Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 375293, 11 pages
doi:10.1155/2010/375293

Research Article

New Types of Almost Countable Dense Homogeneous Space

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Received 28 September 2009; Revised 19 January 2010; Accepted 9 March 2010

Academic Editor: Richard Wilson

In 1972, Bennett studied the countable dense homogeneous (CDH) spaces and in 1992, Fitzpatrick, White, and Zhou proved that every CDH space is a T_1 space. Afterward Bsoul, Fora, and Tallafha gave another proof for the same result, also they defined the almost CDH spaces and almost T_1, T_0 spaces, indeed they prove that every ACDH space is an almost T_1 space. In this paper we introduce a new type of almost CDH spaces called ACDH-1, we characterize the ACDH spaces, the almost T_0 spaces, we also give relations between different types of CDH spaces. We define new type of almost T_1 (AT_1) spaces, and we study the relations between the old and new definitions. By extending the techniques given by Tallafha, Bsoul, and Fora, we prove that every ACDH-1 is an AT_1.

1. Introduction

In 1920, Sierpinski introduced in [1] the notion of homogeneous spaces, saying that a topological space X is a homogeneous space if for any $x \neq y$ in X, then there is a homeomorphism h of X such that $h(x) = y$. Fréchet in [2] and Brouwer in [3] observed that the n-dimensional Euclidean space $\mathbb{R}^n$ has the property that if A, B are countable dense subsets of $\mathbb{R}^n$, then there is a homeomorphism h of $\mathbb{R}^n$ such that $h(A) = B$. Afterward, in 1972 the abstract study was begun by Bennett in [4], who called such spaces the countable dense homogeneous (CDH) spaces.

In 1974, Lauer defined in [5] the densely homogeneous (DH) spaces, and in 1992, Fitzpatrick et al. proved in [6] that DH and CDH spaces are T_1 spaces, and afterward Tallafha et al. in [7] gave another proof for the fact that CDH spaces are T_1 spaces. In [8], Fora et al. defined almost CDH (ACDH) spaces, almost T_0, T_1 (AT_0, AT_1) spaces, and they discussed the relation between such spaces.

In the first part of this paper, we introduce the definitions given by Fora et al. in [8] for almost CDH spaces, almost T_0, T_1 spaces. We give a characterization of almost CDH spaces besides we define a new type of almost CDH (ACDH-1) spaces, Transposition Homogeneous space (TH). Also, we discuss the relation between the new type and the others.

In Section 4 we give a characterization of almost T_0 we introduce new definition of almost T_1 space AST_1, and we study the relation between them and the old ones. We finally prove our main result using the idea of almost closurely ordered sets, more precisely, we show that every almost CDH space of type 1 (ACDH-1) is an AST_1 space.

Finally, the following abbreviations and symbols will be used throughout this paper. For a subset A of a topological space (X, τ), we write $\overline{A}$, or $\mathrm{Cl}(A)$ for the closure of A, for $x \in X$, by $\overline{x}$ which denotes the closure of $\{x\}$ and $|A|$ which denotes the cardinality of A. By $H(X)$ we mean the set of all homeomorphisms of X, Δ refers to the symmetric difference of sets, (X, τ_{ind}), (X, τ_{dis}), (X, τ_{cof}), and (X, τ_{coc}) denote the set X with the indiscrete, discrete, cofinite, and cocountable topologies, and $(\mathbb{R}, \tau_{l\cdot r})$, $(\mathbb{R}, \tau_{r\cdot r})$, and $(\mathbb{R}, \tau_u)$ denote $\mathbb{R}$, with the left ray, right ray, and the usual topologies, respectively.

2. Almost CDH Spaces

Fora et al. in [8] defined almost and strong almost countable dense homogeneous spaces.

Definition 2.1 (see [8]). A space (X, τ) is called an almost countable dense homogeneous (ACDH) space if it is a separable space and for any two countable dense subsets K_1, K_2, there are two finite subsets F_1, F_2, $K_1 \cap F_1 = K_2 \cap F_2 = \emptyset$ and $h \in H(X)$ such that $h(K_1 \cup F_1) = K_2 \cup F_2$ and $h(F_1) \cap F_2 = \emptyset$. In addition, if $|F_1| = |F_2|$, then (X, τ) is called a strong ACDH space which is denoted by SACDH.

Remark 2.2. In the previous definition the condition $h(F_1) \cap F_2 = \emptyset$ is redundant since if there exist finite sets B_1 and B_2 such that $K_1 \cap B_1 = K_2 \cap B_2 = \emptyset$ and $h(K_1 \cup B_1) = K_2 \cup B_2$, then we choose the finite sets F_1 and F_2 as follows $F_1 = B_1 \setminus h^{-1}(C)$, $F_2 = B_2 \setminus C$, where $C = h(B_1) \cap B_2$.

Theorem 2.3 (see [8]). *If X is countable and (X, τ) is SACDH, then τ is the discrete topology.*

Let (X, τ) be any topological space. For $x \in X$, let $A_x = \{y \in X : \overline{y} = \overline{x}\}$, $A = \{x : |A_x| > 1\}$, $B = \{x : |\overline{x}| \geq \aleph_0\}$, $C_x = \{y \in X : x \in \overline{y}\}$, and $C = \{x : |C_x| \geq \aleph_0\}$. And Let $F = \{x \in X : \{x\}$ is not a closed set$\}$.

Definition 2.4 (see [8]). A topological space (X, τ) is called almost T_o (AT_0) if $|A| < \aleph_0$. If, in addition, $|B| < \aleph_0$, then (X, τ) is called strong almost T_o (SAT_o).

Clearly every T_o-space is AT_o.

Definition 2.5 (see [8]). A topological space (X, τ) is called almost T_1 and denoted by AT_1 if it is SAT_o and $|C| < \aleph_0$, that is, $A, B,$ and C are all finite sets.

Definition 2.6 (see [8]). A topological space (X, τ) is called strong almost T_1 space (SAT_1) if $|F| < \aleph_o$. Clearly every T_1 space is SAT_1 and every SAT_1 is AT_1.

Theorem 2.7 (see [8]). *Every ACDH space is AT_0.*

Theorem 2.8 (see [8]). *Let (X, τ) be ACDH space. If $|X| \leq \aleph_0$, then (X, τ) is AT_1.*

Now let us define a new type of almost CDH spaces.

Definition 2.9. A topological space (X,τ) is called an almost CDH of type 1 (ACDH-1) if it is separable space and there exists a finite subset F such that, for any two countable dense subsets A and B of X, there exists $h \in H(X)$ such that $h(A \setminus F) = B \setminus F$. Note that from now on, we will refer to F by a related finite set.

Clearly every finite space is an almost CDH of type 1. Moreover, we have the following result.

Proposition 2.10. *If (X,τ) is ACDH-1, $|X| \leq \aleph_0$, and F is a related finite set, then for all $x \notin F$, we have $\{x\}$ which is an open set in X.*

Proof. Let $x \in X \setminus F$. If $\{x\}$ is not an open set in X, then $\overline{X \setminus \{x\}} = X$, therefore there is $h \in H(X)$ such that $h(X \setminus F) = (X \setminus \{x\}) \setminus F$ which is a contradiction. □

As an application of Proposition 2.10, we have the following corollary.

Corollary 2.11. *If (X,τ) is an ACDH-1, $|X| \leq \aleph_0$, then there exists the smallest related finite set A.*

Proof. Let (X,τ) be a nondiscrete ACDH-1 space and $A = \{x \in X : \{x\} \text{ not open}\}$. As $\tau \neq \tau_{\text{dis}}$, $A \neq \phi$. Also by Proposition 2.10, $A \subseteq F$, for all related finite sets F, so A is finite. If K_1, K_2 are two dense sets and $h \in H(X)$, then $X \setminus A \subseteq K_1 \cap K_2$ and $h(X \setminus A) = X \setminus A$, hence $h(K_1 \setminus A) = h(X \setminus A) = X \setminus A = K_2 \setminus A$; therefore A is the smallest finite related set. Moreover, if $\tau = \tau_{\text{dis}}$, then $A = \emptyset$. □

The following example shows that ACDH-1 space need be a CDH space.

Example 2.12. Let $X = \mathbf{N}$ and $\beta = \{\{1,2\},\{3,4\},\{5\},\{6\},\ldots\}$. Let A, B be two dense subsets of X. Then $\{5,6,7,\ldots\} \subseteq A \cap B$ let $F = \{1,2,3,4\}$, hence $(X,\tau(\beta))$ is ACDH-1 space which is not a CDH space as it is not a T_1 space.

Now let prove the following characterizations of ACDH spaces.

Theorem 2.13. *If (X,τ) is a separable space, then the following are equivalent*

(i) (X,τ) *is an ACDH space*

(ii) *For any two countable dense subsets K_1, K_2, there exist two finite subsets F_1, F_2 of K_1, K_2, respectively, and $h \in H(X)$ such that $h(K_1 \setminus F_1) = K_2 \setminus F_2$.*

(iii) *For any two countable dense subsets K_1, K_2, there exist two equipotent finite subsets F_1, F_2, and $h \in H(X)$ such that $h(K_1 \setminus F_1) = K_2 \setminus F_2$.*

(iv) *For any two countable dense subsets K_1, K_2, there exist two finite subsets F_1, F_2, and $h \in H(X)$ such that $h(K_1 \setminus F_1) = K_2 \setminus F_2$.*

Proof. (i) implies (ii) Suppose that (X,τ) is ACDH space. Let K_1, K_2 be two countable dense sets. Then there exist two finite sets G_1, G_2 and $h \in H(X)$ such that $G_1 \cap K_1 = G_2 \cap K_2 = G_2 \cap h(G_1) = \emptyset$ and $h(K_1 \cup G_1) = K_2 \cup G_2$. Let $F_1 = h^{-1}(G_2)$ and $F_2 = h(G_1)$, so F_1, F_2 are two finite sets and

$$h(K_1 \setminus F_1) = (h(K_1 \cup G_1) \setminus h(G_1)) \setminus G_2 = K_2 \setminus F_2. \tag{2.1}$$

(ii) implies (iii) Let K_1, K_2 be two countable dense sets, then there exist two finite sets F_1, F_2, and $h \in H(X)$ such that $F_1 \subseteq K_1, F_2 \subseteq K_2$, and $h(K_1 \setminus F_1) = K_2 \setminus F_2$. Let $G_1 = F_1 \cup h^{-1}(F_2)$ and $G_2 = h(F_1) \cup F_2$. Clearly $|G_1| = |G_2|$. Moreover,

$$h(K_1 \setminus G_1) = h\Big(K_1 \setminus F_1 \cup h^{-1}(F_2)\Big) = K_2 \setminus F_2. \tag{2.2}$$

But clearly $(h(F_1) \setminus F_2) \cap K_2 = \emptyset$, so $K_2 \setminus F_2 = K_2 \setminus G_2$. (iii) implies that (iv) is clear.
(iv) implies (i) Let K_1, K_2 be two countable dense sets, then there are G_1, G_2 finite sets and $h \in H(X)$ such that $h(K_1 \setminus G_1) = K_2 \setminus G_2$.

Let $F_1 = h^{-1}(G_2 \cap K_2) \setminus (G_1 \cap K_1)$ and $F_2 = h(G_1 \cap K_1) \setminus (G_2 \cap K_2)$. Then F_1, F_2 are two finite sets. If $t \in F_1 \cap K_1$, then $t \in K_1 \setminus G_1$ and $h(t) \in G_2 \cap K_2$ which gives a contradiction. Similarly, $F_2 \cap K_2 = \emptyset$. Moreover,

$$\begin{aligned} h(K_1 \cup F_1) &= h(K_1) \cup [(G_2 \cap K_2) \setminus h(G_1 \cap K_1)] \\ &= h(K_1) \cup (G_2 \cap K_2) \\ &= h((K_1 \setminus G_1) \cup (K_1 \cap G_1)) \cup (G_2 \cap K_2) \\ &= K_2 \setminus G_2 \cup h(K_1 \cap G_1) \cup (G_2 \cap K_2) \\ &= K_2 \cup (h(K_1 \cap G_1) \setminus K_2) \\ &= K_2 \cup F_2. \end{aligned} \tag{2.3}$$

□

Consequently, we have the following result.

Corollary 2.14. *Every ACDH-1 space is an ACDH space.*

Theorem 2.15. *Let (X, τ) be a separable space. Then (X, τ) is an SACDH space if and only if for every two countable dense sets K_1, K_2, there exist two equipotent finite subsets F_1, F_2 of K_1, K_2, respectively, and $h \in H(X)$ such that $h(K_1 \setminus F_1) = K_2 \setminus F_2$.*

Proof. Let K_1, K_2 be two countable dense sets, so there exist $h \in H(X)$, $G_1 \subsetneq K_1$, $G_2 \subsetneq K_2$, and $|G_1| = |G_2| < \aleph_0$ such that $h(K_1 \setminus G_1) = K_2 \setminus G_2$. Let $F_1 = h^{-1}(G_2) \setminus K_1$ and $F_2 = h(G_1) \setminus K_2$. It is clear that $F_1 \cap K_1 = F_2 \cap K_2 = \emptyset$. Also,

$$\begin{aligned} |F_1| &= \left|h^{-1}(G_2) \setminus K_1\right| \\ &= \left|h^{-1}(G_2) \setminus \Big(K_1 \cap h^{-1}(G_2)\Big)\right| \\ &= \left|h^{-1}(G_2)\right| - \left|K_1 \cap h^{-1}(G_2)\right| \\ &= |G_1| - |(h(K_1 \setminus G_1) \cup h(G_1)) \cap G_2| \\ &= |G_1| - |h(G_1) \cap G_2| \\ &= |G_1| - |G_2 \cap h(G_1) \cup ((K_2 \setminus G_2) \cap h(G_1))| \\ &= |h(G_1)| - |K_2 \cap h(G_1)| \\ &= |F_2|. \end{aligned} \tag{2.4}$$

Moreover,

$$
\begin{aligned}
h(K_1 \cup F_1) &= h(K_1) \cup h(F_1) \\
&= h(K_1) \cup (G_2 \setminus h(K_1)) \\
&= h(K_1) \cup G_2 \\
&= h((K_1 \setminus G_1) \cup G_1) \cup G_2 \\
&= K_2 \cup h(G_1) \\
&= K_2 \cup (h(G_1) \setminus K_2) \\
&= K_2 \cup F_2.
\end{aligned}
\tag{2.5}
$$

Hence, (X, τ) is an SACDH space.

Conversely, assume that (X, τ) is an SACDH space. Let K_1, K_2 be two countable dense sets, therefore, there are two finite sets G_1, G_2 with $|G_1| = |G_2|$ and $h \in H(X)$ such that $h(K_1 \cup G_1) = K_2 \cup G_2$; moreover, $K_1 \cap G_1 = G_2 \cap K_2 = h(G_1) \cap G_2 = \emptyset$. Let $F_1 = h^{-1}(G_2)$ and $F_2 = h(G_1)$. Then $|F_1| = |F_2|$, moreover, $F_1 \subseteq K_1$ and $F_2 \subseteq K_2$. We need to prove that $h(K_1 \setminus F_1) = K_2 \setminus F_2$. Now, $h(K_1 \setminus F_1) = h(K_1 \setminus h^{-1}(G_2)) = h(K_1) \setminus G_2$. Claim that $h(K_1) \setminus G_2 = K_2 \setminus F_2$. If $x \in h(K_1) \setminus G_2$, then $x \in (K_2 \cup G_2) \setminus G_2 = K_2$. As $K_1 \cap G_1 = \emptyset$, $x \notin h(G_1) = F_2$, so $x \in K_2 \setminus F_2$. To prove the other inclusion, suppose that $x \in K_2 \setminus F_2 = K_2 \setminus h(G_1) = h(h^{-1}(K_2) \setminus G_1)$. Therefore, $x = h(t)$, for some $t \in h^{-1}(K_2) \setminus G_1$, and then $t \in (K_1 \cup G_1) \setminus G_1$ so $t \in K_1$. Also $t \notin h^{-1}(G_2)$, hence $x \in h(K_1) \setminus G_2$. □

Consequently, we have the following result.

Corollary 2.16. *Every ACDH-1 space is SACDH.*

3. T-Homogeneous Spaces

A transposition on X is a permutation on X which exchanges the places of two elements x, y, while leaving all the other elements unchanged. Now we will define the Transposition-Homogeneous (TH) spaces, and we will show that every TH SACDH space is a CDH space.

Definition 3.1. A space (X, τ) is called Transposition-Homogeneous (TH) space if every transposition on X is a homeomorphism.

Proposition 3.2. *A space (X, τ) is (TH) if and only if, for any two finite subsets F_1, F_2 with the same cardinality and for every $h \in H(X)$, there exists $h' \in H(X)$ such that*

(1) $h'(F_1) = F_2$,

(2) $h'(x) = h(x)$, *for every* $x \notin F_1 \cup h^{-1}(F_2)$,

(3) $h'(h^{-1}(F_2)) = h(F_1)$.

Proof. Let (X, τ) be a TH space and F a finite subset of X. Let σ be a permutation which fixes $X \setminus F$, clearly σ is a composition of finite transpositions which is a homeomorphism. Let

F_1, F_2 be two finite subsets with the same cardinality and $h \in H(X)$, clearly $|F_1 \setminus h^{-1}(F_2)| = |h^{-1}(F_2) \setminus F_1|$ and $F = (F_1 \setminus h^{-1}(F_2)) \sqcup (h^{-1}(F_2) \setminus F_1)$ are a disjoint union of finite sets which is finite. Let $f : F \to F$ be a bijection such that $f(F_1 \setminus h^{-1}(F_2)) = h^{-1}(F_2) \setminus F_1$. For each $x \in F_1 \setminus h^{-1}(F_2)$, let σ_x be a transposition on X which transposes x and $f(x)$. Now let σ be the finite composition of the transpositions $\sigma_x,\ x \in F_1 \setminus h^{-1}(F_2)$. Then $\sigma_x(y) = y$ for all $y \in X \setminus F$, $\sigma(F_1) = h^{-1}(F_2)$, and $\sigma(h^{-1}(F_2)) = F_1$. Now $h' = h \circ \sigma$ is the required function since $h'(x) = h(x)$, for every $x \in X \setminus F$.

The converse is obvious by choosing h to be the identity and $F_1 = \{x\}$, $F_2 = \{y\}$. □

Example 3.3. One can show that the spaces (X, τ_{ind}), (X, τ_{dis}), (X, τ_{cof}), and (X, τ_{coc}) are all TH spaces. However, the spaces $(\mathbb{R}, \tau_{l.r})$ and $(\mathbb{R}, \tau_u)$ are not TH spaces.

The following example shows that ACDH-1, TH space need be CDH space, hence ACDH TH space need be a CDH space.

Example 3.4. Let X be such that $1 < |X| < \aleph_0$, with the indiscrete topology. The space (X, τ) is a TH-space and it is also ACDH-1, but it is not a CDH space as it is not a T_1 space.

Theorem 3.5. *If (X, τ) is SACDH, TH space, then (X, τ) is a CDH space.*

Proof. If A, B are two countable dense subsets of X, then there exist two finite subsets F_1, F_2 of A, B, respectively with $|F_1| = |F_2|$ and there is $h \in H(X)$ such that $h(A \setminus F_1) = B \setminus F_2$. As X is a TH space, there exists $h' \in H(X)$ such that $h'(F_1) = F_2$, $h'(x) = h(x)$, for all $x \notin F_1 \cup h^{-1}(F_2)$ and $h'(h^{-1}(F_2)) = h(F_1)$. To show that $h'(A) = B$ we show first that $A \cap h^{-1}(F_2) \subseteq F_1$. Suppose that there is $x \in (A \cap h^{-1}(F_2)) \setminus F_1$. Then $h(x) \in B \setminus F_2$ which gives a contradiction. Hence $h'(A \setminus F_1) = h(A \setminus F_1) = B \setminus F_2$, so $h'(A) = h'((A \setminus F_1) \cup F_1) = (B \setminus F_2) \cup F_2 = B$. □

Let (X, τ) be an ACDH-1 space and $\mathfrak{F} = \{F : F \text{ is a related set}\}$ and define $D = \{A \subsetneq X : A \text{ is acountable dense subset of } X\}$, and also we define the relation $\sim$ on D by: $A \sim B$ if and only if there is $F \in \mathfrak{F}$ such that $|A \cap F| = |B \cap F|$. Let $D_1 = \{(A, B) \in D \times D : A \sim B\}$.

Now we have the following result.

Theorem 3.6. *If (X, τ) is an ACDH-1, TH space, and $(A, B) \in D_1$, then there is $h \in H(X)$ such that $h(A) = B$.*

Proof. Suppose that $(A, B) \in D_1$, so there are $F \in \mathfrak{F}$ and $h \in H(X)$ such that $h(A \setminus F) = B \setminus F$ with $|A \cap F| = |B \cap F|$. If $A \cap F = \emptyset$, then we are done. In general let $F_1 = F \cap A$ and $F_2 = F \cap B$, therefore $|F_1| = |F_2|$. As X is a TH space, there exists $h' \in H(X)$ such that $h'(F_1) = F_2$ and $h'(x) = h(x)$, for all $x \notin F_1 \cup h^{-1}(F_2)$ and $h'(h^{-1}(F_2)) = h(F_1)$, also $A \cap h^{-1}(F_2) \subseteq F$. Now $h'(A \setminus \mathrm{F}_1) = h'(A \setminus F) = h(A \setminus F) = B \setminus F = B \setminus F_2$, so $h'(A) = h'((A \setminus F_1) \cup F_1) = (B \setminus F_2) \cup F_2 = B$. □

Consequently, we have the following Corollary.

Corollary 3.7. *If (X, τ) is an ACDH-1, TH space, and $D_1 = D \times D$, then (X, τ) is a CDH space.*

4. Almost CDH Spaces, and New Separation Axioms

We know that almost CDH space is not a T_0 space. In this section we will give a characterization of almost T_0 space, also we will give a new definition of almost T_1 space.

Theorem 4.1. *Let (X,τ) be a topological space. Then (X,τ) is AT_0 space if and only if there exists a finite subset F of X, such that, for all $x \neq y$ and $\{x,y\} \cap F = \emptyset$, there is an open set containing only one of x, y. We will refer to F by a related finite set F.*

Proof. Assume that (X,τ) is an AT_0 space. Let $F = A$. Then for all $x \neq y$ with $\{x,y\} \cap F = \emptyset$, we have that $|A_x| = 1$ and $|A_y| = 1$, hence $y \notin \overline{x}$ or $x \notin \overline{y}$. Conversely, suppose that there exists a finite set F such that for all $x \neq y$ with $\{x,y\} \cap F = \emptyset$, there is an open set containing only one of x, y. If $|A|$ is an infinite, then there exists a denumerable subset of A, say $\{x_1, x_2, \ldots\}$, and $|A_{x_n}| > 1$, for all $n \in \mathbb{N}$, so there exist $y_n \in A_{x_n}$ and $y_n \neq x_n$ for all $n \in \mathbb{N}$. Therefore there is $n_0 \in \mathbb{N}$ such that $x_{n_0} \neq y_{n_0}$ are both not in F and $\overline{x}_{n_0} = \overline{y}_{n_0}$, which gives a contradiction. □

Definition 4.2. A space (X,τ) is called an almost strong T_1 (AST_1) space if there is a finite subset F of X such that, for all $x \neq y$ and $\{x,y\} \cap F = \emptyset$, there are two open subsets u_1, u_2 of X, such that $x \in u_1 \setminus u_2$ and $y \in u_2 \setminus u_1$. F is called the related finite set.

One may easily prove the following proposition.

Proposition 4.3. *Let (X,τ) be a topological space. If (X,τ) is an AST_1 space, then for all $x \notin F$, we have $\overline{x} \subseteq \{x\} \cup F$; where F is a related finite set. Conversely, if there is a finite set F such that for all $x \notin F$, $\overline{x} \subseteq \{x\} \cup F$, then (X,τ) is an AST_1.*

In the following results we show that the new separation axiom AST_1 is stronger than the one defined by Fora et al. in [8].

Proposition 4.4. *Every AST_1 space is AT_1 space.*

Proof. Suppose that (X,τ) is an AST_1 space, and let F be a related finite set. By Theorem 4.1 it is an AT_0, therefore, $|A| < \aleph_0$. By Proposition 4.3, for all $x \notin F$, we have $\overline{x} \subseteq \{x\} \cup F$, so that $B \subseteq F$. If $|C| \geq \aleph_0$, then there is $x_n \in C$ such that for all $m \neq n$, $x_n \neq x_m$ and $|C_{x_n}| \geq \aleph_0$, for all $n \in \mathbb{N}$. Let n_1 be such that $x_{n_1} \notin F$. Therefore $|C_{x_{n_1}}| \geq \aleph_0$, then there is $t \in C_{x_{n_1}} \setminus F$, so $x_{n_1} \in \bar{t}$, which gives a contradiction, and hence the proposition is proved. □

The following example shows that the converse of the previous proposition need not be true.

Example 4.5. Let $X = \mathbb{N}$,: $\beta = \{\{1\},\{1,2\},\{3\},\{3,4\},\{5\},\{5,6\},\ldots\}$ so β is a base for some topology on X. Note that for $n \in \mathbb{N}$, we have

$$\overline{n} = \begin{cases} \{n, n+1\}; & n \text{ is odd}, \\ \{n\}; & n \text{ is even}. \end{cases} \tag{4.1}$$

Therefore, for all $x \in \mathbb{N}$, $A_x = \{x\}$, hence $A = \emptyset$, $B = \emptyset$ as for all $n \in \mathbb{N}$, $|\overline{n}| \leq 2$. Now for $n \in \mathbb{N}$

$$C_n = \begin{cases} \{n-1, n\}; & n \text{ is even}, \\ \{n\}; & n \text{ is odd}. \end{cases} \tag{4.2}$$

Therefore, $C = \emptyset$, and then $(X, \tau(\beta))$ is AT_1 space. Let F be any finite subset of X. Let $m = \sup : F$, as $2m+2 \in \mathrm{Cl}\{2m+1\}$ and $2m+2,\ 2m+1$ are both not in F, $(X, \tau(\beta))$ is not an AST_1 space.

One may easily prove the following proposition.

Proposition 4.6. *Every SAT_1 space is AST_1 space.*

Fitzpatrick et al. proved in [6] that every CDH space is a T_1 space. Indeed, Tallafha et al. in [7] gave us another proof for the same argument by using the idea of closurely ordered sets. Now, we will prove that every ACDH-1 space is AST_1 space by using the idea of almost closurely ordered sets.

Definition 4.7 (see [7]). Let (X, τ) be a topological space. A countable subset K of X is said to have the closurely ordered property if there exists a numeration of K, say $K = \{x_1, x_2, \ldots\}$ such that for all $n \geq 2$, $x_n \notin \mathrm{Cl}\{x_1, x_2, \ldots, x_{n-1}\}$. The numeration $\{x_1, x_2, \ldots\}$ is called closurely ordered countable set.

Definition 4.8 (see [7]). A countable collection $\mathfrak{A}$ of subsets of X is said to have the closurely ordered countable property if $\mathfrak{A}$ can be written as $\mathfrak{A} = \{A_1, A_2, \ldots\}$, where $A_n \cap \mathrm{Cl}\{\bigcup_{i=1}^{n-1} A_i\} = \emptyset$. The form $\{A_1, A_2, \ldots\}$ is called closurely ordered countable family.

Theorem 4.9 (see [7]). *Let (X, τ) be a topological space and let K be any countable dense subset of X. Then there exists a countable dense subset K_1 of K, such that K_1 is closurely ordered countable set.*

Theorem 4.10 (see [7]). *Let (X, τ) be a topological space, then,*

(i) *if $h : X \to Y$ is an injective open function and K has the closurely ordered property in X, then $h(K)$ has the closurely ordered property in Y,*

(ii) *having closurely ordered property, is a topological property,*

(iii) *every subset of a set having closurely ordered property must have closurely ordered property.*

Now let us define the following.

Definition 4.11. A countable set K in (X, τ) is said to have the almost closurely ordered property if there is a finite set F in X such that $K \setminus F$ has the closurely ordered property. If $K \setminus F = \{x_1, x_2, \ldots\}$ is a closurely ordered set, then K is called almost closurely ordered set.

Proposition 4.12. *If (X, τ) is an AST_1 space and F is a related finite set, then each doubleton $\{x, y\} \subseteq X \setminus F$ has the closurely ordered property. Conversely, in a topological space (X, τ) if there exists a finite set F all doubletons $\{x, y\} \subseteq X \setminus F$ have the closurely ordered property, then (X, τ), is an AT_0 space.*

Proof. The first part is clear. To prove the converse, assume that there is such a finite set F. Let x, y be such that $x \neq y$ and $\{x, y\} \subseteq X \setminus F$. So $x \notin \overline{y}$ or $y \notin \overline{x}$, by Theorem 4.1 (X, τ) is an AT_0. □

Theorem 4.13. *Every ACDH-1 space is AT_0 space.*

Proof. Assume that (X, τ) is an ACDH-1 space and F is a related finite set. We want to show that F is the desired set. If $x \neq y$ with $\{x, y\} \cap F = \emptyset$ and K is a countable dense subset of X, by Theorem 4.9, we may assume that K has the closurely ordered property, also by

Theorem 4.10, $K \setminus F$ has the closurely ordered property. Now $\{x, y\} \cup K$ is also a countable dense subset of X, therefore there is $h \in H(X)$ such that $h(K \setminus F) = (K \cup \{x, y\}) \setminus F$. So $\{x, y\}$ has the closurely ordered property, the result follows by Theorem 4.10. □

Theorem 4.14 (see [7]). *If (X, τ) a topological space and K is a countable dense subset, then there exists a countable collection of subsets $A_1, A_2, \ldots$ of K such that*

(i) $A = \bigcup_{n=1}^{m} A_n \subseteq K : m \leq \aleph_0$,

(ii) $\{A_1, A_2, \ldots\}$ *is closurely ordered countable family,*

(iii) *each A_k has the closurely ordered property,*

(iv) *each A_k is either a singleton or an infinite set,*

(v) $\overline{A} = \overline{K}$,

(vi) *if A_k is a singleton, say $\{a_k\}$, then $a_k \notin \overline{x}$ and $x \notin Cl\{a_k\}$, for all $x \in A \setminus \{a_k\}$,*

(vii) *if $A_k = \{a_1^k, a_2^k, \ldots\}$ is infinite set, then $a_i^k \in Cl\{a_{1+i}^k\}$, for all i.*

It is easy to prove the following result.

Proposition 4.15. *The properties (i)–(vii) in the last theorem are all preserved under homeomorphisms.*

We now prove the following theorem, that will be used to prove our main result.

Theorem 4.16. *Let (X, τ) be an ACDH-1 space and let F be a related finite set. If $x \neq y$ and $\{x, y\} \cap F = \emptyset$ and $x \in \overline{y}$, then $\overline{x}$ is an infinite set.*

Proof. Suppose that $x \neq y, \{x, y\} \cap F = \emptyset$ and $x \in \overline{y}$. Suppose that $\overline{x}$ is a finite set, then there is $N \in \mathbb{N}$ such that $|\overline{x}| = N$. Let K be any countable dense set in X, then by Theorem 4.14, there is a countable collection of subsets of K say $A_1, A_2, \ldots$ satisfing the conditions (i)–(vii). So, $A = \bigcup_{n=1}^{m} A_n \subseteq K$ and $\overline{A} = \overline{K} = X$. Let $I = \{i : |A_i| = \aleph_o\}$. For $i \in I$, define $B_i = A_i \setminus \{a_1^i, a_2^i, \ldots, a_N^i\}$ and for $i \notin I B_i = A_i = \{a_i\}$, also define $B = \bigcup_{i=1}^{m} B_i$. To show that $\overline{B} = \overline{A}$. If $i \in I$, then $\mathrm{Cl}\{a_1^k, a_2^k, \ldots, a_N^k\} \subseteq \mathrm{Cl}\{a_{N+1}^i\} \subseteq \overline{B}_i$, therefore $\overline{A}_i \subseteq \overline{B}_i \cup \mathrm{Cl}\{a_1^i, a_2^i, \ldots, a_N^i\} \subseteq \overline{B}_i$. Then B is a countable dense set in X and so is $B \cup \{x, y\}$. Therefore, there exists $h \in H(X)$ such that $h((B \cup \{x, y\} \setminus F)) = B \setminus F$. As $x \notin F$, we have $h(x) \in B_i$ for some $i \in \mathbb{N}$. If $|B_i| = 1$, then $B_i = \{b^i\}$ and $h(x) = b^i$, so $b^i \in h(\overline{y})$ where $y \notin F$; therefore, $h(y) \in B \setminus \{b^i\}$ which is a contradiction by Theorem 4.14(vi). If $|B_i| = \aleph_0$ and $h(x) = b_{n_o}^i$ for some $n_o > N$, so all $b_i^k, b_2^i, \ldots, b_n^i$ are in $\overline{h(x)}$. As $|\overline{h(x)}| = |\overline{x}| > N$, which is impossible, so $\overline{x}$ is an infinite set. □

Recall that in ACDH-1 space, if K is a countable dense subset of X, then by Theorem 4.14 there are countable subsets $A_1, A_2, \ldots$ of K satisfying (i)–(vii) of the pointed theorem.

Moreover, $A = \bigcup_{i=1}^{m} A_i \subseteq K$, $\overline{A} = \overline{K} = X$. Therefore, there is $h \in H(X)$ such that $h(A \setminus F) = K \setminus F$. By Proposition 4.15, $K \setminus F$ can be decomposed in the same way as $A \setminus F$.

The following theorem shows that all the above A_is are singletons.

Theorem 4.17. *Let (X, τ) be an ACDH-1 space, K any countable dense subset of X, and F a related finite set. Then $K \setminus F = \bigcup_{i=1}^{\infty} A_i$ and $|A_i| = 1$.*

Proof. Let $I = \{i : |A_i| = \aleph_0\}$. If $I \neq \emptyset$, then $A_i = \{a_1^i, a_2^i, \ldots\}$, for some $i \in I$. We have $a_1^i \in \mathrm{Cl}\{a_2^i\}$ and $\{a_1^i, a_2^i\} \cap F = \emptyset$, so, by Theorem 4.16, we have $\mathrm{Cl}\{a_1^i\}$ which is an infinite set. Let $a_0^i \notin F$ with $a_0^i \in \mathrm{Cl}\{a_1^i\} \setminus \{a_1^i\}$. In a similar way, let $a_{-1}^i \notin F$ with $a_{-1}^i \in \mathrm{Cl}\{a_0^i\} \setminus \{a_0^i\}$. By the same argument, we have a sequence $\ldots, a_{-n}^i\ a_{-n+1}^i, \ldots, a_{-1}^i, a_0^i$ and $a_{-k}^i \in \mathrm{Cl}\{a_{-k+1}^i\} \setminus \{a_{-k+1}^i\}$. Now we claim that for all $k, n \in \mathbb{N}, a_{-k}^i \neq a_n^i$. If $a_{-k}^i = a_n^i$ for some $n \geq 2$ and $k \geq 0$, then $a_n^i \in \mathrm{Cl}\{a_1^i\}$ which contradicts the fact that $\{a_1^i, a_2^i, \ldots\}$ is a closurely ordered set. Also $a_1^i \neq a_{-k}^i$ for all $k > 0$, since $a_1^i \notin \mathrm{Cl}\{a_0^i\}$ and $a_{-k}^i \in \mathrm{Cl}\{a_0^i\}$, so we proved our claim. Let $K_1 = [\bigcup_{i \notin I} A_i] \cup [\bigcup_{i \in I}\{\ldots, a_{-2}^i\ a_{-1}^i, a_0^i, a_1^i, a_1^i, \ldots\}]$. Then $X = \overline{K} \subseteq \overline{K_1 \cup F}$, hence $K_1 \cup F$ is a countable dense subset of X. Then there is $h \in H(X)$ such that $h(K \setminus F) = (K_1 \cup F) \setminus F = K_1$. For $i \notin I, A_i = \{a^i\}$ and $a^i \notin \overline{x}$, for all $x \in (K \setminus F) \setminus \{a^i\}$. Then by Proposition 4.15, $h(A_i) = \{a^j\} = A_j$, $j \notin I$. Now define $i_0 = \inf(I)$, therefore $A_{i_0} = \{a_1^{i_0}, a_2^{i_0}, \ldots\}$. Moreover, $h(a_1^{i_0}) \in A_{j_0}$, for some $j_o \in I$, where $A_{j_0} = \{\ldots, a_{-2}^{j_0}, a_{-1}^{j_0}, \cdots, a_1^{j_0}, a_2^{j_0}, \ldots\}$. Then $h(a_1^{i_0}) = a_k^{j_0}$, for some $k \in \mathbb{Z}$. Also $x \in \mathrm{Cl}\{a_1^{i_0}\}$, where $x \in K \setminus F$ and $x = h^{-1}(a_{k-1}^{j_0})$. If $x \in A_r$, for some $r \in I$, then $A_r \cap \mathrm{Cl}\{\bigcup_{j=1}^{r-1} A_j\} \neq \emptyset$ which is a contradiction, so $I = \phi$. □

As a consequence of the previous theorem, we have the following results.

Corollary 4.18. *If (X, τ) is an ACDH-1 space, F is a related finite set, and K is a countable dense subset of X, then $K \setminus F$ has the closurely ordered property.*

Proof. If K is a countable dense subset of X, then by using Theorem 4.17, we have that $K \setminus F = \bigcup_{i=1}^{\infty} A_i$ and $|A_i| = 1$, for all $i \in \mathbb{N}$. Therefore, $K \setminus F = \bigcup_{i=1}^{\infty} a^i$ indeed, $a^i \notin \mathrm{Cl}\{a^j\}$, for all $i \neq j$. □

Corollary 4.19. *Every ACDH-1 space is an AST_1 space.*

Proof. Let F be a related finite set and $x \neq y$ with $\{x, y\} \cap F = \emptyset$. If K is a countable dense subset of X, say $K = \{x_1, x_2, \ldots\}$, then the set $K_1 = \{x, y, x_1, x_2, \ldots\}$ is also a countable dense subset of X, therefore by Corollary 4.18, we have that $K_1 \setminus F$ has the closurely ordered property and $\{x, y\} \subseteq K_1 \setminus F$, therefore $y \notin \overline{x}$. Similarly, $x \notin \overline{y}$. □

Acknowledgment

This paper is financially supported by the Deanship of Academic Research at the University of Jordan, Amman, Jordan.

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