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Research Article

Equivalence of the Apollonian and Its Inner Metric

Peter Hästö,[1] S. Ponnusamy,[2] and S. K. Sahoo[2]

[1] *Department of Mathematical Sciences, University of Oulu, P.O. Box 3000, 90014 Oulu, Finland*
[2] *Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India*

Correspondence should be addressed to Peter Hästö, peter.hasto@helsinki.fi

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We show that the equivalence of the Apollonian metric and its inner metric remains unchanged by the removal of a point from the domain. For this we need to assume that the complement of the domain is not contained in a hyperplane. This improves a result of the authors wherein the same conclusion was reached under the stronger assumption that the domain contains an exterior point.

1. Introduction and the Main Result

The Apollonian metric was first introduced by Barbilian [1] in 1934-35 and then rediscovered by Beardon [2] in 1995. This metric has also been considered in [3–14]. It should also be noted that the same metric has been studied from a different perspective under the name of the Barbilian metric, for instance, in [1, 15–20]; compare, for example, [21] for a historical overview and more references. One interesting historical point, made in [21], is that Barbilian himself proposed the name “Apollonian metric” in 1959, which was later independently coined by Beardon [2]. Recently, the Apollonian metric has also been studied with certain group structures [22].

In this paper we mainly study the equivalence of the Apollonian metric and its inner metric proving a result which is a generalization of Theorem 5.1 in [12]. In addition, we also consider the j_D metric and its inner metric, namely, the quasihyperbolic metric. Inequalities among these metrics (see Table 1) and the geometric characterization of these inequalities in certain domains have been studied in [12, 13]. We start by defining the above metrics and stating our main result. The notation used mostly is from the standard books by Beardon [23] and Vuorinen [24].

We will be considering domains (open connected nonempty sets) D in the Möbius space $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$. The “Apollonian metric” is defined for $x, y \in D \subsetneq \overline{\mathbb{R}^n}$ by

Table 1: Inequalities between the metrics α_D, j_D, $\tilde{\alpha}_D$, and k_D. The subscripts are omitted for clarity with the understanding that every metric is defined in the same domain. The A-column refers to whether the inequality can occur in simply connected planar domains, the B-column refers to whether it can occur in proper subdomains of $\mathbb{R}^n$.

No.	Inequality	A	B	No.	Inequality	A	B
(1)	$\alpha \approx j \approx \tilde{\alpha} \approx k$	+	+	(7)	$\alpha \approx j \ll \tilde{\alpha} \ll k$	−	−
(2)	$\alpha \ll j \approx \tilde{\alpha} \approx k$	−	−	(8)	$\alpha \ll j \ll \tilde{\alpha} \ll k$	−	−
(3)	$\alpha \approx j \approx \tilde{\alpha} \ll k$	−	−	(9)	$\alpha \approx \tilde{\alpha} \ll j \approx k$	−	+
(4)	$\alpha \ll j \approx \tilde{\alpha} \ll k$	−	−	(10)	$\alpha \ll \tilde{\alpha} \ll j \approx k$	−	+
(5)	$\alpha \approx j \ll \tilde{\alpha} \approx k$	+	+	(11)	$\alpha \approx \tilde{\alpha} \ll j \ll k$	−	−
(6)	$\alpha \ll j \ll \tilde{\alpha} \approx k$	+	+	(12)	$\alpha \ll \tilde{\alpha} \ll j \ll k$	−	−

the formula

$$\alpha_D(x,y) := \sup_{a,b\in\partial D} \log \frac{|a-y||b-x|}{|a-x||b-y|}, \tag{1.1}$$

(with the understanding that $|\infty - x|/|\infty - y| = 1$) where ∂D denotes the boundary of D. It is in fact a metric if and only if the complement of D is not contained in a hyperplane and a pseudometric otherwise, as was noted in [2, Theorem 1.1]. Some of the main reasons for the interest in the metric are that

(i) the formula has a very nice geometric interpretation, see Section 2.2,

(ii) it is invariant under Möbius map,

(iii) it equals the hyperbolic metric in balls and half-spaces.

Now we define the inner metric as follows. Let $\gamma : [0,1] \to D \subset \mathbb{R}^n$ be a path, that is, a continuous function. If d is a metric in D, then the d-length of γ is defined by

$$d(\gamma) := \sup \sum_{i=0}^{k-1} d(\gamma(t_i), \gamma(t_{i+1})), \tag{1.2}$$

where the supremum is taken over all $k < \infty$ and all sequences $\{t_i\}$ satisfying $0 = t_0 < t_1 < \cdots < t_k = 1$. All the paths in this paper are assumed to be rectifiable, that is, to have finite Euclidean length. The inner metric of the metric d is defined by the formula

$$\tilde{d}(x,y) := \inf_{\gamma} d(\gamma), \tag{1.3}$$

where the infimum is taken over all paths γ connecting x and y in D. We denote the inner metric of the Apollonian metric by $\tilde{\alpha}_D$ and call it the "Apollonian inner metric". Strictly speaking, the Apollonian inner metric is only a pseudometric in a general domain $D \subsetneq \mathbb{R}^n$; it is a metric if and only if the complement of D is not contained in an $(n-2)$-dimensional plane [10, Theorem 1.2]. We say that a path γ joining x, y is a *geodesic* (of the metric d) if $d(x,y) = d(\gamma)$; there always exists a geodesic path γ for the Apollonian inner metric $\tilde{\alpha}_D$ connecting x and y in D such that $\tilde{\alpha}_D(\gamma) = \tilde{\alpha}_D(x,y)$ [10].

Let $D \subsetneq \mathbb{R}^n$ be a domain and $x, y \in D$. The j_D metric [25], which is a modification of a metric from [26], is defined by

$$j_D(x,y) := \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial D), d(y,\partial D)\}}\right), \tag{1.4}$$

where $d(x,\partial D)$ denotes the shortest Euclidean distance from x to the boundary ∂D of D. The quasihyperbolic metric from [27] is defined by

$$k_D(x,y) := \inf_\gamma \int \frac{|dz|}{d(z,\partial D)}, \tag{1.5}$$

where the infimum is taken over all paths γ joining x and y in D. Note that the quasihyperbolic metric is the inner metric of the j_D metric.

We now recall some relations on the set of metrics in D for an overview of our previous work in [12].

Definition 1.1. Let d and d' be metrics on D.

(i) We write $d \lesssim d'$ if there exists a constant $K > 0$ such that $d \le Kd'$, similarly for the relation $d \gtrsim d'$.

(ii) We write $d \approx d'$ if $d \lesssim d'$ and $d \gtrsim d'$.

(iii) We write $d \ll d'$ if $d \lesssim d'$ and $d \not\gtrsim d'$.

Let us first of all note that the following inequalities hold in every domain $D \subsetneq \mathbb{R}^n$:

$$\alpha_D \lesssim j_D \lesssim k_D, \qquad \alpha_D \lesssim \tilde{\alpha}_D \lesssim k_D. \tag{1.6}$$

The first two are from [2, Theorem 3.2] and the second two are from [7, Remark 5.2, Corollary 5.4]. We see that, of the four metrics to be considered, the Apollonian is the smallest and the quasihyperbolic is the largest.

In this paper we are especially concerned with the relation $\alpha_D \approx \tilde{\alpha}_D$, that is, the question whether or not the Apollonian metric is quasiconvex. We note that this always holds in simply connected uniform planar domains [7, Theorem 1.10, Lemma 6.4]. Also, in convex uniform domains this relation always holds: from [6, Theorem 4.2] we know that $\alpha_D \approx j_D$ in convex domains; additionally, $j_D \approx k_D$ if D is uniform; hence $\tilde{\alpha}_D \lesssim k_D \lesssim j_D \lesssim \alpha_D \lesssim \tilde{\alpha}_D$. On the other hand, there are also domains in which $\alpha_D \ll \tilde{\alpha}_D$, for example, the infinite strip. Finally, we note that in [13, Corollary 1.4] it was shown that $\alpha_D \approx \tilde{\alpha}_D$ implies that D is uniform.

In [12], we have undertaken a systematic study of which of the inequalities in (1.6) can hold in the strong form with $\ll$ and which of the relations $j_D \ll \tilde{\alpha}_D$, $j_D \approx \tilde{\alpha}_D$, and $j_D \gg \tilde{\alpha}_D$ can hold. Thus we are led to twelve inequalities, which are given along with the results in Table 1, where we have indicated in column A whether the inequality can hold in simply connected planar domains and in column B whether it can hold in arbitrary proper subdomains of $\mathbb{R}^n$. Two entries, 11B and 12B, could not be dealt with at that time, but they have meanwhile been resolved in [13]. From the table we see that most of the cases cannot occur, which means that there are many restrictions on which inequalities can occur together.

One ingredient in the proofs of some of the inequalities in [12] was the following result, which shows that removing a point from the domain (i.e., adding a boundary point) does not affect the inequality $\alpha_D \approx \tilde{\alpha}_D$.

Theorem 1.2. *Let $D \subsetneq \mathbb{R}^n$ be a domain with an exterior point. Let $p \in D$ and $G := D \setminus \{p\}$. If $\alpha_D \approx \tilde{\alpha}_D$, then $\alpha_G \approx \tilde{\alpha}_G$ as well.*

Note that by Möbius invariance, one may assume that the exterior point is in fact ∞, in which case the domain is bounded, as was the assumption in the original source. This assumption was of a technical nature, and in this article we show that indeed it can be replaced by a much weaker assumption that the complement of D is not contained in a hyperplane. Note that this is a minimal assumption for α_D to be a metric in the first place, as noted above.

Theorem 1.3 (Main Theorem). *Let $D \subsetneq \mathbb{R}^n$ be a domain whose boundary is not contained in a hyperplane. Let $p \in D$ and $G := D \setminus \{p\}$. If $\alpha_D \approx \tilde{\alpha}_D$, then $\alpha_G \approx \tilde{\alpha}_G$ as well.*

The structure of the rest of this paper is as follows. We start by reviewing the notation and terminology. These tools will be applied in later sections to prove the new results of this article. The main problem in this paper is the inequality $\alpha_G \gtrsim \tilde{\alpha}_G$ where the integral representation [10, Theorem 1.4] of the Apollonian inner metric plays a crucial rule. The main result shows that if the boundary of the domain contains $n+1$ points which form extreme points of an n-simplex, then the equivalence of the Apollonian metric and its inner metric will remain unchanged even if we remove a point from the original domain.

2. Background

2.1. Notation

The notation used conforms largely to that in [23, 24], as was mentioned in Section 1.

We denote by $\{e_1, e_2, \ldots, e_n\}$ the standard basis of $\mathbb{R}^n$ and by n the dimension of the Euclidean space under consideration and assume that $n \geq 2$. For $x \in \mathbb{R}^n$ we denote by x_i its ith coordinate. The following notation is used for Euclidean balls and spheres:

$$\begin{gathered} B^n(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}, \qquad S^{n-1}(x,r) := \{y \in \mathbb{R}^n : |x-y| = r\}, \\ B^n := B^n(0,1), \qquad S^{n-1} := S^{n-1}(0,1). \end{gathered} \tag{2.1}$$

For $x, y, z \in \mathbb{R}^n$ we denote by $\widehat{xyz}$ the smallest angle between the vectors $x-y$ and $z-y$ at 0.

We use the notation $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$ for the one-point compactification of $\mathbb{R}^n$, equipped with the chordal metric. Thus an open ball of $\overline{\mathbb{R}^n}$ is an open Euclidean ball, an open half-space, or the complement of a closed Euclidean ball. We denote by ∂G, G^c, and $\overline{G}$ the boundary, complement, and closure of G, respectively, all with respect to $\overline{\mathbb{R}^n}$.

We also need some notation for quantities depending on the underlying Euclidean metric. For $x \in G \subsetneq \mathbb{R}^n$ we write

$$\delta(x) := d(x, \partial G) := \min\{|x-z| : z \in \partial G\}. \tag{2.2}$$

For a path γ in $\mathbb{R}^n$ we denote by $\ell(\gamma)$ its Euclidean length.

2.2. The Apollonian Balls Approach

In this subsection we present the Apollonian balls approach which gives a geometric interpretation of the Apollonian metric.

For $x, y \in G \subsetneq \overline{\mathbb{R}^n}$ we define

$$q_x := \sup_{a \in \partial G} \frac{|a-y|}{|a-x|}, \qquad q_y := \sup_{b \in \partial G} \frac{|b-x|}{|b-y|}. \tag{2.3}$$

The numbers q_x and q_y are called the Apollonian parameters of x and y (with respect to G) and by the definition

$$\alpha_G(x, y) = \log(q_x q_y). \tag{2.4}$$

The balls (in $\overline{\mathbb{R}^n}$!),

$$B_x := \left\{ z \in \overline{\mathbb{R}^n} : \frac{|z-x|}{|z-y|} < \frac{1}{q_x} \right\}, \qquad B_y := \left\{ z \in \overline{\mathbb{R}^n} : \frac{|z-y|}{|z-x|} < \frac{1}{q_y} \right\}, \tag{2.5}$$

are called the *Apollonian balls* about x and y, respectively. We collect some immediate results regarding these balls; similar results obviously hold with x and y interchanged.

(1) $x \in B_x \subset G$ and $\overline{B_x} \cap \partial G \neq \emptyset$.

(2) If i_x and i_y denote the inversions in the spheres ∂B_x and ∂B_y, then

$$y = i_x(x) = i_y(x). \tag{2.6}$$

(3) If $\infty \notin G$, we have $q_x \geq 1$. If, moreover, $\infty \notin \overline{G}$, then $q_x > 1$.

2.3. Uniformity

Uniform domains were introduced by Martio and Sarvas in [28, 2.12], but the following definition is an equivalent form from [26, equation (1.1)]. In the paper in [29] there is a survey of characterizations and implications of uniformity; as an example we mention that a Sobolev mapping can be extended from G to the whole space if G is uniform; see [30].

Definition 2.1. A domain $G \subsetneq \mathbb{R}^n$ is said to be *uniform* with constant K if for every $x, y \in G$ there exists a path γ, parameterized by arc-length, connecting x and y in G, such that

(1) $\ell(\gamma) \leq K|x-y|$,

(2) $K\delta(\gamma(t)) \geq \min\{t, \ell(\gamma) - t\}$.

The relevance of uniformity to our investigation comes from [26, Corollary 1] which states that a domain is uniform if and only if $k_G \approx j_G$. This condition is also equivalent to $\tilde{\alpha}_G \lesssim j_G$; see [13, Theorem 1.2]. Thus we have a geometric characterization of domains satisfying these inequalities as well.

2.4. Directed Density and the Apollonian Inner Metric

We start by introducing some concepts which allow us to calculate the Apollonian inner metric. First we define a directed density of the Apollonian metric as follows:

$$\overline{\alpha}_G(x;r) = \lim_{t\to 0}\frac{1}{t}\alpha_G\left(x, x+t\frac{r}{|r|}\right), \tag{2.7}$$

where $r \in \mathbb{R}^n \setminus \{0\}$. If $\overline{\alpha}_G(x;r)$ is independent of r in every point of G, then the Apollonian metric is isotropic and we may denote $\overline{\alpha}_G(x) := \overline{\alpha}_G(x;e_1)$ and call this function the density of α_G at x. In order to present an integral formula for the Apollonian inner metric we need to relate the density of the Apollonian metric with the limiting concept of the Apollonian balls, which we call the Apollonian spheres.

Definition 2.2. Let $G \subsetneq \overline{\mathbb{R}^n}$, $x \in G$ and $\theta \in S^{n-1}$.

(i) If $B^n(x+s\theta, s) \subset G$ for every $s > 0$ and $\infty \notin G$, then let $r_+ = \infty$.

(ii) If $B^n(x+s\theta, s) \subset G$ for every $s > 0$ and $\infty \in G$, then let r_+ to be the largest negative real number such that $G \subset B^n(x+r_+\theta, |r_+|)$.

(iii) Otherwise let $r_+ > 0$ to be the largest real number such that $B^n(x+r_+\theta, r_+) \subset G$.

Define r_- in the same way but using the vector $-\theta$ instead of θ. We define the *Apollonian spheres through x in direction θ* by

$$S_+ := S^{n-1}(x+r_+\theta, r_+), \qquad S_- := S^{n-1}(x-r_-\theta, r_-) \tag{2.8}$$

for finite radii and by the limiting half-space for infinite radii.

Using these spheres, we can present a useful result from [7].

Lemma 2.3 (see [7, Lemma 5.8]). *Let $G \subsetneq \overline{\mathbb{R}^n}$ be open, $x \in G \setminus \{\infty\}$ and $\theta \in S^{n-1}$. Let $r_\pm$ be the radii of the Apollonian spheres $S_\pm$ at x in the direction θ. Then*

$$\overline{\alpha}_G(x;\theta) = \frac{1}{2r_+} + \frac{1}{2r_-}, \tag{2.9}$$

where one understands $1/\infty = 0$.

The following result shows that we can find the Apollonian inner metric by integrating over the directed density, as should be expected. This is also used as a main tool for proving our main result. Piecewise continuously differentiable means continuously differentiable except at a finite number of points.

Lemma 2.4 (see [10, Theorem 1.4]). *If $x, y \in G \subsetneq \mathbb{R}^n$, then*

$$\tilde{\alpha}_G(x,y) = \inf_\gamma \int \overline{\alpha}_G(\gamma(t);\gamma'(t))\left|\gamma'(t)\right| dt, \tag{2.10}$$

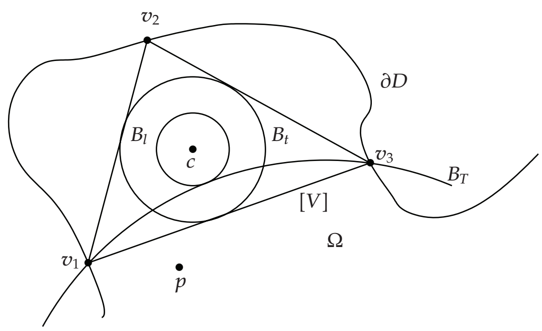

Figure 1: The largest ball B_T tangent to B_l and contained in $\Omega = \mathbb{R}^n \setminus V$, where $V = \{v_1, v_2, v_3\}$.

where the infimum is taken over all paths connecting x and y in G that are piecewise continuously differentiable (with the understanding that $\overline{\alpha}_G(z; 0) = 0$ for all $z \in G$, even though $\overline{\alpha}_G(z; 0)$ is not defined).

3. The Proof of the Main Theorem

Proof of Main Theorem. In this proof we denote by δ the distance to the boundary of D, not of $G = D \setminus \{p\}$. It is enough to prove the inequality $\alpha_G \gtrsim \tilde{\alpha}_G$, because other way inequality always holds. Let $x, y \in G$ and denote $B := B^n(p, \delta(p)/2)$. Let γ_{xy} be a path connecting x and y such that $\alpha_D(\gamma_{xy}) = \tilde{\alpha}_D(x, y)$; note that such a length-minimizing path exists by [10, Theorem 1.5].

Case 1. (a) $x, y \in D \setminus B$ and $\gamma_{xy} \cap B = \emptyset$.

Let $z \in \partial D$ be such that $\delta(p) = |p - z|$. Let V be the collection of $n + 1$ boundary points of D where they form the vertices of an n-simplex. Denote by $B_t := B^n(c, t)$ the largest ball with radius t and centered at c such that B_t is inside the n-simplex $[V]$; see Figure 1. Define $l = t/2$. Denote by $B_l := B^n(c, l)$ the ball with radius l and centered at c. Define $\Omega = \mathbb{R}^n \setminus V$. Let $B_T \subset \Omega$ be a ball tangent to B_l with maximal radius, denoted by T.

Choose $L = 5 \max\{|p - c|, T\}$. Consider the ball $B^n(c, L)$ centred at c with radius L and denote it by B_L. Then we see that $V \cup \{\infty\} \subset \overline{\mathbb{R}^n} \setminus D$. Since

$$\partial\Omega = \partial(\mathbb{R}^n \setminus V) = V \cup \{\infty\} \subset \overline{\mathbb{R}^n} \setminus D, \tag{3.1}$$

we see that

$$\overline{\alpha}_D(w; r) \geq \overline{\alpha}_\Omega(w; r) \tag{3.2}$$

for $r \in S^{n-1}$.

We now estimate the density of the Apollonian spheres (see Definition 2.2) in Ω passing through $w \in \gamma_{xy}$ and in the direction $r \in S^{n-1}$. In order to compare the density $\overline{\alpha}_\Omega(w; r)$ with the densities $\overline{\alpha}_G(w; r)$ and $\overline{\alpha}_D(w; r)$, we consider two possibilities of the choice of $w \in \gamma_{xy}$ w.r.t. B_L.

We first assume that $w \in \mathbb{R}^n \setminus B_L$. Denote by F the ray from w along r. Consider a sphere S_1 with radius R_1 and centered at $x \in F$ such that S_1 is tangent to B_l. Denote $\theta := \widehat{xwc}$. Construction of B_T gives that, for $|\theta| < \pi/2$, the Apollonian spheres passing through w and in the direction r are smaller in size than the sphere S_1.

This gives

$$\overline{\alpha}_\Omega(w;r) \geq \frac{1}{2R_1} = \frac{l + (d+l)\cos\theta}{(d+l)^2 - l^2}, \tag{3.3}$$

where $d := d(w, B_l)$ and R_1 is obtained using the cosine formula in the triangle $\triangle\, xwc$.

Now the sphere with radius R_2 and centre at q passing through w and p gives

$$\frac{|w-p|}{2} = R_2 \cos(\theta - \psi), \tag{3.4}$$

where $\psi = \widehat{cwp}$ and $q \in F$. If the Apollonian spheres (passing through w and in the direction r) are affected by the boundary point p, then by Lemma 2.3 we have

$$\begin{aligned} \overline{\alpha}_G(w;r) &= \frac{1}{R_2} + \frac{1}{r_+} \\ &\leq \frac{1}{R_2} + \overline{\alpha}_D(w;r) \\ &= \frac{2\cos(\theta-\psi)}{|w-p|} + \overline{\alpha}_D(w;r), \end{aligned} \tag{3.5}$$

where r_+ denotes the radius of the smaller Apollonian sphere which touches ∂D. Denote $\phi := \widehat{wpc}$. Since $p \in B_L$, using the sine formula in the triangle $\triangle\, wpc$ we get

$$\sin\psi = \frac{|p-c|}{|w-p|}\sin\phi \leq \frac{|p-c|}{|w-p|}. \tag{3.6}$$

Then we see that

$$\cos(\theta-\psi) \leq \cos\theta + \sin\psi \leq \cos\theta + \frac{|p-c|}{|w-p|}. \tag{3.7}$$

Thus, from (3.5) we get

$$\begin{aligned} \overline{\alpha}_G(w;r) &\leq \frac{2(\cos\theta + |p-c|/|w-p|)}{|w-p|} + \overline{\alpha}_D(w;r) \\ &= \frac{2\cos\theta}{|w-p|} + \frac{2|p-c|}{|w-p|^2} + \overline{\alpha}_D(w;r). \end{aligned} \tag{3.8}$$

Since $w \notin B_L$, we notice that the Euclidean triangle inequalities of the triangle $\triangle\ wpc$ give $|w-p| \approx d$. We then obtain

$$\begin{aligned}\overline{\alpha}_G(w;r) &\lesssim \frac{\cos\theta}{d} + \frac{l}{d^2} + \overline{\alpha}_D(w;r) \\ &\approx \frac{l+(d+l)\cos\theta}{(d+l)^2 - l^2} + \overline{\alpha}_D(w;r).\end{aligned} \tag{3.9}$$

We next assume that $w \in \overline{B}_L$. It is clear that if $\overline{\alpha}_\Omega(w;r) = 0$ then $\partial\Omega$ is contained in a hyperplane, which contradicts our assumption. Thus if $w \in \overline{B_L}$, then $\overline{\alpha}_\Omega(w;r) > 0$, and since the density function is continuous it has a greatest lower bound; namely, there exists a constant $k > 0$ such that for $r \in S^1$ we have

$$\overline{\alpha}_\Omega(w;r) \geq k. \tag{3.10}$$

Therefore, (3.3) and (3.10) together give

$$\overline{\alpha}_\Omega(w;r) \geq \min\left\{\frac{l+(d+l)\cos\theta}{(d+l)^2 - l^2}, k\right\}. \tag{3.11}$$

Since $\gamma_{xy} \cap B = \emptyset$, we note that $|w-p| \geq \delta(p)/2$ for all $w \in \gamma_{xy}$. Thus, if the Apollonian spheres passing through w and in the direction $r \in S^{n-1}$ are affected by the boundary point p, then by Lemma 2.3

$$\begin{aligned}\overline{\alpha}_G(w;r) &\leq \frac{1}{|w-p|} + \frac{1}{2r_+} \leq \frac{1}{|w-p|} + \overline{\alpha}_D(w;r) \\ &\leq \frac{2}{\delta(p)} + \overline{\alpha}_D(w;r) \approx k + \overline{\alpha}_D(w;r)\end{aligned} \tag{3.12}$$

hold, where r_+ denotes the radius of the smaller Apollonian sphere which touches ∂D. Then (3.2), (3.9), (3.11), and (3.12) together give

$$\begin{aligned}\overline{\alpha}_G(w;r) &\lesssim \min\left\{\frac{l+(d+l)\cos\theta}{(d+l)^2 - l^2}, k\right\} + \overline{\alpha}_D(w;r) \\ &\lesssim \overline{\alpha}_D(w;r).\end{aligned} \tag{3.13}$$

Thus, by the definition of the inner metric and Lemma 2.4, we get the relation

$$\alpha_G(\gamma_{xy}) \leq K\alpha_D(\gamma_{xy}) = K\tilde{\alpha}_D(x,y) \tag{3.14}$$

for some constant K. This gives

$$\tilde{\alpha}_G(x,y) \lesssim \tilde{\alpha}_D(x,y) \approx \alpha_D(x,y) \leq \alpha_G(x,y), \tag{3.15}$$

where the second inequality holds by assumption and the third holds trivially, as G is a subdomain of D.

(b) $x, y \in D \setminus B$ and γ_{xy} intersects B.

Let γ be an intersecting part of γ_{xy} from x_1 to x_2 (if there are more intersecting parts, we proceed similarly). Let γ' be the shortest circular arc on ∂B from x_1 to x_2, as shown in Figure 2.

Using the density bounds (3.2) and (3.10), we get $k \le \overline{\alpha}_D(u; r) \le 2/\delta(u)$ for every $u \in \gamma'$. Then we see that the inequalities

$$\alpha_D(\gamma) \ge \frac{\ell(\gamma)}{k}, \qquad \alpha_D(\gamma') \le \frac{4\ell(\gamma')}{\delta(p)} \tag{3.16}$$

hold. But since $\ell(\gamma) \ge |x_1 - x_2|$ and $\ell(\gamma') \le (\pi/2)|x_1 - x_2|$, we have $\ell(\gamma') \lesssim \ell(\gamma)$. This shows that $\alpha_D(\gamma'_{xy}) \lesssim \alpha_D(\gamma_{xy})$, where the path γ'_{xy} is obtained from γ_{xy} by modifying γ with the circular arc γ' joining x_1 to x_2. Since $\gamma'_{xy} \subset G \setminus B$, (3.13) implies that $\alpha_G(\gamma'_{xy}) \lesssim \alpha_D(\gamma'_{xy})$. So we get

$$\alpha_G(x, y) \ge \alpha_D(x, y) \approx \tilde{\alpha}_D(x, y) = \alpha_D(\gamma_{xy}) \gtrsim \alpha_D\left(\gamma'_{xy}\right) \gtrsim \alpha_G\left(\gamma'_{xy}\right) \ge \tilde{\alpha}_G(x, y). \tag{3.17}$$

Thus we have shown that $\alpha_G(x, y) \gtrsim \tilde{\alpha}_G(x, y)$ holds for all $x, y \in D \setminus B$.

Case 2 ($x, y \in B^n(p, (3/4)\delta(p))$). Without loss of generality we assume that $|y - p| \le |x - p|$. Since $\partial G = \partial D \cup \{p\}$, it is clear by the definition and the monotonicity property of the Apollonian metric that

$$\overline{\alpha}_G(x, y) \ge \max\left\{\log \frac{|x - p|}{|y - p|}, \alpha_D(x, y)\right\}. \tag{3.18}$$

Let $\gamma := \gamma_1 \cup \gamma_2$, where γ_1 is the path which is circular about the point p from y to $(|y - p|(x - p)/|x - p|) + p$ and γ_2 is the radial part from $(|y - p|(x - p)/|x - p|) + p$ to x, as shown in Figure 3.

Since the Apollonian spheres are not affected by the boundary point p in the circular part, we have

$$\begin{aligned} \overline{\alpha}_G\left(\gamma_1(t); \gamma_1'(t)\right) &\le \overline{\alpha}_{B^n(p, \delta(p))}\left(\gamma_1(t); \gamma_1'(t)\right) \\ &= \frac{1}{\sigma(p) - |y - p|} + \frac{1}{\delta(p) + |y - p|} \\ &= \frac{2\sigma(p)}{\sigma(p)^2 - |y - p|^2}, \end{aligned} \tag{3.19}$$

where the first equality holds since the Apollonian metric equals the hyperbolic metric in a ball. For $\gamma_2(t)$, by monotonicity in the domain of definition, we see that

$$\begin{aligned}\overline{\alpha}_G(\gamma_2(t);\gamma_2'(t)) &\leq \overline{\alpha}_{B^n(p,\delta(p))\setminus\{p\}}(\gamma_2(t);\gamma_2'(t))\\ &= \frac{1}{|p-\gamma_2(t)|}+\frac{1}{\delta(p)-|p-\gamma_2(t)|}.\end{aligned}\tag{3.20}$$

Hence, by Lemma 2.4 we have

$$\begin{aligned}\tilde{\alpha}_G(x,y)\leq\alpha_G(\gamma)&\leq\int_{\gamma_1}\frac{2\delta(p)}{\delta(p)^2-|y-p|^2}\,|dy|+\int_{|y-p|}^{|x-p|}\left(\frac{1}{t}+\frac{1}{\delta(p)-t}\right)dt\\ &=\frac{2\delta(p)\ell(\gamma_1)}{\delta(p)^2-|y-p|^2}+\log\left(\frac{|x-p|}{|y-p|}\frac{\delta(p)-|y-p|}{\delta(p)-|x-p|}\right)\\ &\leq\frac{32}{7}\frac{\ell(\gamma_1)}{\delta(p)}+\log\left(\frac{|x-p|}{|y-p|}\frac{\delta(p)-|y-p|}{\delta(p)-|x-p|}\right).\end{aligned}\tag{3.21}$$

Since $u\mapsto u^3(\delta(p)-u)$ is increasing for $0<u<3\delta(p)/4$ and we have

$$|y-p|\leq|x-p|\leq\frac{3\delta(p)}{4},\tag{3.22}$$

for the choice $u=|x-p|$, the inequality

$$|x-p|^3(\delta(p)-|x-p|)\geq|y-p|^3(\delta(p)-|y-p|)\tag{3.23}$$

holds. This inequality is equivalent to

$$\log\left(\frac{|x-p|}{|y-p|}\frac{\delta(p)-|y-p|}{\delta(p)-|x-p|}\right)\leq 4\log\frac{|x-p|}{|y-p|}.\tag{3.24}$$

Using $\alpha_D\approx\tilde{\alpha}_D$, we easily get $\alpha_D(x,y)\gtrsim\ell(\gamma_1)/\delta(p)$. We have thus shown that

$$\tilde{\alpha}_G(x,y)\leq K\alpha_D(x,y)+4\log\frac{|x-p|}{|y-p|}\leq(K+4)\alpha_G(x,y),\tag{3.25}$$

for some constant K.

Case 3. $x\notin B^n(p,3\delta(p)/4)$ and $y\in B$.

Let $w\in S^{n-1}(p,(3/4)\delta(p))$ be such that

$$|y-w|=d\left(y,S^{n-1}\left(p,\frac{3}{4}\delta(p)\right)\right).\tag{3.26}$$

Let $\gamma := \gamma_1 \cup \gamma_2$, where $\gamma_1 = [y, w]$ and γ_2 is a path connecting w and x such that

$$\alpha_G(\gamma_2) = \tilde{\alpha}_G(w, x). \tag{3.27}$$

As we discussed in the previous case, we have

$$\alpha_G(\gamma_1) \leq 4 \log \frac{3\delta(p)}{4|y-p|} \leq 4 \log \frac{|x-p|}{|y-p|} \leq 4\alpha_G(x, y). \tag{3.28}$$

Since $x, w \notin B$, it follows by Case 1 that

$$\alpha_G(\gamma_2) = \tilde{\alpha}_G(w, x) \lesssim \tilde{\alpha}_D(w, x) \approx \alpha_D(w, x) \leq 2 j_D(w, x). \tag{3.29}$$

It is now sufficient to see that $j_D(w, x) \lesssim \alpha_G(x, y)$.

If $\delta(w) \leq \delta(x)$, then the triangle inequality $|w - x| \leq |w - p| + |x - p|$ and the fact $\delta(w) \geq \delta(p)/4$ together give

$$j_D(w, x) \leq \log\left(1 + \frac{4|w-p| + 4|x-p|}{\delta(p)}\right) = \log\left(4 + \frac{4|x-p|}{\delta(p)}\right), \tag{3.30}$$

where the equality holds due to the fact that $w \in S^{n-1}(p, (3/4)\delta(p))$. But for $s \geq 3/2$, we have $\log(4 + 2s) \leq 5 \log s$. For the choice $s = |x - p|/|y - p|$, the inequality (3.30) reduces to

$$j_D(w, x) \leq \log\left(4 + \frac{2|x-p|}{|y-p|}\right) \leq 5 \log \frac{|x-p|}{|y-p|} \leq 5\alpha_G(x, y), \tag{3.31}$$

where the first inequality holds since $|y - p| \leq \delta(p)/2$ and the last holds by the definition of the Apollonian metric.

We next move on to the case $\delta(w) \geq \delta(x)$. If $|x - y| \geq 3\delta(x)$, we see (by the triangle inequality $|b - y| \geq |x - y| - |b - x|$) that

$$\alpha_G(x, y) \geq \sup_{b \in \partial D} \log \frac{|b-y|}{|b-x|} \geq \log\left(\frac{|x-y|}{\delta(x)} - 1\right) \tag{3.32}$$

holds. Using (3.32) and the fact that $|x - p|/|y - p| \geq 3/2$, we get

$$\alpha_G(x, y) \geq \begin{cases} \log\left(\dfrac{|x-y|}{\delta(x)} - 1\right) & \text{for } \dfrac{|x-y|}{\delta(x)} \geq 3, \\[2ex] \log \dfrac{3}{2} & \text{otherwise.} \end{cases} \tag{3.33}$$

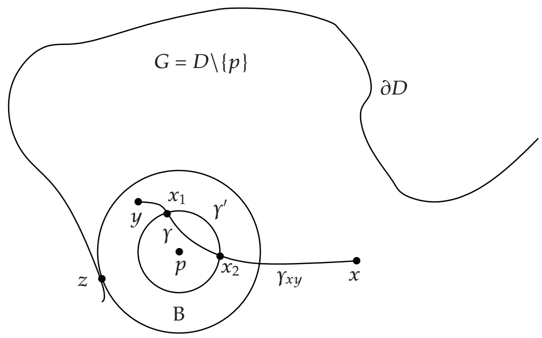

Figure 2: The geodesic path γ_{xy} (w.r.t. the Apollonian inner metric $\tilde{\alpha}_D$) connecting x and y intersects B, and its modification γ' from x_1 to x_2 along the circular part.

Since $|x-y| \geq \delta(p)/4$, we get the following upper bound for $j_D(w,x)$:

$$j_D(w,x) \lesssim \log\left(1+\frac{|x-y|+\delta(p)}{\delta(x)}\right) \leq \log\left(1+\frac{5|x-y|}{\delta(x)}\right), \tag{3.34}$$

where the first inequality follows by the triangle inequality $|w-x| \leq |x-y|+|y-w|$ and the fact that $|y-w| \leq (3/4)\delta(p)$. We see that the function $f(s) = (s-1)^4 - (1+5s)$ is increasing for $s \geq 3$, so $f(s) \geq f(3) = 0$. Thus, for the choice $s = |x-y|/\delta(x) \geq 3$, we get

$$j_D(w,x) \lesssim \log\left(1+\frac{5|x-y|}{\delta(x)}\right) \leq 4\log\left(\frac{|x-y|}{\delta(x)}-1\right) \leq 4\alpha_G(x,y), \tag{3.35}$$

where the last inequality holds by (3.32). On the other hand, if $|x-y|/\delta(x) < 3$, then $j_D(w,x)$ is bounded above by $4\log 2$ and $\alpha_G(x,y)$ is bounded below by $\log(3/2)$, so the inequality $j_D(w,x) \lesssim \alpha_G(x,y)$ is clear. Thus for the choice of w, x, and y we obtain

$$\alpha_G(\gamma_2) \lesssim j_D(w,x) \lesssim \alpha_G(x,y), \tag{3.36}$$

which concludes that

$$\tilde{\alpha}_G(x,y) \leq \alpha_G(\gamma) \lesssim \alpha_G(x,y). \tag{3.37}$$

We have now verified the inequality in all of the possible cases, so the proof is complete. □

Of course, we can iterate the main result, to remove any finite set of points from our domain. Like in [12], we get the following.

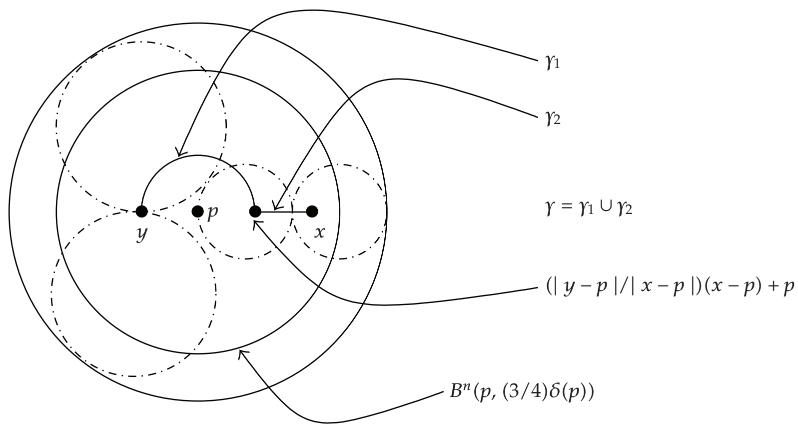

Figure 3: A short path $\gamma = \gamma_1 \cup \gamma_2$ connecting x and y in $B^n(p,(3/4)\sigma(p))$.

Corollary 3.1. *Let $D \subsetneq \mathbb{R}^n$ be a domain whose boundary does not lie in a hyperplane. Suppose that $(p_i)_{i=1}^k$ is a finite nonempty sequence of points in D and define $G := D \setminus \{p_1, p_2, \ldots, p_k\}$. Assume that $\alpha_D \approx \widetilde{\alpha}_D$ and $j_D \approx k_D$. Then Inequality (9) in Table 1, $\alpha_G \approx \widetilde{\alpha}_G \ll j_G \approx k_G$, holds.*

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