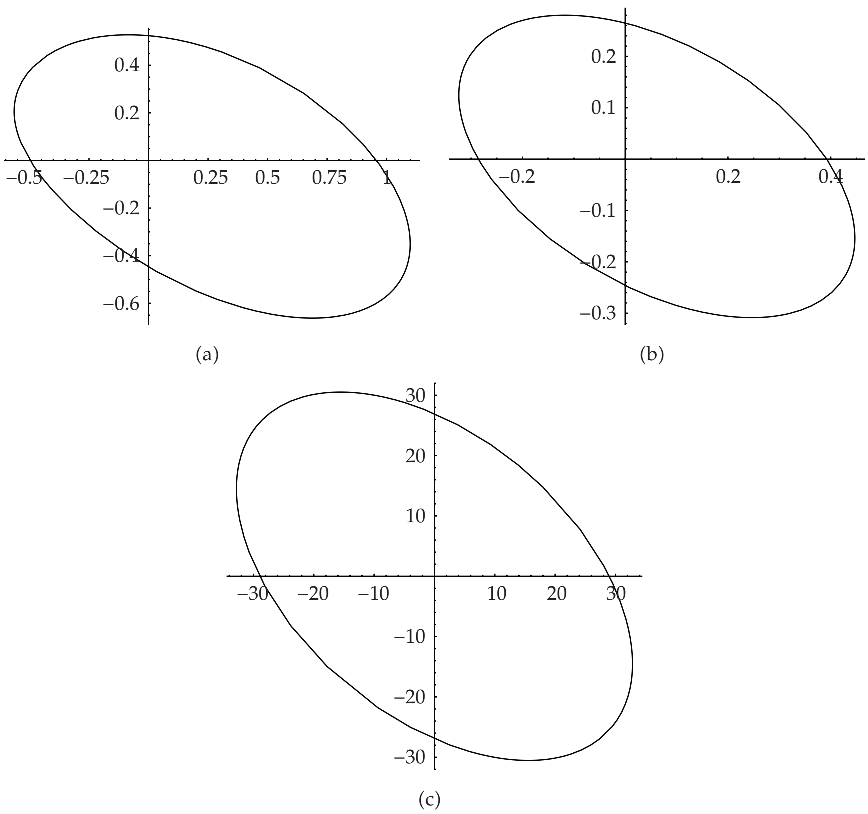


Figure 3

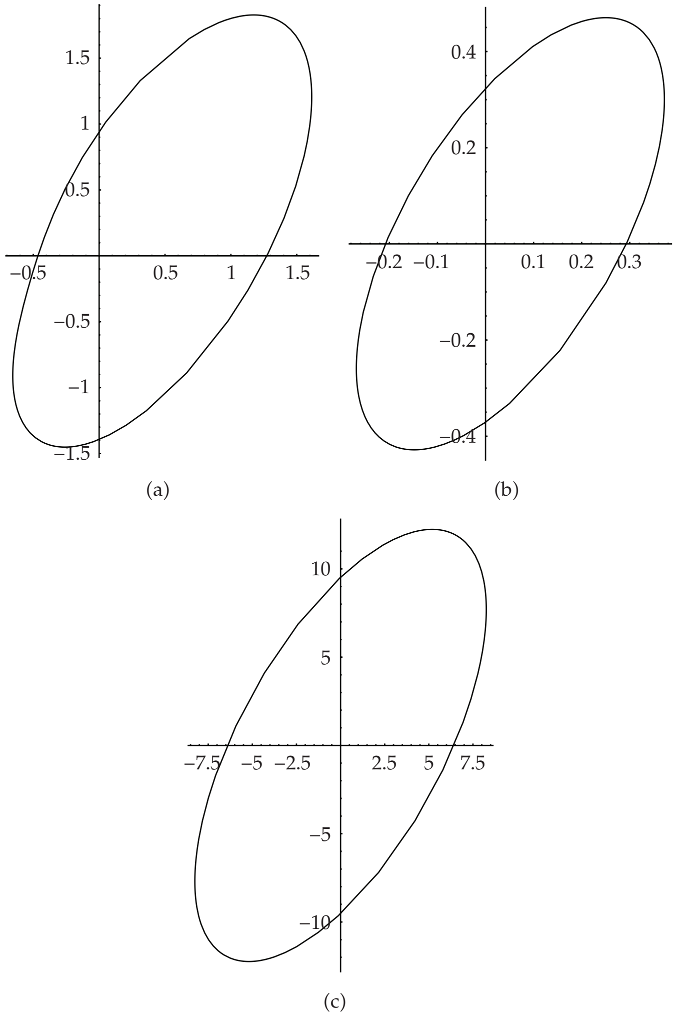


Figure 4

Then $T(0) = 0$. We see that the Möbius transformation $w(z)$ maps the open unit disk $\mathbb{D}$ conformally onto the half-plane

$$\left\{w = u + iv : u \sin\left(\frac{\pi a}{a+b}\right) + v \cos\left(\frac{\pi a}{a+b}\right) > 0\right\} \tag{29}$$

and so, we easily obtain that T maps $\mathbb{D}$ conformally onto the vertical strip $\{w : -a < \operatorname{Re} \ w < b\}$. This observation shows that $T \in \mathcal{S}_{a,b,u}$ and is in fact an extremal function for this class.

Next, we choose an arbitrary $f \in \mathcal{S}_{a,b,u} \setminus \{0\}$. Then we have $f \prec T$ and so, there exists a Schwarz function $\omega \in \mathcal{B}_0 \setminus \{0\}$ such that $f(z) = T(\omega(z))$. Note that both f and T are univalent in $\mathbb{D}$ and so, $\omega = T^{-1} \circ f$ is univalent in $\mathbb{D}$ with $\omega(0) = 0$. It follows from the classical Schwarz

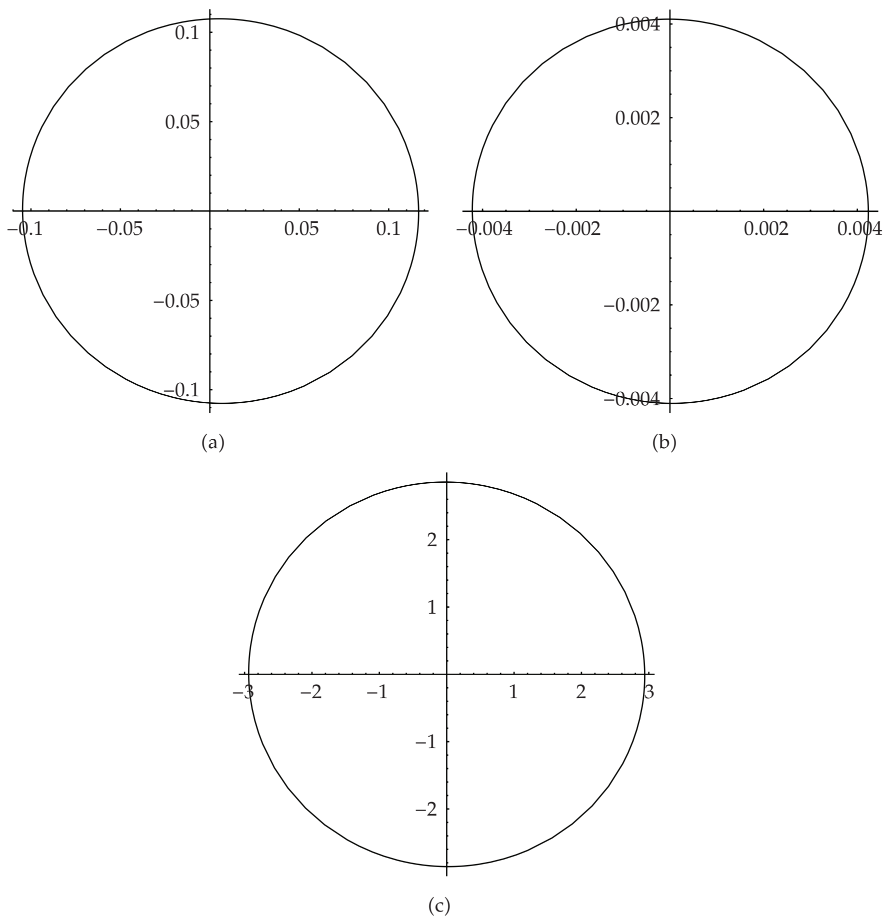


Figure 5

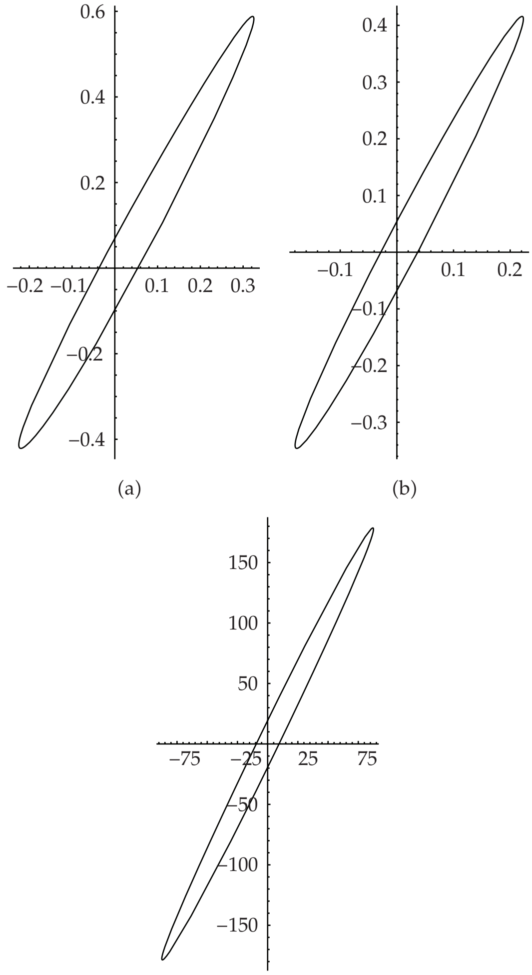
lemma that $|\omega(z)| \leq |z|$ in $\mathbb{D}$. Because ω is also univalent in $\mathbb{D}$, we obtain that the region of variability of

$$V_{\omega,u}(z_0) = \{\omega(z_0) : \omega \in (\mathcal{B}_0 \cap \mathcal{S}_0) \cup \{0\}\} \tag{30}$$

coincides with the set $\{z : |z| \leq |z_0|\}$. Hence the region of variability $V_{\mathcal{S}_{a,b,u}}(z_0)$ coincides with the set

$$\left\{ \frac{a+b}{i\pi} \log\left(\frac{1 - ze^{-2\pi ai/(a+b)}}{1-z} \right) : |z| \leq |z_0| \right\}. \tag{31}$$

The proof of Theorem 3 is complete. □



(a) (b) (c)

Figure 6

Theorem 4. *The boundary* $\partial V_{\mathscr{L},\mathcal{S}_{a,b}}(z_0,\lambda)$ *of* $V_{\mathscr{L},\mathcal{S}_{a,b}}(z_0,\lambda)$ *is the Jordan curve given by*

$$(-\pi,\pi]\ni\theta\longmapsto\frac{a+b}{i\pi}\left[\log\left(\frac{1-z_0\delta(z_0e^{i\theta},\lambda)e^{-2\pi ai/(a+b)}}{1-z_0\delta(z_0e^{i\theta},\lambda)}\right)\right.$$
$$\left.-\alpha\,\overline{\log\left(\frac{1-z_0\delta(z_0e^{i\theta},\lambda)e^{-2\pi ai/(a+b)}}{1-z_0\delta(z_0e^{i\theta},\lambda)}\right)}\right], \tag{32}$$

where $\delta(cz,\lambda)$ *is given by* (17).

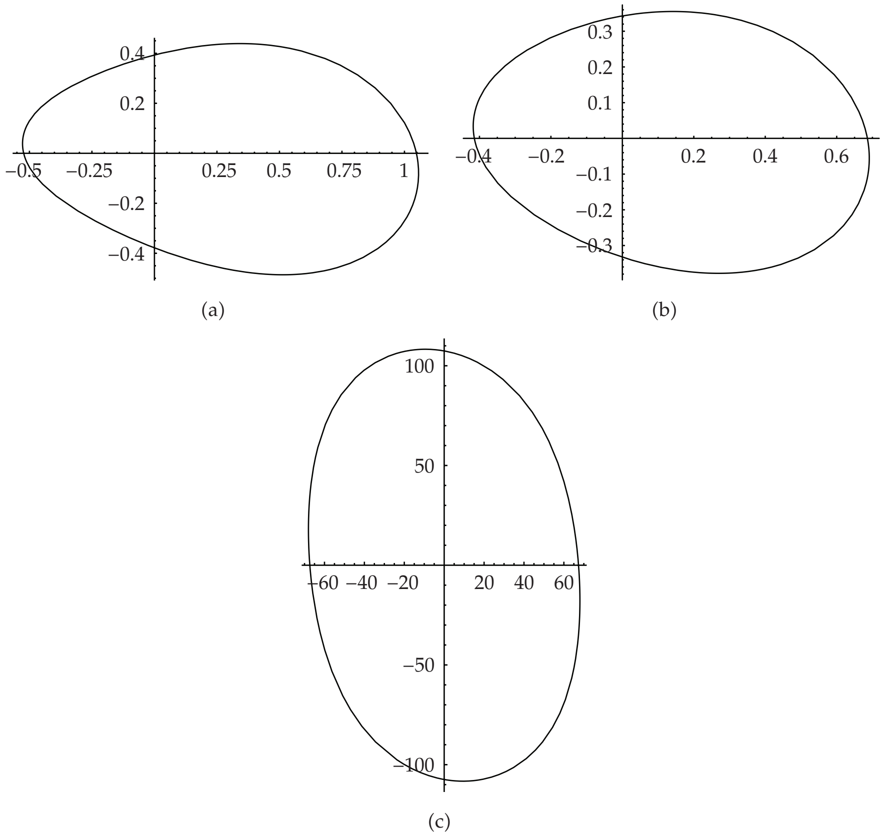


Figure 7

Proof. For convenience, we let $p = (a+b)/(i\pi)$ and $q = e^{-2\pi ai/(a+b)}$ and consider

$$V_{\mathcal{S}_{a,b}}(z_0, \lambda) = \{ f(z_0) : f \in \mathcal{S}_{a,b},\ f'(0) = p(1-q)\lambda \}. \tag{33}$$

As before, it suffices to prove the theorem for $V_{\mathcal{S}_{a,b}}(z_0, \lambda)$. Let $f \in \mathcal{S}_{a,b}$ with $f'(0) = p(1-q)\lambda$. Define

$$g(z) = \frac{f(z)}{p}, \qquad h(z) = e^z, \qquad \phi(z) = \frac{z-1}{z-q}. \tag{34}$$

Then, by the mapping properties of these functions, it can be easily seen that the composed mapping

$$\omega_f(z) = (\phi \circ h \circ g)(z) = \frac{e^{f(z)/p} - 1}{e^{f(z)/p} - q} \tag{35}$$

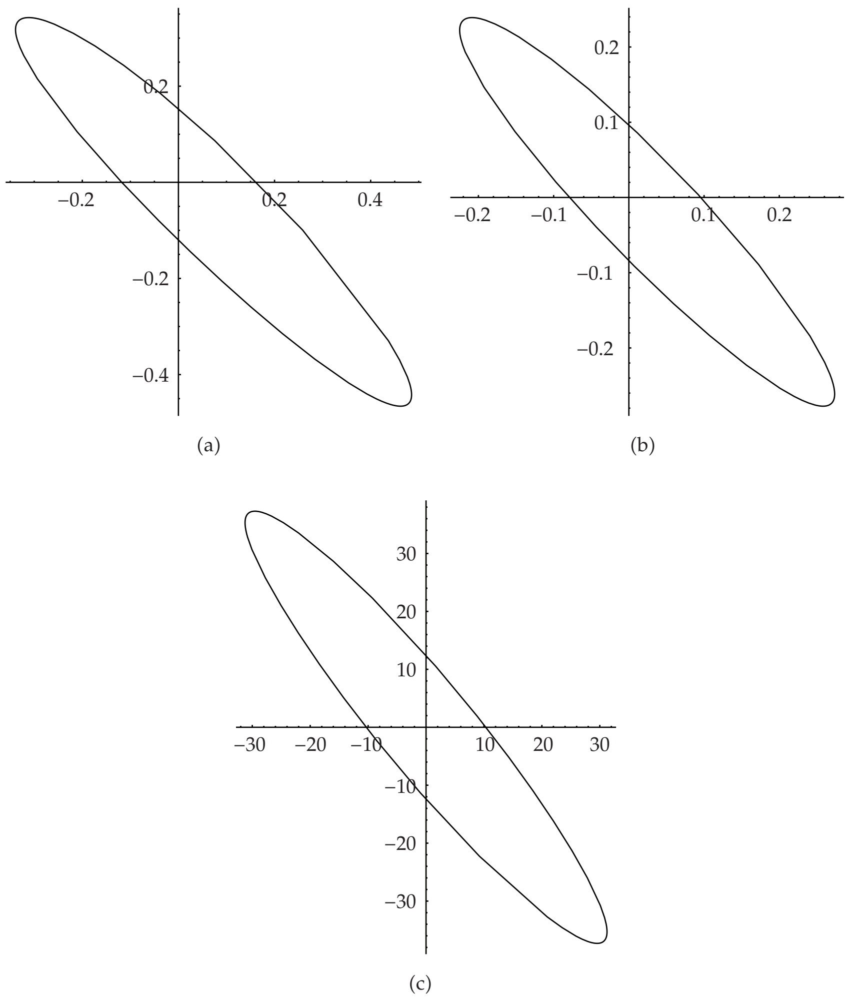


Figure 8

is analytic in $\mathbb{D}$ and maps unit disk $\mathbb{D}$ into $\mathbb{D}$ such that $\omega_f(0) = 0$ and $\omega'_f(0) = \lambda$. Next, we introduce $Q_f : \mathbb{D} \to \mathbb{D}$ by

$$Q_f(z) = \frac{\omega_f(z)/z - \lambda}{1 - \overline{\lambda}(\omega_f(z)/z)}. \tag{36}$$

Clearly, $Q_f \in \mathcal{B}_0$. If we let

$$\begin{aligned} S_{a,b,\omega_f,\lambda} &= \left\{ Q_f : \ \omega_f \in \mathcal{B}_0, \ \omega'_f(0) = \lambda \right\}, \\ V_{Q_f}(z_0) &= \left\{ \omega_f(z_0) : \ \omega_f \in S_{a,b,\omega_f,\lambda} \right\}, \end{aligned} \tag{37}$$

Table 1: Subordination of harmonic mappings.

Figure	z_0	α	β	$a = b$
1	$0.0928916 + 0.0656754i$	$0.343308 + 0.551846i$	1.25961	58.9326
2	$-0.495149 - 0.48309i$	$0.377474 + 0.363979i$	1.14901	81.0473
3	$-0.210195 - 0.485306i$	$0.126883 - 0.247013i$	0.57185	45.4015
4	$-0.117278 + 0.329628i$	$-0.183041 + 0.337725i$	1.44013	21.5077
5	$0.0370762 + 0.00949962i$	$0.0147993 - 0.00392657i$	1.42167	59.4649
6	$0.312303 + 0.721208i$	$-0.524227 + 0.716229i$	0.187546	82.7409
7	$0.315822 - 0.788402i$	$0.0532365 - 0.057638i$	0.276943	71.5991
8	$-0.660899 + 0.013848i$	$-0.0571237 - 0.691304i$	0.234536	31.2565

then, by the Schwarz lemma, we have $|Q_f(z)| \le |z|$. The region of variability $V_{Q_f}(z_0)$ coincides with the set $\{z : |z| \le |z_0|\}$. Equation (36) implies that

$$\omega_f(z) = \frac{z(Q_f(z) + \lambda)}{1 + Q_f(z)\overline{\lambda}}. \tag{38}$$

It follows from (35) and (38) that $V_{\mathcal{S}_{a,b}}(z_0, \lambda)$ coincides with the set

$$\left\{p \log \frac{1 - z_0\delta(z,\lambda)q}{1 - z_0\delta(z,\lambda)} : |z| \le |z_0|, \text{ where } \delta(z,\lambda) = \frac{z + \lambda}{1 + z\overline{\lambda}}\right\}. \tag{39}$$

The proof of Theorem 4 is complete. □

Geometric View of the Jordan Curves: (15), (26), *and* (27)

Table 1 gives the list of these parameter values corresponding to Figures 1–8 which concern the regions of variability for $\partial V_{\phi_0,\mathcal{L}}(z_0)$, $\partial V_{\phi_0,\mathcal{L}}(z_0, 0)$, and $\partial V_{\mathcal{L},\mathcal{S}_{a,a,u}}(z_0)$, respectively.

Using Mathematica (see [13]), we describe the boundary sets $\partial V_{\phi_0,\mathcal{L}}(z_0)$, $\partial V_{\phi_0,\mathcal{L}}(z_0, 0)$, and $\partial V_{\mathcal{L},\mathcal{S}_{a,a,u}}(z_0)$ described by the Jordan curve given by (15), (26), and (27), respectively. In the program below, "z0 stands for z_0," "[Alpha] for α," and "[Beta] for β.''

In Table 1, the parameter values of z_0 and α are common for all the three cases, namely, $\partial V_{\phi_0,\mathcal{L}}(z_0)$, $\partial V_{\phi_0,\mathcal{L}}(z_0, 0)$, and $\partial V_{\mathcal{L},\mathcal{S}_{a,a,u}}(z_0)$, whereas the β value is applicable only for the first two cases and the $a = b$ values listed in the last column is meant only for the last case.

```
(* Geometric view the main Theorem.... *)
Remove["Global`*"];
z0 = Random [] Exp[I Random[Real, {-Pi, Pi}]]
\[Alpha] = Random[]Exp[I Random[Real, {-Pi, Pi}]]
\[Beta] = Random[Real, {0, 2}]
a = Random[Real, {0,100}]
Print["z0=", z0]
Print["\[Alpha]=", \[Alpha]]
Print["\[Beta]=", \[Beta]]
Print["a", a]

myf1[the_, \[Alpha]_, \[Beta]_, z0_]:=
((1+Exp[I*the]*z0)/(1-Exp[I*the]*z0))^\[Beta] +
\[Alpha]*Conjugate[((1+Exp[I*the]*z0)/(1-Exp[I*the]*z0))^\[Beta]]
- 1-\[Alpha];

myf2[the_, \[Alpha]_, \[Beta]_,z0_]:=
((1+Exp[I*the]*z0*z0)/(1-Exp[I*the]*z0*z0))^\[Beta]+
\[Alpha]*Conjugate[((1+Exp[I*the]*z0*z0)/(1-Exp[I*the]*z0*z0))^\[Beta]]
-1-\[Alpha];

myf3[the_, \[Alpha]_, a_,z0_]:=
(2a)/(I*Pi)(Log((1-Exp[I*the]*Exp[-I*Pi]*z0)/(1-Exp[I*the]*z0))-
\[Alpha]*Conjugate[Log((1-Exp[I*the]*Exp[-I*Pi]*z0)/(1-Exp[I*the]*z0))])

image1 = ParametricPlot[{Re[myf1[the, \[Alpha], \[Beta], z0]],
Im[myf1[the, \[Alpha], \[Beta], z0]]}, {the, -Pi, Pi},
AspectRatio -> Automatic,DisplayFunction -> $DisplayFunction,
TextStyle -> {FontFamily -> "Times", FontSize -> 14},
AxesStyle -> {Thickness[0.0035]}];

image2 =ParametricPlot[{Re[myf2[the, \[Alpha], \[Beta], z0]],
Im[myf2[the, \[Alpha], \[Beta], z0]]}, {the, -Pi, Pi},
AspectRatio -> Automatic,DisplayFunction -> $DisplayFunction,
TextStyle -> {FontFamily -> "Times", FontSize -> 14},
AxesStyle -> {Thickness[0.0035]}];

image3 =ParametricPlot[{Re[myf3[the, \[Alpha], a, z0]],
Im[myf3[the, \[Alpha], a, z0]]}, {the, -Pi, Pi},
AspectRatio -> Automatic,DisplayFunction -> $DisplayFunction,
TextStyle -> {FontFamily -> "Times", FontSize -> 14},
AxesStyle -> {Thickness[0.0035]}];

Clear[the, z0, \[Alpha], \[Beta], a, myf1, myf2, myf3];
```

Acknowledgment

The research was partly supported by NSFs of china (No. 10771059).

References

[1] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiae Scientiarum Fennicae. Series A*, vol. 9, pp. 3–25, 1984.

[2] P. Duren, *Harmonic Mappings in the Plane*, vol. 156 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2004.

[3] M. Chuaqui, P. Duren, and B. Osgood, "Ellipses, near ellipses, and harmonic Möbius transformations," *Proceedings of the American Mathematical Society*, vol. 133, no. 9, pp. 2705–2710, 2005.

[4] S. H. Chen, S. Ponnusamy, and X. Wang, "Some properties and regions of variability of affine harmonic mappings and affine biharmonic mappings," *International Journal of Mathematics and Mathematical Sciences*. In press.

[5] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, Germany, 1975.

[6] P. L. Duren, *Univalent Functions*, vol. 259 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Berlin, Germany, 1983.

[7] S. S. Miller and P. T. Mocanu, *Differential Subordinations, Theory and Applications*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.

[8] J. M. Anderson, J. Clunie, and C. Pommerenke, "On Bloch functions and normal functions," *Journal für die Reine und Angewandte Mathematik*, vol. 270, pp. 12–37, 1974.

[9] S. Yamashita, "Gap series and α-Bloch functions," *Yokohama Mathematical Journal*, vol. 28, no. 1-2, pp. 31–36, 1980.

[10] L. E. Schaubroeck, "Subordination of planar harmonic functions," *Complex Variables. Theory and Application*, vol. 41, no. 2, pp. 163–178, 2000.

[11] S. Ponnusamy and A. Vasudevarao, "Region of variability of two subclasses of univalent functions," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1323–1334, 2007.

[12] S. Ponnusamy, A. Vasudevarao, and H. Yanagihara, "Region of variability of univalent functions $f(z)$ for which $zf'(z)$ is spirallike," *Houston Journal of Mathematics*, vol. 34, no. 4, pp. 1037–1048, 2008.

[13] H. Ruskeepää, *Mathematica Navigator: Mathematics, Statistics, and Graphics*, Elsevier Academic, Burlington, Mass, USA, 2nd edition, 2004.