Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 865705, 15 pages
doi:10.1155/2009/865705

Research Article

Engel Series and Cohen-Egyptian Fraction Expansions

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Received 11 September 2009; Accepted 2 December 2009

Recommended by Stéphane Louboutin

Two kinds of series representations, referred to as the Engel series and the Cohen-Egyptian fraction expansions, of elements in two different fields, namely, the real number and the discrete-valued non-archimedean fields are constructed. Both representations are shown to be identical in all cases except the case of real rational numbers.

1. Introduction

It is well known [1] that each nonzero real number can be uniquely written as an Engel series expansion, or ES expansion for short, and an ES expansion represents a rational number if and only if each digit in such expansion is identical from certain point onward. In 1973, Cohen [2] devised an algorithm to uniquely represent each nonzero real number as a sum of Egyptian fractions, which we refer to as its Cohen-Egyptian fraction (or CEF) expansion. Cohen also characterized the real rational numbers as those with finite CEF expansions. At a glance, the shapes of both expansions seem quite similar. This naturally leads to the question whether the two expansions are related. We answer this question affirmatively for elements in two different fields. In Section 2, we treat the case of real numbers and show that for irrational numbers both kinds of expansion are identical, while for rational numbers, their ES expansions are infinite, periodic of period 1, but their CEF expansions always terminate. In Section 3, we treat the case of a discrete-valued non-archimedean field. After devising ES and CEF expansions for nonzero elements in this field, we see immediately that both expansions are identical. In Section 4, we characterize rational elements in three different non-archimedean fields.

2. The Case of Real Numbers

Recall the following result, see, for example, Kapitel IV of [1], which asserts that each nonzero real number can be uniquely represented as an infinite ES expansion and rational numbers have periodic ES expansions of period 1.

Theorem 2.1. *Each $A \in \mathbb{R} \setminus \{0\}$ is uniquely representable as an infinite series expansion, called its Engel series (ES) expansion, of the form*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n}, \tag{2.1}$$

where

$$a_0 = \begin{cases} [A], & \text{if } A \notin \mathbb{Z}, \\ A - 1 & \text{if } A \in \mathbb{Z}, \end{cases} \quad a_1 \geq 2, \ a_{n+1} \geq a_n \quad (n \geq 1). \tag{2.2}$$

Moreover, $A \in \mathbb{Q}$ if and only if $a_{n+1} = a_n (\geq 2)$ for all sufficiently large n.

Proof. Define $A_1 = A - a_0$, then $0 < A_1 \leq 1$. If $A_n \neq 0 \ (n \geq 1)$ is already defined, put

$$a_n = 1 + \left[\frac{1}{A_n}\right], \tag{2.3}$$

$$A_{n+1} = a_n A_n - 1. \tag{2.4}$$

Observe that a_n is the least integer $> 1/A_n$ and

$$\frac{1}{a_n} < A_n \leq \frac{1}{a_n - 1}. \tag{2.5}$$

We now prove the folowing.

Claim. We have $0 < \cdots \leq A_{n+1} \leq A_n \leq \cdots \leq A_2 \leq A_1 \leq 1$.

Proof of the Claim. First, we show that $A_n > 0$ for all $n \geq 1$ by induction. If $n = 1$, we have seen that $A_1 > 0$. Assume now that $A_n > 0$ for $n \geq 1$. By (2.3), we see that $a_n \in \mathbb{N}$. Since

$$A_{n+1} = a_n A_n - 1 = \left(A_n - \frac{1}{a_n}\right) a_n \tag{2.6}$$

and $1/a_n < A_n$, we have $A_{n+1} > 0$. If there exists $k \in \mathbb{N}$ such that $A_{k+1} > A_k$, then

$$a_k A_k - 1 = A_{k+1} > A_k \tag{2.7}$$

and so $a_k - 1 > 1/A_k$, contradicting the minimal property of a_n and the Claim is proved. □

From the Claim and (2.3), we deduce that $a_1 \geq 2$ and $a_{n+1} \geq a_n$ $(n \geq 1)$. Iterating (2.4), we get

$$A_1 = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \cdots a_n} + \frac{A_{n+1}}{a_1 a_2 \cdots a_n}. \tag{2.8}$$

To establish convergence, let

$$B_n = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \cdots a_n} \quad (n \geq 1). \tag{2.9}$$

Since $A_n > 0$ and $a_n \in \mathbb{N}$ for all $n \geq 1$, the sequence of real numbers (B_n) is increasing and bounded above by A_1. Thus, $\lim_{n\to\infty} B_n$ exists and so

$$\frac{1}{a_1 a_2 \cdots a_n} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.10}$$

By the Claim,

$$0 < \frac{A_{n+1}}{a_1 a_2 \cdots a_n} \leq \frac{1}{a_1 a_2 \cdots a_n} \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{2.11}$$

showing that any real number has an ES expansion. To prove uniqueness, assume that we have two infinite such expansions such that

$$a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n} = b_0 + \sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \cdots b_n}, \tag{2.12}$$

with the restrictions $a_0 \in \mathbb{Z}$, $a_1 \geq 2$, $a_{n+1} \geq a_n$ $(n \geq 1)$ and the same restrictions also for the b_n's. From the restrictions, we note that

$$0 < A_1 := \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n} \leq 1. \tag{2.13}$$

If $A_1 = 1$, then by (2.12) we also have $\sum_{n\geq 1} 1/b_1 b_2 \cdots b_n = 1$, forcing $a_0 = b_0$. If $0 < A_1 < 1$, then (2.12) shows that $0 < \sum_{n\geq 1} 1/b_1 b_2 \cdots b_n < 1$, forcing again $a_0 = b_0$. In either case, cancelling out the terms a_0, b_0 in (2.12) we get

$$A_1 := \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n} = \sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \cdots b_n}. \tag{2.14}$$

Since $a_{n+1} \geq a_n$, then

$$a_1 A_1 - 1 = \frac{1}{a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_2 a_3 a_4} + \cdots \leq \frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \cdots = A_1, \tag{2.15}$$

so $0 < a_1 - 1/A_1 \leq 1$. But there is exactly one integer a_1 satisfying these restrictions. Thus, $a_1 = b_1$. Cancelling out the terms a_1 and b_1 in (2.14) and repeating the arguments we see that $a_i = b_i$ for all i.

Concerning the rationality characterization, if its ES expansion is infinite periodic of period 1, it clearly represents a rational number. To prove its converse, let $A = a/b \in \mathbb{Q} \setminus \{0\}$. Since

$$A_1 = A - a_0 = \frac{a - ba_0}{b}, \tag{2.16}$$

we see that A_1 is a rational number in the interval $(0, 1]$ whose denominator is b. In general, from (2.4), we deduce that each A_n $(n \geq 1)$ is a rational number in the interval $(0, 1]$ whose denominator is b. But the number of rational numbers in the interval $(0, 1]$ whose denominator is b is finite implying that there are two least suffixes $h, k \in \mathbb{N}$ such that $A_{h+k} = A_h$. Thus, by (2.3), we have $a_{h+k} = a_h$. From (2.2), we know that the sequence $\{a_n\}$ is increasing. We must then have $k = 1$ and the assertion follows. □

Remarks. In passing, we make the following observations.

(a) For $n \geq 1$, we have

$$a_{n+1} = a_n \Longleftarrow A_{n+1} = A_n \Longleftrightarrow a_n A_n - 1 = A_n \Longleftrightarrow a_n = 1 + \frac{1}{A_n} \Longleftrightarrow \frac{1}{A_n} \in \mathbb{Z}. \tag{2.17}$$

(b) If $A \in \mathbb{Q}^c$, then $A_n \in \mathbb{Q}^c$ and so $1/A_n \notin \mathbb{Q}$ for all $n \geq 1$.

(c) If $A \in \mathbb{Z}$, then its ES expansion is

$$A = A - 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots . \tag{2.18}$$

To construct a Cohen-Egyptian fraction expansion, we proceed as in [2] making use of the following lemma.

Lemma 2.2. *For any $y \in (0, 1)$, there exist a unique integer $n \geq 2$ and a unique $r \in \mathbb{R}$ such that*

$$1 = ny - r, \quad 0 \leq r < y. \tag{2.19}$$

Proof. Let $n = [1/y] \in \mathbb{N}$ and $r = ny - 1$. Put $\langle 1/y \rangle := n - 1/y \in [0, 1)$ and so

$$r = ny - 1 = y\left\langle \frac{1}{y} \right\rangle \in [0, y). \tag{2.20}$$

To prove uniqueness, assume $ny - r = 1 = my - s$ so that

$$1 + \frac{1}{y} > n = \frac{1 + r}{y} \geq \frac{1}{y}. \tag{2.21}$$

Since there is only one integer with this property, we deduce $n = m$ and consequently, $r = s$ proving the lemma. □

Theorem 2.3. *Each $A \in \mathbb{R} \setminus \{0\}$ is uniquely representable as a CEF expansion of the form*

$$A = n_0 + \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k}, \tag{2.22}$$

subject to the condition

$$n_0 \in \mathbb{Z}, \quad n_1 \geq 2, \quad n_{k+1} \geq n_k \quad (k \geq 1), \tag{2.23}$$

and no term of the sequence appears infinitely often. Moreover, each CEF expansion terminates if and only if it represents a rational number.

Proof. To construct a CEF expansion for $A \in \mathbb{R} \setminus \{0\}$, define

$$r_0 = A - n_0 \in [0, 1). \tag{2.24}$$

If $r_0 = 0$, then the process stops and we write $A = n_0$. If $r_0 \neq 0$, by Lemma 2.2, there are unique $n_1 \in \mathbb{N}$ and $r_1 \in \mathbb{R}$ such that

$$1 = n_1 r_0 - r_1, \quad 0 \leq r_1 < r_0, \ n_1 \geq 2. \tag{2.25}$$

Thus,

$$A = n_0 + r_0 = n_0 + \frac{1}{n_1} + \frac{r_1}{n_1}. \tag{2.26}$$

If $r_1 = 0$, then the process stops and we write $A = n_0 + 1/n_1$. If $r_1 \neq 0$, by Lemma 2.2, there are $n_2 \in \mathbb{N}$ and $r_2 \in \mathbb{R}$ such that

$$1 = n_2 r_1 - r_2, \quad 0 \leq r_2 < r_1, \ n_2 \geq n_1, \tag{2.27}$$

the last inequality being followed from $n_1 = \lceil 1/r_0 \rceil$, $n_2 = \lceil 1/r_1 \rceil$, and $r_1 < r_0$. Observe also that

$$A = n_0 + \frac{1}{n_1} + \frac{1}{n_1 n_2} + \frac{r_2}{n_1 n_2}. \tag{2.28}$$

Continuing this process, we get

$$A = n_0 + \frac{1}{n_1} + \frac{1}{n_1 n_2} + \cdots + \frac{1}{n_1 n_2 \cdots n_k} + \frac{r_k}{n_1 n_2 \cdots n_k}, \tag{2.29}$$

with

$$1 = n_i r_{i-1} - r_i, \quad 1 > r_{i-1} > r_i \geq 0, \ 2 \leq n_i \leq n_{i+1} \ (i = 1, 2, \ldots). \tag{2.30}$$

If some $r_k = 0$, then the process stops, otherwise the series convergence follows at once from

$$\left| \frac{r_k}{n_1 n_2 \cdots n_k} \right| \longrightarrow 0 \quad (k \longrightarrow \infty). \tag{2.31}$$

To prove uniqueness, let

$$n_0 + \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} = A = m_0 + \sum_{k=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_k}, \tag{2.32}$$

with the restrictions (2.23) on both digits n_i and m_j. Now

$$\sum_{k \geq 1} \frac{1}{n_1 n_2 \cdots n_k} \leq \sum_{k \geq 1} \frac{1}{2^k} = 1. \tag{2.33}$$

It is clear that the restrictions (2.23) imply the strict inequality in (2.33). This also applies to the right-hand sum in (2.32). Equating integer and fractional parts in (2.32), we get

$$n_0 = m_0, \quad \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} = \sum_{k=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_k} =: w, \ \text{say}. \tag{2.34}$$

Since $n_{k+1} \geq n_k$, then

$$n_1 w - 1 = \frac{1}{n_2} + \frac{1}{n_2 n_3} + \frac{1}{n_2 n_3 n_4} + \cdots \leq \frac{1}{n_1} + \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} + \cdots = w, \tag{2.35}$$

so $0 < n_1 - 1/w \leq 1$. But there is exactly one integer n_1 satisfying these restrictions. Then $n_1 = m_1$ and

$$\sum_{k \geq 2} \frac{1}{n_2 \cdots n_k} = \sum_{k \geq 2} \frac{1}{m_2 \cdots m_k}. \tag{2.36}$$

Proceeding in the same manner, we conclude that $n_i = m_i$ for all i.

Finally, we look at its rationality characterization. If $A \in \mathbb{Q}$, then $r_0 \in \mathbb{Q}$, say $r_0 := p/q$, where $p, q \in \mathbb{N}$. From (2.30), we see that each r_i is a rational number whose denominator is q. Using this fact and the second inequality condition in (2.30), we deduce that $r_j = 0$ for some $j \leq p$, that is, the expansion terminates. On the other hand, it is clear that each terminating

CEF expansion represents a rational number. Now suppose that A is irrational and there is a j and integer n such that $n_i = n$ for all $i \geq j$. Then

$$A = n_0 + \sum_{k=1}^{j} \frac{1}{n_1 n_2 \cdots n_k} + \frac{1}{n_1 n_2 \cdots n_j} \sum_{k=1}^{\infty} \frac{1}{n^k}. \tag{2.37}$$

Since $\sum_{k\geq 1} 1/n^k = 1/(n-1)$, it follows that A is rational, which is impossible. □

The connection and distinction between ES and CEF expansions of a real number are described in the next theorem.

Theorem 2.4. *Let $A \in \mathbb{R} \setminus \{0\}$ and the notation be as set out in Theorems 2.1 and 2.3.*

(i) *If $A \in \mathbb{Q}$, then its ES expansion is infinite periodic of period 1, while its CEF expansion is finite. More precisely, for $A \in \mathbb{Q} \setminus \mathbb{Z}$, let its ES and CEF expansions be, respectively,*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n} = n_0 + \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k}. \tag{2.38}$$

If m is the least positive integer such that $1/A_m \in \mathbb{Z}$, then

$$a_0 = n_0, \quad a_1 = n_1, \ldots, \quad a_{m-1} = n_{m-1}, \quad a_m = n_m + 1, \quad a_m = a_{m+i} \quad (i \geq 1), \tag{2.39}$$

and the digits n_i terminate at n_m.

(ii) *If $A \in \mathbb{R} \setminus \mathbb{Q}$, then its ES and its CEF expansions are identical.*

Proof. Both assertions follow mostly from Theorems 2.1, 2.3, and Remark (b) except for the result related to the expansions in (2.38) which we show now.

Let $A \in \mathbb{Q} \setminus \mathbb{Z}$ and let m be the least positive integer such that $1/A_m \in \mathbb{Z}$. We treat two seperate cases.

Case 1 ($m = 1$). In this case, we have $1/A_1 \in \mathbb{Z}$ and $a_1 = 1 + [1/A_1] = 1 + 1/A_1$. Since $r_0 = A - n_0 = A - [A] = A - a_0 = A_1$, we get $n_1 = 1/A_1$ and so $a_1 = n_1 + 1$. We have $r_1 = n_1 r_0 - 1 = 0$, and so the CEF expansion terminates. On the other hand, by Remark (a) after Theorem 2.1, we have $a_1 = a_i$ $(i \geq 2)$.

Case 2 ($m > 1$). Thus, $1/A_1 \notin \mathbb{Z}$ and $A_1 = r_0$. By Lemma 2.2, we have $a_1 = n_1$. For $1 \leq i \leq m-2$, assume that $A_i = r_{i-1}$ and $a_i = n_i$. Then

$$A_{i+1} = a_i A_i - 1 = n_i r_{i-1} - 1 = r_i. \tag{2.40}$$

Since $1/A_{i+1} \notin \mathbb{Z}$, again by Lemma 2.2, $a_{i+1} = n_{i+1}$. This shows that $a_1 = n_1, \ldots, a_{m-1} = n_{m-1}$. Since $1/A_m \in \mathbb{Z}$, we have $a_m = 1 + [1/A_m] = 1 + 1/A_m$ and thus

$$A_m = a_{m-1} A_{m-1} - 1 = n_{m-1} r_{m-2} - 1 = r_{m-1}. \tag{2.41}$$

From the construction of CEF, we know that $n_m = [1/r_{m-1}]$. Thus, $n_m = 1/A_m$ showing that $a_m = n_m + 1$. Furthermore, $r_m = n_m r_{m-1} - 1 = 0$, implying that the CEF terminates at n_m, and by Remark (a) after Theorem 2.1, $a_m = a_{m+i}$ $(i \geq 1)$. □

3. The Non-Archimedean Case

We recapitulate some facts about discrete-valued non-archimedean fields taken from [3, Chapter 4]. Let K be a field complete with respect to a discrete non-archimedean valuation $|\cdot|$ and $\mathcal{O} := \{A \in K; |A| \leq 1\}$ its ring of integers. The set $\mathcal{P} := \{A \in K; |A| < 1\}$ is an ideal in $\mathcal{O}$ which is both a maximal ideal and a principal ideal generated by a prime element $\pi \in K$. The quotient ring $\mathcal{O}/\mathcal{P}$ is a field, called the residue class field. Let $\mathcal{A} \subset \mathcal{O}$ be a set of representatives of $\mathcal{O}/\mathcal{P}$. Every $A \in K \setminus \{0\}$ is uniquely of the shape

$$A = \sum_{n=N}^{\infty} b_n \pi^n \quad (b_n \in \mathcal{A},\ b_N \neq 0) \tag{3.1}$$

for some $N \in \mathbb{Z}$, and define the *order* $v(A)$ of A by $|A| = 2^{-v(A)} = 2^{-N}$, with $v(0) := +\infty$. The *head part* $\langle A \rangle$ of A is defined as the finite series

$$\langle A \rangle = \sum_{n=v(A)}^{0} b_n \pi^n \quad \text{if } v(A) \leq 0, \text{ and } 0 \text{ otherwise.} \tag{3.2}$$

Denote the set of all head parts by

$$S := \{\langle A \rangle; A \in K\}. \tag{3.3}$$

The Knopfmachers' series expansion algorithm for series expansions in K [4] proceeds as follows. For $A \in K$, let

$$a_0 := \langle A \rangle \in S. \tag{3.4}$$

Define

$$A_1 := A - a_0. \tag{3.5}$$

If $A_n \neq 0$ $(n \geq 1)$ is already defined, put

$$a_n = \left\langle \frac{1}{A_n} \right\rangle, \qquad A_{n+1} = \left(A_n - \frac{1}{a_n} \right) \frac{s_n}{r_n} \tag{3.6}$$

if $a_n \neq 0$, where r_n and $s_n \in K \setminus \{0\}$ which may depend on $a_1, \ldots, a_n$. Then for $n \geq 1$

$$A = a_0 + A_1 = \cdots = a_0 + \frac{1}{a_1} + \frac{r_1}{s_1}\frac{1}{a_2} + \cdots + \frac{r_1 \cdots r_{n-1}}{s_1 \cdots s_{n-1}}\frac{1}{a_n} + \frac{r_1 \cdots r_n}{s_1 \cdots s_n} A_{n+1}. \tag{3.7}$$

The process ends in a finite expansion if some $A_{n+1} = 0$. If some $a_n = 0$, then A_{n+1} is not defined. To take care of this difficulty, we impose the condition

$$v(s_n) - v(r_n) \geq 2v(a_n) - 1. \tag{3.8}$$

Thus

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{r_1 \cdots r_n}{s_1 \cdots s_n} \cdot \frac{1}{a_{n+1}}. \tag{3.9}$$

When $r_n = 1$, $s_n = a_n$, the algorithm yields a well-defined (with respect to the valuation) and unique series expansion, termed *non-archimedean Engel series expansion*. Summing up, we have the following.

Theorem 3.1. *Every $A \in K \setminus \{0\}$ has a finite or an infinite convergent non-archimedean ES expansion of the form*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n}, \tag{3.10}$$

where the digits a_n are subject to the restrictions

$$a_0 = \langle A \rangle \in S, \quad a_n \in S, \quad v(a_n) \le -n, \quad v(a_{n+1}) \le v(a_n) - 1 \quad (n \ge 1). \tag{3.11}$$

Now we turn to the construction of a non-archimedean Cohen-Egyptian fraction expansion, in the same spirit as that of the real numbers, that is, by way of Lemma 2.2. To this end, we start with the following lemma.

Lemma 3.2. *For any $y \in K \setminus \{0\}$ such that $v(y) \ge 1$, there exist a unique $n \in S$ such that $v(n) \le -1$ and a unique $r \in K$ such that*

$$1 = ny - r, \quad v(r) \ge v(y) + 1 \quad (\textit{i.e.}, \ 0 \le |r| < |y|). \tag{3.12}$$

Proof. Let $n = \langle 1/y \rangle$. Then $v(n) = v(1/y) = -v(y) \le -1$. Putting $r = ny - 1$, we show now that $v(r) \ge v(y) + 1$. Since $n = \langle 1/y \rangle$, we have

$$\frac{1}{y} = n + \sum_{k \ge 1} c_k \pi^k, \tag{3.13}$$

where $c_k \in \mathcal{A}$, and so

$$ny - 1 = -y \sum_{k \ge 1} c_k \pi^k. \tag{3.14}$$

Thus

$$v(r) = v(ny - 1) = v(-y) + v\left(\sum_{k \ge 1} c_k \pi^k\right) \ge v(y) + 1 > v(y). \tag{3.15}$$

To prove the uniqueness, assume that there exist $n_1 \in S$ such that $v(n_1) \leq -1$ and $r_1 \in K$ such that

$$1 = n_1 y - r_1, \quad 0 \leq |r_1| < |y|. \tag{3.16}$$

From $ny - r = 1 = n_1 y - r_1$, we get $(n - n_1)y = r - r_1$. If $n \neq n_1$, since $n, n_1 \in S$ we have $|n - n_1| \geq 1$. Using $|y| > |r - r_1|$, we deduce that

$$|r - r_1| < |y| \leq |n - n_1||y| = |r - r_1|, \tag{3.17}$$

which is a contradiction. Thus, $n = n_1$ and so $r = r_1$. □

For a non-archimedean CEF expansion, we now prove the following.

Theorem 3.3. *Each $y \in K \setminus \{0\}$ has a non-archimedean CEF expansion of the form*

$$y = n_0 + \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k}, \tag{3.18}$$

where

$$n_k \in S, \quad v(n_k) \leq -k, \quad v(n_{k+1}) \leq v(n_k) - 1 \quad (k \geq 1). \tag{3.19}$$

This series representation is unique subject to the digit condition (3.19).

Proof. Define $n_0 = \langle y \rangle$ and $r_0 = y - n_0$. Then $v(r_0) \geq 1$. If $r_0 = 0$, the process stops and we write $y = n_0$. If $r_0 \neq 0$, by Lemma 3.2, there are $n_1 \in S$ and $r_1 \in K$ such that

$$n_1 = \left\langle \frac{1}{r_0} \right\rangle, \quad r_1 = n_1 r_0 - 1, \tag{3.20}$$

where $v(n_1) \leq -1$ and $v(r_1) \geq v(r_0) + 1$. So

$$y = n_0 + r_0 = n_0 + \frac{1}{n_1} + \frac{r_1}{n_1}. \tag{3.21}$$

If $r_1 = 0$, the process stops and we write $y = n_0 + 1/n_1$. If $r_1 \neq 0$, by Lemma 3.2, there are $n_2 \in S$ and $r_2 \in K$ such that

$$n_2 = \left\langle \frac{1}{r_1} \right\rangle, \quad r_2 = n_2 r_1 - 1, \tag{3.22}$$

where $v(n_2) \leq -1$ and $v(r_2) \geq v(r_1) + 1$. So

$$y = n_0 + \frac{1}{n_1} + \frac{1}{n_1 n_2} + \frac{r_2}{n_1 n_2}. \tag{3.23}$$

Continuing the process, in general,

$$n_k = \left\langle \frac{1}{r_{k-1}} \right\rangle, \quad r_k = n_k r_{k-1} - 1,$$
$$y = n_0 + \frac{1}{n_1} + \frac{1}{n_1 n_2} + \cdots + \frac{1}{n_1 n_2 \cdots n_k} + \frac{r_k}{n_1 n_2 \cdots n_k}, \tag{3.24}$$

where

$$n_k \in S, \quad v(n_k) \le -1, \quad v(r_k) \ge v(r_{k-1}) + 1 \quad (k \ge 1). \tag{3.25}$$

Thus,

$$v(n_{k+1}) = -v(r_k) \le -v(r_{k-1}) - 1 = v(n_k) - 1 \quad (k \ge 1). \tag{3.26}$$

We observe that the process terminates if $r_k = 0$. Next, we show that $v(n_k) \le -k$ $(k \ge 1)$. By construction, we have $v(n_1) \le -1$. Assume that $v(n_k) \le -k$, then

$$v(n_{k+1}) \le v(n_k) - 1 \le -k - 1. \tag{3.27}$$

Regarding convergence, consider

$$\begin{aligned} v\left(\frac{r_k}{n_1 n_2 \cdots n_k}\right) &= -v(n_1) - v(n_2) - \cdots - v(n_k) + v(r_k) \\ &= -v(n_1) - v(n_2) - \cdots - v(n_k) - v(n_{k+1}) \\ &\ge 1 + 2 + \cdots + k + (k+1) \longrightarrow \infty \quad (k \longrightarrow \infty). \end{aligned} \tag{3.28}$$

It remains to prove the uniqueness. Suppose that $x \in K \setminus \{0\}$ has two such expansions

$$x = n_0 + \sum_j \frac{1}{n_1 n_2 \cdots n_j} = m_0 + \sum_i \frac{1}{m_1 m_2 \cdots m_i}. \tag{3.29}$$

Since $v(\sum_j 1/n_1 n_2 \cdots n_j) = v(1/n_1) \ge 1$ and $n_0 \in S$, we have $n_0 = \langle y \rangle$. Similarly, $m_0 = \langle y \rangle$ yielding by uniqueness $n_0 = m_0$ and $\sum_{j\ge 1} 1/n_1 n_2 \cdots n_j = \sum_{i \ge 1} 1/m_1 m_2 \cdots m_i$. Putting

$$\omega := \sum_{j \ge 1} \frac{1}{n_1 n_2 \cdots n_j} = \sum_{i \ge 1} \frac{1}{m_1 m_2 \cdots m_i}, \tag{3.30}$$

we have $n_1\omega = 1 + \sum_{j\geq 2} 1/n_2 \cdots n_j$ and so

$$1 = n_1\omega - \sum_{j\geq 2} \frac{1}{n_2 \cdots n_j}, \qquad v\left(\sum_{j\geq 2} \frac{1}{n_2 \cdots n_j}\right) = v\left(\frac{1}{n_2}\right) = -v(n_2) \geq -v(n_1) + 1 = v(\omega) + 1. \tag{3.31}$$

By Lemma 3.2, since n_1 is the unique element in S with such property, we deduce $n_1 = m_1$. Continuing in the same manner, we conclude that the two expansions are identical. □

It is clear that the construction of non-archimedean ES and CEF expansions is identical which implies at once that the two representations are exactly the same in the non-archimedean case.

4. Rationality Characterization in the Non-Archimedean Case

In the case of real numbers, we have seen that both ES and CEF expansions can be used to characterize rational numbers with quite different outcomes. In the non-archimedean situation, though ES and CEF expansions are identical, their use to characterize rational elements depend significantly on the underlying nature of each specific field. We end this paper by providing information on the rationality characterization in three different non-archimedean fields, namely, the field of p-adic numbers and the two function fields, one completed with respect to the degree valuation and the other with respect to a prime-adic valuation.

The following characterization of rational numbers by p-adic ES expansions is due to Grabner and Knopfmacher [5].

Theorem 4.1. *Let $x \in p\mathbb{Z}_p \setminus \{0\}$. Then x is rational, $x = \alpha/\beta$, if and only if either the p-adic ES expansion of x is finite, or there exist an m and an $s \geq m$, such that*

$$a_{m+j} = \frac{p^{s+j+1} - \gamma}{p^{s+j}} \quad (j = 0, 1, 2, \ldots), \tag{4.1}$$

where $\gamma \mid \beta$.

Now for function fields, we need more terminology. Let $\mathbb{F}$ denote a field and $\pi(x)$ an irreducible polynomial of degree d over $\mathbb{F}$. There are two types of valuation in the field of rational functions $\mathbb{F}(x)$, namely, the $\pi(x)$-adic valuation $|\cdot|_\pi$, and the degree valuation $|\cdot|_{1/x}$ defined as follows. From the unique representation in $\mathbb{F}(x)$,

$$\frac{f(x)}{g(x)} = \pi(x)^m \frac{r(x)}{s(x)}, \quad f(x), g(x), r(x), s(x) \in \mathbb{F}[x] \setminus \{0\};\ \pi(x) \nmid r(x)s(x),\ m \in \mathbb{Z}, \tag{4.2}$$

set

$$|0|_\pi = 0, \quad \left|\frac{f(x)}{g(x)}\right|_\pi = 2^{-md}; \quad |0|_{1/x} = 0, \quad \left|\frac{f(x)}{g(x)}\right|_{1/x} = 2^{\deg f(x) - \deg g(x)}. \tag{4.3}$$

Let $\mathbb{F}((\pi))$ and $\mathbb{F}((1/x))$ be the completions of $\mathbb{F}(x)$, with respect to the $\pi(x)$-adic and the degree valuations, respectively. The extension of the valuations to $\mathbb{F}((\pi))$ and $\mathbb{F}((1/x))$ is also denoted by $|\cdot|_\pi$ and $|\cdot|_{1/x}$.

For a characterization of rational elements, we prove the following.

Theorem 4.2. *The CEF of $y \in \mathbb{F}((\pi))$ or in $\mathbb{F}((1/x))$ terminates if and only if $y \in \mathbb{F}(x)$.*

Proof. Although the assertions in both fields $\mathbb{F}((\pi))$ and $\mathbb{F}((1/x))$ are the same, their respective proofs are different. In fact, when the field $\mathbb{F}$ has finite characteristic, both results have already been shown in [6] and the proof given here is basically the same.

We use the notation of the last section with added subscripts π or $1/x$ to distinguish their corresponding meanings.

If the CEF of y in either field is finite, then y is clearly rational. It remains to prove the converse and we begin with the field $\mathbb{F}((\pi))$. Assume that $y \in \mathbb{F}(x) \setminus \{0\}$. By construction, each $r_k \in \mathbb{F}(x)$ and so can be uniquely represented in the form

$$r_k = \pi(x)^{v(r_k)} \frac{p_k(x)}{q_k(x)}, \tag{4.4}$$

where $p_k(x),\ q_k(x)\ (\neq 0) \in \mathbb{F}[x]$ with gcd $(p_k(x), q_k(x)) = 1,\ \pi(x) \nmid p_k(x)q_k(x)$. Since $n_k = \langle 1/r_{k-1} \rangle \in S_\pi$ and $v(n_k) \leq -k$, it is of the form

$$\begin{aligned} n_k &= s_{v(n_k)}(x)\pi(x)^{v(n_k)} + s_{v(n_k)+1}(x)\pi(x)^{v(n_k)+1} + \cdots + s_{-1}(x)\pi(x)^{-1} + s_0(x) \\ &=: m_k(x)\pi(x)^{v(n_k)}, \end{aligned} \tag{4.5}$$

where $s_{v(n_k)}(x), \ldots, s_0(x)$ are polynomials over $\mathbb{F}$, not all 0, of degree $< d$ and $m_k(x) \in \mathbb{F}[x]$. Thus,

$$\begin{aligned} |n_k|_{1/x} \leq \max\Big\{ &\left|s_{v(n_k)}(x)\pi(x)^{v(n_k)}\right|_{1/x}, \left|s_{v(n_k)+1}(x)\pi(x)^{v(n_k)+1}\right|_{1/x}, \ldots, \\ &\left|s_{-1}(x)\pi(x)^{-1}\right|_{1/x}, |s_0(x)|_{1/x} \Big\} \\ \leq 2^{d-1}&, \end{aligned} \tag{4.6}$$

yielding

$$|m_k(x)|_{1/x} \leq 2^{d-dv(n_k)-1}. \tag{4.7}$$

By construction, we have

$$r_k = n_k r_{k-1} - 1. \tag{4.8}$$

Substituting (4.4) and (4.5) into (4.8) and using $v(r_{k-1}) = -v(n_k)$ lead to

$$\pi(x)^{-v(n_{k+1})} p_k(x) q_{k-1}(x) = q_k(x)\big(m_k(x) p_{k-1}(x) - q_{k-1}(x)\big). \tag{4.9}$$

Since gcd $(\pi(x)^{-v(n_{k+1})} p_k(x), q_k(x)) = 1$, it follows that $q_k(x) \mid q_{k-1}(x)$, and so successively, we have

$$\left|q_k(x)\right|_{1/x} \le \left|q_{k-1}(x)\right|_{1/x} \le \cdots \le \left|q_1(x)\right|_{1/x}, \tag{4.10}$$

which together with (4.9) yield

$$\left|p_k(x)\right|_{1/x} \le |\pi(x)|_{1/x}^{v(n_{k+1})} \max\left\{ \left|m_k(x) p_{k-1}(x)\right|_{1/x}, \left|q_1(x)\right|_{1/x} \right\}. \tag{4.11}$$

Using (3.19) and (4.7), we consequently have

$$\begin{aligned} \left|p_k(x)\right|_{1/x} &\le 2^{d(v(n_k)-1)} \max\left\{ 2^{d-dv(n_k)-1} \left|p_{k-1}(x)\right|_{1/x}, \left|q_1(x)\right|_{1/x} \right\} \\ &\le \max\left\{ \frac{1}{2}\left|p_{k-1}(x)\right|_{1/x}, \frac{\left|q_1(x)\right|_{1/x}}{2^{d(k+1)}} \right\}. \end{aligned} \tag{4.12}$$

This shows that $|p_k(x)|_{1/x} \le (1/2)|p_{k-1}(x)|_{1/x}$ for all large k which implies that from some k onwards, $p_k(x) = 0$, and so $r_k = 0$, that is, the expansion terminates.

Finally for the field $\mathbb{F}((1/x))$, assume that $y = p(x)/q(x) \in \mathbb{F}(x) \setminus \{0\}$. Without loss of generality, assume $\deg p(x) \ge \deg q(x)$. By the Euclidean algorithm, we have

$$y = \frac{p(x)}{q(x)} = N_0(x) + \frac{R_0(x)}{q(x)} := n_0 + r_0, \tag{4.13}$$

where

$$n_0 := N_0(x) = \langle y \rangle_{1/x}, \quad R_0(x) \in \mathbb{F}[x], \quad 0 \le \deg R_0 < \deg q, \quad r_0 = \frac{R_0(x)}{q(x)}. \tag{4.14}$$

From the Euclidean algorithm,

$$\frac{q(x)}{R_0(x)} = N_1(x) + \frac{R_1(x)}{R_0(x)}, \quad N_1(x), R_1(x) \in \mathbb{F}[x], \quad 0 \le \deg R_1 < \deg R_0 < \deg q, \tag{4.15}$$

which is, in the terminology of Lemma 3.2,

$$1 = r_0 N_1 + \frac{R_1}{q} = r_0 n_1 - r_1. \tag{4.16}$$

Again, from the Euclidean algorithm,

$$\frac{-q(x)}{R_1(x)} = N_2(x) + \frac{R_2(x)}{R_1(x)}, \quad N_2(x), R_2(x) \in \mathbb{F}[x], \quad 0 \le \deg R_2 < \deg R_1 < \deg R_0 < \deg q, \tag{4.17}$$

which is, in the terminology of Lemma 3.2,

$$1 = r_1 N_2 - \frac{R_2}{q} = r_1 n_2 - r_2. \tag{4.18}$$

Proceeding in the same manner, in general we have

$$r_j = (-1)^j \frac{R_j}{q}, \quad 0 \le \deg R_j < \deg R_{j-1} < \cdots < \deg R_1 < \deg q. \tag{4.19}$$

There must then exist $k \in \mathbb{N}$ such that $\deg R_k = 0$, that is, $R_k \in \mathbb{F} \setminus \{0\}$. Thus, the CEF of y is

$$y = n_0 + \frac{1}{n_1} + \cdots + \frac{1}{n_1 \cdots n_k} + \frac{r_k}{n_1 \cdots n_k} = n_0 + \frac{1}{n_1} + \cdots + \frac{1}{n_1 \cdots n_k} + \frac{1}{n_1 \cdots n_k n_{k+1}}, \tag{4.20}$$

where $n_{k+1} = (-1)^k R_k^{-1} q \in \mathbb{F}[x]$, which is a terminating CEF. □

Acknowledgments

This work was supported by the Commission on Higher Education and the Thailand Research Fund RTA5180005 and by the Centre of Excellence In Mathematics, the Commission on Higher Education.

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