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Research Article

Fixed Points for w-Contractive Multimaps

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Using the generalized Caristi's fixed point theorems we prove the existence of fixed points for self and nonself multivalued weakly w-contractive maps. Consequently, Our results either improve or generalize the corresponding fixed point results due to Latif (2007), Bae (2003), Suzuki, and Takahashi (1996) and others.

1. Introduction

It is well known that Caristi's fixed point theorem [1] is equivalent to Ekland variational principle [2], which is nowadays is an important tool in nonlinear analysis. Most recently, many authors studied and generalized Caristi's fixed point theorem to various directions. For example, see [3–6] and references therein.

Using the concept of Hausdorff metric, Nadler Jr. [7] has proved multivalued version of the Banach contraction principle which states that each closed bounded valued contraction map on a complete metric space, has a fixed point. Recently, Bae [4] introduced a notion of multivalued weakly contractive maps and applying generalized Caristi's fixed point theorems he proved several fixed point results for such maps in the setting of metric and Banach spaces. Many authors have been using the Hausdorff metric to obtain fixed point results for multivalued maps on metric spaces, but, in fact for most cases the existence part of the results can be proved without using the concept of Hausdorff metric.

Recently, using the concept of w-distance [8], Suzuki and Takahashi [9] introduced a notion of multivalued weakly contractive(in short, w-contractive) maps and improved the Nadler's fixed point result without using the concept of Hausdorff metric. Most recently, Latif [10] generalized the fixed point result of Suzuki and Takahashi [9, Theorem 1]. Some interesting examples and fixed point results concerning w-distance can be found in [6, 11–15] and references therein.

In this paper, introducing a notion of multivalued weakly w-contractive maps, we prove some fixed point results for self and nonself multivalued maps. Our results either improve or generalize the corresponding results due to Latif [10], Bae [4], Mizoguchi and Takahashi [16], Suzuki and Takahashi [9], Husain and Latif [17], Kaneko [18] and many others.

2. Preliminaries

Let X be a metric space with metric d. We use 2^X to denote the collection of all nonempty subsets of X and $\mathrm{Cl}(X)$ for the collection of all nonempty closed subsets of X. Recall that a real-valued function φ defined on X is said to be *lower (upper) semicontinuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $\varphi(x) \leq \liminf_{n\to\infty}\varphi(x_n)$ ($\varphi(x) \geq \limsup_{n\to\infty}\varphi(x_n)$).

Introducing the following notion of w–distance, Kada et al. [8] improved the Caristi's fixed point theorem, Ekland variational principle, and Takahashi existence theorem.

A function $\omega : X \times X \to [0,\infty)$ is called a *w-distance* on X if it satisfies the following for any $x, y, z \in X$:

(w_1) $\omega(x,z) \leq \omega(x,y) + \omega(y,z)$;

(w_2) a map $\omega(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous;

(w_3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z,x) \leq \delta$ and $\omega(z,y) \leq \delta$ imply $d(x,y) \leq \epsilon$.

Note that, in general for $x, y \in X$, $\omega(x,y) \neq \omega(y,x)$ and not either of the implications $\omega(x,y) = 0 \Leftrightarrow x = y$ necessarily hold. Clearly, the metric d is a w-distance on X. Let $(Y, \|\cdot\|)$ be a normed space. Then the functions $\omega_1, \omega_2 : Y \times Y \to [0,\infty)$ defined by $\omega_1(x,y) = \|y\|$ and $\omega_2(x,y) = \|x\| + \|y\|$ for all $x, y \in Y$ are w-distances [8].

Let M be a nonempty subset of X. A multivalued map $T : M \to 2^X$ is called *w-contractive* [9] if there exist a w-distance ω on X and a constant $h \in (0,1)$ such that for any $x, y \in X$ and $u \in T(x)$ there is $v \in T(y)$ satisfying

$$\omega(u,v) \leq h\omega(x,y). \tag{2.1}$$

In particular, if we take $\omega = d$, then w-contractive map is a contractive type map [17].

We say T is *weakly w-contractive* if there exists a w-distance ω on X such that for any $x, y \in X$ and $u \in T(x)$ there is $v \in T(y)$ with

$$\omega(u,v) \leq \omega(x,y) - \varphi(\omega(x,y)), \tag{2.2}$$

where φ is a function from $[0,\infty)$ to $[0,\infty)$ such that φ is positive on $(0,\infty)$ and $\varphi(0) = 0$.

In particular, if we take $\varphi(t) = (1-h)t$ for a constant h with $0 < h < 1$, then a weakly w-contractive map is w-contractive. If we define $k(t) = 1 - \varphi(t)/t$ for $t > 0$ and $k(0) = 0$, then k is a function from $(0,\infty)$ to $[0,1)$ with $\limsup_{r\to t^+} k(r) < 1$, for every $t \in [0,\infty)$. Also we get

$$\omega(u,v) \leq k(\omega(x,y))\omega(x,y), \tag{2.3}$$

that is, the weakly w-contractive map is generalized w-contraction [10].

We say a multivalued map $T : M \to 2^X$ is w-inward if for each $x \in M$, $T(x) \subset$ w-$I_M(x)$, where w-$I_M(x)$ is the w-inward set of M at x, which consists all the elements $z \in X$ such that either $z = x$ or there exists $y \in M$ with $y \neq x$ and $\omega(x,z) = \omega(x,y) + \omega(y,z)$.

In particular, if we take $\omega = d$, then w-inward set is known as metrically inward set [4].

A point $x \in M$ is called a fixed point of $T : M \to 2^X$ if $x \in T(x)$ and the set of all fixed points of T is denoted by $\mathrm{Fix}(T)$.

In the sequel, otherwise specified, we will assume that $\psi : X \to [0,\infty)$ is lower semicontinuous function, $\varphi : [0,\infty) \to [0,\infty)$ is positive function on $(0,\infty)$ and $\varphi(0) = 0$ and ω is a w-distance on X.

Using the concept of w-distance, Kada et al. [8] have generalized Caristi's fixed point theorem as follows.

Theorem 2.1. *Let (X,d) be a complete metric space. Let $f : X \to X$ be a map such that for each $x \in X$,*

$$\psi(f(x)) + \omega(x, f(x)) \leq \psi(x). \tag{2.4}$$

Then, there exists $x_o \in X$ such that $f(x_o) = x_o$ and $\omega(x_o, x_o) = 0$.

Now, we state generalized Caristi's fixed point theorems which are variant to the results of Bae [4, Theorem 2.1 and Corollary 2.5].

Theorem 2.2. *Let (X,d) be a complete metric space. Let $f : X \to X$ be a map such that for each $x \in X$,*

$$\omega(x, f(x)) \leq \max\{c(\psi(x)), c(\psi(f(x)))\}(\psi(x) - \psi(f(x))), \tag{2.5}$$

where $c : [0,\infty) \to (0,\infty)$ is an upper semicontinuous function from the right. Then, f has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Theorem 2.3. *Let (X,d) be a complete metric space. Let $\varphi : [0,\infty) \to [0,\infty)$ be lower semicontinuous function such that $\varphi(t) > 0$ for $t > 0$ and*

$$\limsup_{t \to 0^+} \frac{t}{\varphi(t)} < \infty. \tag{2.6}$$

Let $f : X \to X$ be a map such that for each $x \in X$, $\omega(x, f(x)) \leq \psi(x)$ and

$$\varphi(\omega(x, f(x))) \leq \psi(x) - \psi(f(x)). \tag{2.7}$$

Then, f has a fixed point $x_0 \in X$ such that $\omega(x_0, x_0) = 0$.

Suzuki and Takahashi [9] have proved the following fixed point result which is an improved version of the multivalued contraction principle due to Nadler Jr. [7].

Theorem 2.4. *Let (X,d) be a complete metric space. Then each multivalued w-contractive map $T : X \to \mathrm{Cl}(X)$ has a fixed point.*

3. Main Results

Without using the Hausdorff metric, we prove the following fixed point result for multivalued self map.

Theorem 3.1. *Let (X,d) be a complete metric space and let $T : X \to \mathrm{Cl}(X)$ be a weakly w-contractive map for which φ is lower semicontinuous from the right and $\limsup_{t\to 0^+}(t/\varphi(t)) < \infty$. Then T has a fixed point.*

Proof. Let $G = \{(x,y) : x \in X, y \in T(x)\}$ be the graph of T. Clearly, G is a closed subset of $X \times X$. Define a metric ρ on G by

$$\rho\big((x,y),(u,v)\big) = \max\{d(x,u), d(y,v)\}. \tag{3.1}$$

Then (G,ρ) is a complete metric space and ρ is w-distance on G. Now, define $\psi : G \to [0,\infty)$ by $\psi(x,y) = \omega(x,y) = d(x,y)$ for all $(x,y) \in G$ and $c : [0,\infty) \to [0,\infty)$ by

$$c(t) = \begin{cases} \dfrac{t}{\varphi(t)}, & \text{if } t > 0, \\ \limsup\limits_{t\to 0^+}\left(\dfrac{t}{\varphi(t)}\right), & \text{if } t = 0. \end{cases} \tag{3.2}$$

Then ψ is lower semicontinuous and c is upper semicontinuous from the right because φ is lower semicontinuous from the right. Define $p : G \times G \to [0,\infty)$ by

$$p\big((x,y),(u,v)\big) = \max\{\psi(x,y), \rho\big((x,y),(u,v)\big)\}. \tag{3.3}$$

Then p is a w-distance on G (see [14, page 47]. Now, suppose $\mathrm{Fix}(T)=\varnothing$. Then for each $(x,y) \in G$, we have $x \neq y$. Since $y \in T(x)$ there is $z \in T(y)$ such that

$$\omega(y,z) \le \omega(x,y) - \varphi\big(\omega(x,y)\big). \tag{3.4}$$

Since $(x,y),(y,z) \in G$, we have

$$p\big((x,y),(y,z)\big) = \rho\big((x,y),(y,z)\big) = \omega(x,y) = \psi(x,y), \tag{3.5}$$

also, note that

$$p\big((x,y),(y,z)\big) = \omega(x,y) \le \frac{\omega(x,y)}{\varphi\big(\omega(x,y)\big)}\big[\omega(x,y) - \omega(y,z)\big]. \tag{3.6}$$

Define a function $f : G \to G$ by $f(x,y) = (y,z)$, then we get

$$p\big((x,y), f(x,y)\big) \le c\big(\psi(x,y)\big)\big[\psi(x,y) - \psi\big(f(x,y)\big)\big]. \tag{3.7}$$

Thus, by Theorem 2.2, f has a fixed point, which is impossible. Hence, T must has a fixed point. This completes the proof. □

As a consequence, we obtain the following recent fixed point result of Latif [10, Theorem 2.2].

Corollary 3.2. *Let (X,d) be a complete metric space. Let $T : X \to \mathrm{Cl}(X)$ be a map such that for any $x, y \in X$ and $u \in T(x)$ there is $v \in T(y)$ with*

$$\omega(u,v) \leq k(\omega(x,y))\omega(x,y), \tag{3.8}$$

where k is function from $[0,\infty)$ to $[0,1)$ with $\limsup_{r\to t^+} k(r) < 1$, for every $t \in [0,\infty)$. Then T has a fixed point.

Proof. Define $\varphi : [0,\infty) \to [0,\infty)$ by

$$\varphi(t) = \min\left\{t(1-k(t)), \liminf_{r\to t^+} r(1-k(r))\right\} \quad \forall t \geq 0. \tag{3.9}$$

Then $\varphi(t) > 0$ for all $t > 0$, φ is lower semicontinuous from the right (see [19]). Also note that

$$\limsup_{t\to 0^+} \frac{t}{\varphi(t)} < \infty, \tag{3.10}$$

and for each $x, y \in X$, we have

$$\varphi(\omega(x,y)) \leq \omega(x,y)(1-k(\omega(x,y))). \tag{3.11}$$

It follows from (3.8) and (3.11) that

$$\omega(u,v) \leq \omega(x,y) - \varphi(\omega(x,y)). \tag{3.12}$$

Thus T is weakly w-contractive map for which φ is lower semicontinuous from the right and $\limsup_{t\to 0^+}(t/\varphi(t)) < \infty$. Therefore, by Theorem 3.1, T has a fixed point. □

Remark 3.3. (a) Theorem 3.1 generalizes Theorem 2.4 of Suzuki and Takahashi [9]. Indeed, consider $\varphi(\omega(x,y)) = (1-h)\omega(x,y)$ for a constant h with $0 < h < 1$. Theorem 3.1 also generalizes and improves the fixed point result of Bae [4, Theorem 3.1] .

(b) Corollary 3.2 generalizes fixed point result of Husain and Latif [17, Theorem 2.3] and improves [16, Theorem 5]. Moreover, it improves and generalizes [18, Theorem 1].

Without using the Hausdorff metric, we prove the following fixed point result for nonself multivalued maps with respect to w-distance.

Theorem 3.4. *Let M be a closed subset of a complete metric space (X,d) and let $T : M \to \mathrm{Cl}(X)$ be a weakly w-contractive map for which φ is lower semicontinuous and $\limsup_{t\to 0^+}(t/\varphi(t)) < \infty$. Then T has a fixed point provided T is w-inward on M.*

Proof. Let G, ρ, p, and ψ be the same as in the proof of Theorem 3.1. Suppose $\mathrm{Fix}(T) = \emptyset$. Then, for each $(x, y) \in G$ we have $x \neq y$. Since $y \in T(x) \subset$ w-$I_M(x)$ there exists $u \in M$ with $u \neq x$ and

$$\omega(x, y) = \omega(x, u) + \omega(u, y). \tag{3.13}$$

Since the map T is weakly w-contractive, there exists $v \in T(u)$ such that

$$\omega(y, v) \leq \omega(x, u) - \varphi(\omega(x, u)), \tag{3.14}$$

where φ is lower semicontinuous and $\limsup_{t \to 0^+}(t/\varphi(t)) < \infty$. From (3.13) and (3.14), we get

$$\varphi(\omega(x, u)) \leq \omega(x, u) - \omega(y, v) = \omega(x, y) - [\omega(u, y) + \omega(y, v)]. \tag{3.15}$$

Thus,

$$\varphi(\omega(x, u)) \leq \omega(x, y) - \omega(u, v). \tag{3.16}$$

Since $(x, y), (u, v) \in G$, we have

$$\rho((x, y), (u, v)) = \max\{\omega(x, u), \omega(y, v)\}, \tag{3.17}$$

and hence, we get

$$p((x, y), (u, v)) = \omega(x, u) \leq \omega(x, y) = \psi(x, y). \tag{3.18}$$

Now, define a function $f : G \to G$ by $f(x, y) = (u, v)$. Then from (3.18) we get

$$p((x, y), f(x, y)) \leq \psi(x, y), \tag{3.19}$$

and using (3.16), we obtain

$$\varphi(p((x, y), f(x, y))) \leq \psi(x, y) - \psi(f(x, y)). \tag{3.20}$$

Thus by Theorem 2.3, f has a fixed point, which is impossible. Hence, it follows that T must has a fixed point. □

Using the same method as in the proof of Corollary 3.2, we can obtain the following fixed point result for nonself generalized w-contractions.

Corollary 3.5. *Let M be a closed subset of a complete metric space (X, d) and let $T : M \to \mathrm{Cl}(X)$ be a map satisfying inequality (3.8) for which $k : [0, \infty) \to [0, 1)$ is upper semicontinuous. Then T has a fixed point provided T is w-inward on M.*

Remark 3.6. (a) Our Theorem 3.4 and Corollary 3.5 improve the results of Bae [4, Theorem 3.3 and Corollary 3.4], respectively.

(b) The analogue of all the results of this section can be established with respect to τ-distance [20].

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