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Research Article

On Hyperbolic 3-Manifolds Obtained by Dehn Surgery on Links

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We study the algebraic and geometric structures for closed orientable 3-manifolds obtained by Dehn surgery along the family of hyperbolic links with certain surgery coefficients and moreover, the geometric presentations of the fundamental group of these manifolds. We prove that our surgery manifolds are 2-fold cyclic covering of 3-sphere branched over certain link by applying the Montesinos theorem in Montesinos-Amilibia (1975). In particular, our result includes the topological classification of the closed 3-manifolds obtained by Dehn surgery on the Whitehead link, according to Mednykh and Vesnin (1998), and the hyperbolic link L_{d+1} of $d+1$ components in Cavicchioli and Paoluzzi (2000).

1. Introduction

All manifolds will be assumed to be connected, orientable, and PL (Piecewise Linear). In [1, 2], theorems state that any closed orientable 3-manifold can be obtained by Dehn surgeries on the components of an oriented link in the 3-sphere. Considering the hyperbolic link, the Thurston-Jorgensen theory in [3] of hyperbolic surgery implies that the resulting manifolds are hyperbolic for almost all surgery coefficients. Another method for describing closed 3-manifolds says that any closed 3-manifold can be represented as a branched covering of some link in the 3-sphere [2]. As the above, if the link is hyperbolic, the construction yields hyperbolic manifolds for branching indices sufficiently large.

According to the algorithm in [4], any manifold obtained by Dehn surgeries on a strongly invertible link can be presented as a 2-fold covering of the 3-sphere branched over some link. Thus we can construct many classes of closed orientable 3-manifolds by considering its branched coverings or by performing Dehn surgery along it. Moreover, the branched covering and Dehn surgery are nice methods for representing closed orientable

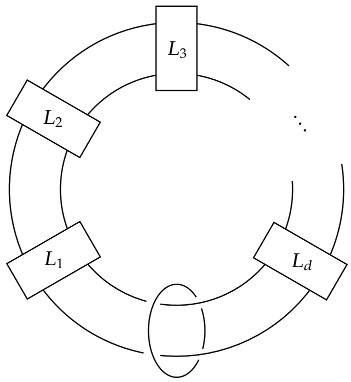

Figure 1: The oriented links $\mathcal{L}_{(m,d)}$ in S^3.

3-manifolds by combinatorial tools. See [5] for the many faces of cyclic branched coverings of 2-bridge links.

In this paper, we consider a family of links $\mathcal{L}_{(m,d)}$ for positive integers m and d as in Figure 1, where each L_i in box denotes the $1/m$-rational tangle. In fact the link $\mathcal{L}_{(m,d)}$ has two component links if d and m are odd, and three components if d and m is even and odd, respectively. Moreover, $\mathcal{L}_{(m,d)}$ has $(d+1)$-component links if m is even. Actually $\mathcal{L}_{(1,1)}$ is the double link and $\mathcal{L}_{(2,1)}$ is the Whitehead link which was considered in [6], and $\mathcal{L}_{(2,d)}$ is the hyperbolic link L_{d+1} as considered in [7]. We note that $\mathcal{L}_{(m,1)}$ is the hyperbolic link for $m > 1$ [8] and that $\mathcal{L}_{(m,d)}$ is the hyperbolic link for $m > 1$, which act by isometries.

Lastly, for positive integers $m > 1$, $n \geq 3$, and $k \geq 1$, it was proved that a family of closed 3-manifold $\mathcal{M}(2m+1,n,k)$ as the identification space of certain polyhedron $\mathcal{P}(2m+1,n,k)$ whose finitely many boundary faces are glued together in pair and which is another method to construct 3-manifolds, is the (n/d)-fold strongly cyclic covering of the 3-sphere branched over the link $\mathcal{L}_{(m,d)}$, where $\gcd(n,k) = d$ [8]. Since our link $\mathcal{L}_{(m,d)}$ is hyperbolic link for $m > 1$, it is clear that $\mathcal{M}(2m+1,n,k)$ is the closed hyperbolic 3-manifold.

In this paper, we study the closed hyperbolic 3-manifolds obtained by Dehn surgeries on the components of these links. Moreover, we show that our surgery manifolds are 2-fold cyclic covering of 3-sphere branched over certain link as Figure 6. In particular, our result includes the topological classification of the closed 3-manifolds obtained by Dehn surgery on the Whitehead link, due to Mednykh and Vesnin [9], and a hyperbolic link L_{d+1} of $(d+1)$ components in [7], which extends the Whitehead link in case of $d = 1$. See [5] for similar results obtained by Dehn surgery on the 2-bridge links.

2. Dehn Surgery on the Link $\mathcal{L}_{(m,d)}$

We now consider the oriented link $\mathcal{L}_{(m,d)}$ in the 3-sphere illustrated in Figure 2, which is formed by a chain of lines K_i between L_i and L_{i+1} for $i = 1,\ldots,d-1$, and a chain of K_d, plus a further circle Λ transversally linked to K_d. Let p_i/r_i be the surgery coefficient along the ith K_i of the chain for $i = 1,\ldots,d$, and let a/b be the surgery coefficient along the transversal component Λ, where $\gcd(p_i,r_i) = \gcd(a,b) = 1$.

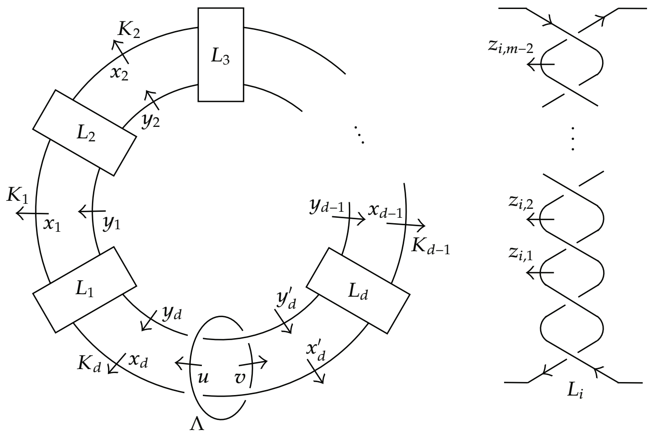

Figure 2: $\mathcal{L}_{(m,d)}$ with surgery coefficients and L_i with $(1/m)$-rational tangle

On the other hand, we obtain the fundamental group $\pi_1(\mathbb{S}^3 \setminus \mathcal{L}_{(m,d)})$ for m even. Let x_i, y_i, $z_{i,j}$, x'_d, y'_d, u, and v be the generators of a Wirtinger presentation of $\pi_1(\mathbb{S}^3 \setminus \mathcal{L}_{(m,d)})$ according to Figure 2. Then we have

$$\pi_1\left(\mathbb{S}^3 \setminus \mathcal{L}_{(m,d)}\right) \simeq \langle G \mid R_1, R_2, R_3 \rangle, \tag{2.1}$$

where $G = \{x_1, \ldots, x_d, x'_d, y_1, \ldots, y_d, y'_d, u, v, z_{i,j} \mid 1 \leq i \leq d, 1 \leq j \leq m-2\}$ and R_1, R_2, and R_3 are as follows under mod$n$;

$$R_1 = \begin{cases} x_{i-1}x_i^{-1}x_{i-1}^{-1}z_{i,1}^{-1} = 1, \\ z_{i,1}x_{i-1}z_{i,1}^{-1}z_{i,2}^{-1} = 1, \\ z_{i,2}z_{i-1}z_{i,2}^{-1}z_{i,3}^{-1} = 1, \\ \qquad \vdots \\ z_{i,m-2}z_{i,m-3}z_{i,m-2}^{-1}y_i^{-1} = 1, \\ y_i z_{i,m-2}y_i^{-1}y_{i-1} = 1, \end{cases} \qquad \text{for } i = 1, \ldots, d-1,$$

$$R_2 = \begin{cases} x_{d-1}\left(x'_d\right)^{-1}x_{d-1}^{-1}z_{d,1}^{-1} = 1, \\ z_{d,1}x_{d-1}z_{d,1}^{-1}z_{d,2}^{-1} = 1, \\ z_{d,2}z_{d-1}z_{d,2}^{-1}z_{d,3}^{-1} = 1, \\ \qquad \vdots \\ z_{d,m-2}z_{d,m-3}z_{d,m-2}^{-1}\left(y'_d\right)^{-1} = 1, \\ y'_d z_{d,m-3}\left(y'_d\right)^{-1}y_{d-1} = 1, \end{cases}$$

$$R_3 = \begin{cases} x_d^{-1}u^{-1}x'_d u = 1, \\ y_d^{-1}u^{-1}y'_d u = 1, \\ u y'_d v^{-1}\left(y'_d\right)^{-1} = 1, \\ \left(x'_d\right)^{-1} v x'_d u^{-1} = 1. \end{cases} \tag{2.2}$$

For our simplicity, we write

$$C_{m+1}(i) = C_{m+1}\left(x_i^{-1}, x_{i-1}\right), \qquad D_m = C_m\left(x'_d, x_{d-1}\right), \tag{2.3}$$

where, for $i \geq 3$,

$$\begin{gathered} C_1(a,b) = a^b = b\underline{a}b^{-1}, \\ C_2(a,b) = (b)^{(a^b)} = ba\underline{b}a^{-1}b^{-1}, \\ C_i(a,b) = (C_{i-2}(a,b))^{C_{i-1}(a,b)} = \begin{cases} \beta a \beta^{-1} \text{ and } \beta = \underbrace{baba\cdots ab}_{i\text{-factors}} & \text{if } i \text{ is odd,} \\ \gamma b \gamma^{-1} \text{ and } \gamma = \underbrace{baba\cdots ba}_{i\text{-factors}} & \text{if } i \text{ is even.} \end{cases} \end{gathered} \tag{2.4}$$

Then since $(x_i^{-1})^{x_{i-1}} = z_{i,1}$, $z_{i,2} = x_{i-1}^{z_{i,1}}$, $z_{i,3} = z_{i-1}^{z_{i,2}}$ in R_1, R_1 reduces to

$$R'_1 = \begin{cases} y_i = C_{m+1}, \\ y_{i-1}^{-1} = y_i C_m y_i^{-1}, \end{cases} \qquad 1 \leq i \leq d-1. \tag{2.5}$$

Similarly R_2 reduces to

$$R'_2 = \begin{cases} y'_d = D_{m+1}, \\ y_{d-1}^{-1} = y'_d D_m \left(y'_d\right)^{-1}. \end{cases} \tag{2.6}$$

Hence we have

$$\pi_1\left(\mathbb{S}^3 \setminus \mathcal{L}_{(m,d)}\right) \simeq \langle G \mid R'_1, R'_2, R_3 \rangle, \tag{2.7}$$

where $G = \{x_1, \ldots, x_d, x'_d, y_1, \ldots, y_d, y'_d, u, v\}$ and R'_1, R'_2, and R_3 are as above. We denote by $\mathcal{M}(p_1/r_1, \ldots, p_d/r_d; a/b)$ the closed connected orientable 3-manifold obtained by Dehn surgeries along the components $K_1, K_2, \ldots, K_d, \Lambda$ of $\mathcal{L}_{(m,d)}$ with surgery coefficients $p_i/r_i, p_2/r_2, \ldots, p_d/r_d$, and a/b, respectively, where $(p_i, r_i) = (a, b) = 1$.

We now obtain finite presentations of the fundamental group of these surgery manifolds as follows.

The meridians m_i and the longitude l_i of each component K_i and the meridian m and the longitude l of Λ are as follows:

$$R_4 = \begin{cases} m_i = x_i \quad (i = 1, \dots, d-1), \\ m = u, \end{cases}$$

$$R_5 = \begin{cases} l_i = C_m(i)C_{m-2}(i)\cdots C_2(i)x_{i-1}C_1(i+1)C_3(i+1)\cdots C_{m-1}(i+1)y_{i+1} \quad (i = 1, \dots, d-2), \\ l_{d-1} = C_m(d-1)C_{m-2}(d-1)\cdots C_2(d-1)x_{d-2}D_1D_3\cdots D_{m-1}y'_d, \\ l_d = u^{-1}D_mD_{m-2}\cdots D_2x_{d-1}C_1(1)C_3(1)C_5(1)\cdots C_{m-1}(1)y_1, \\ l = (y'_d)^{-1}x'_d. \end{cases} \tag{2.8}$$

A presentation of the fundamental group of $\mathcal{M}(p_1/r_1, \dots, p_d/r_d; a/b)$ is obtained from that of the link group of $\mathcal{L}_{(m,d)}$ by adding relations:

$$R_6 = \begin{cases} m_i^{p_i} l_i^{r_i} = 1 \quad (i = 1, \dots, d), \\ m^a l^b = 1. \end{cases} \tag{2.9}$$

Since p_i and r_i (resp., a and b) are coprime, there exist integers s_i and q_i (resp. s and q) such that $r_is_i - p_iq_i = 1$, for any $i = 1, \dots, d$, and $bs - aq = 1$.

Summarizing we obtain the following result.

Theorem 2.1. *The fundamental group of the closed connected orientable 3-manifold $\mathcal{M}(p_1/r_1, \dots, p_d/r_d; a/b)$ obtained by Dehn surgery along the link $\mathcal{L}_{(m,d)}$ with surgery coefficients p_i/r_i and a/b admits the finite presentation $\langle G \mid R_1, \dots, R_6 \rangle$ where G and R_i are as above.*

We note that the link $\mathcal{L}_{(m,d)}$ is hyperbolic in the sense that it has hyperbolic complement. So the Thurston-Jorgensen theory in [3] of hyperbolic surgery yields the following.

Corollary 2.2. *For any integer $d \geq 1$, and for almost all pairs of surgery coefficients p_i/r_i and a/b, the closed connected orientable 3-manifolds $\mathcal{M}(p_1/r_1, \dots, p_d/r_d; a/b)$ is hyperbolic.*

We now describe $\mathcal{M}(p_1/r_1, \dots, p_d/r_d; a/b)$ as 2-fold branched coverings of the 3-sphere in the following. We note that a link $\mathcal{L}$ is *strongly invertible* if there is an orientation-preserving involution of $\mathbb{S}^3$ which induces on each component of $\mathcal{L}$ an involution with two fixed points. The above mentioned involution is called a *strongly invertible involution* of the link. The following theorem of Montesinos relates to two different approaches for describing closed orientable 3-manifolds, which is Dehn surgery and branched coverings (see [1, 10–13] for manuscripts), and moreover, gives us an effective algorithm for describing the branch set of $\mathcal{M}(p_1/r_1, \dots, p_d/r_d; a/b)$ as 2-fold branched coverings of the 3-sphere.

Theorem 2.3 (see [4]). *Let M be a closed orientable 3-manifold obtained by Dehn surgery on a strongly invertible link L of n components. Then M is a 2-fold covering of the 3-sphere branched over a link of at most $n+1$ components. Conversely, every 2-fold cyclic branched covering of the 3-sphere can be obtained in this fashion.*

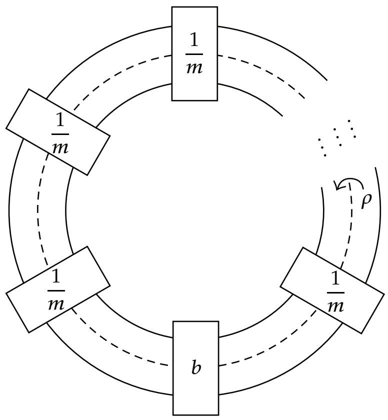

Figure 3: The strongly invertible links $\mathcal{L}''(m,d)$ and its involution.

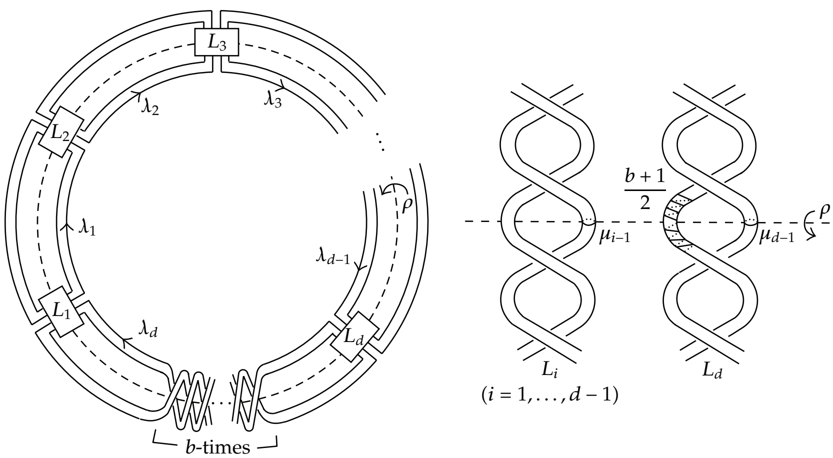

Figure 4: Regular neighborhood of $\mathcal{L}''_{(m,d)}$ and strongly involution.

We now apply Dehn surgery on the link $\mathcal{L}_{(m,d)} = K_1 \cup \cdots \cup K_d \cup \Lambda$ with surgery coefficients $p_1/r_1, \ldots, p_d/r_d$ and a/b for $a = 1$, respectively. Twist the solid torus $\mathbb{S}^3 \setminus \mathrm{int}(N)$, where N is a tabular neighborhood of the transversal circle Λ. The meridean m of N is carried to $\tau\ell + m$ (ℓ being the longitude of N), where τ represents the number of twists which is positive (resp. negative) if the twist is in the right-hand (resp. left-hand) sense. Let $\mathcal{L}'_{(m,d)} = K'_1 \cup \cdots \cup K'_d \cup \Lambda'$ be the link obtained from $\mathcal{L}_{(m,d)}$ by twisting around Λ. Then the surgery coefficients on components Λ' and K'_i of $\mathcal{L}'_{(m,d)}$ are $1/(\tau + b)$ and p_i/r_i for $lk(\Lambda, K_i) = 0$. By setting $\tau := -b$, we can delete the component Λ' from $\mathcal{L}'_{(m,d)}$ obtaining the link $\mathcal{L}''_{(m,d)}$ of d components illustrated in Figure 3. This link is strongly invertible, and the axis of a strongly invertible involution ρ of $\mathcal{L}''_{(m,d)}$ is given by the dotted line in Figure 3.

We choose meridean μ_i and longitude λ_i according to Figure 4. Let V be a regular neighborhood of L''_d in $\mathbb{S}^3$. Without loss of generality, we can choose neighborhood V, meridean μ_i, and longitudes λ_i on ∂V to be invariant under the involution ρ. The image of V under the canonical projection $\pi : S^3$ to the quotient space S^3/ρ of S^3 under ρ consists of d 3-balls B_i. Let θ denote the axis of the involution ρ in S^3. For each 3-ball B_i, the set

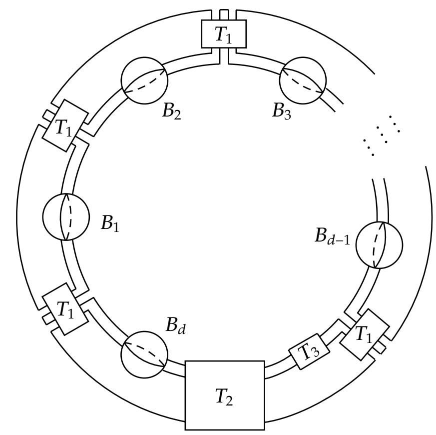

Figure 5: Tangle decompositions.

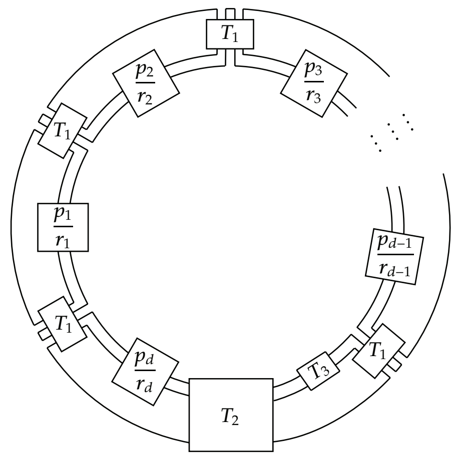

Figure 6: The branched links obtained by the Montesinos algorithm.

$B_i \cap \pi(\theta)$ consists of two arcs. By isotopy of B_i along the image $\pi(\lambda_i)$ of longitude λ_i for any $i = 1, \ldots, d$, we get Figure 5. Each 3-ball B_i with arcs $B_i \cap \pi(\theta)$ is a trivial tangle. By the Montesinos algorithm, we replace these trivial tangles B_i by (p_i/r_i)-rational tangles for any $i = 1, \ldots, d$.

For the simplicity, we now define some series of links. We recall that any link can be obtained as the closure of some braid. Given coprime integers p and q, denote by $\sigma_2^{p/q}$, the rational link (p/q)-tangle whose incoming arc are the ith link and $(i+1)$th strings, where T_1, T_2 denotes a 4-strings braid $(\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1})^m$ and a 3-strings braid $(\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1)^b$ respectively, and T_3 is a rational $((b+1)/2)$-tangle.

Summarizing we have proved the following.

Theorem 2.4. *Let $\mathcal{M}(p_1/r_1, \ldots, p_d/r_d; a/b)$, $d \geq 2$, $a = \pm 1$, be the closed orientable 3-manifold obtained by Dehn surgery on the link $\mathcal{L}_{(m,d)}$ with surgery coefficients p_i/r_i and a/b. Then $\mathcal{M}(p_1/r_1, \ldots, p_d/r_d; a/b)$ is a 2-fold covering of the 3-sphere $\mathbb{S}^3$ branched over the link in Figure 6.*

For $a/b = 1$ and $d = 2$, our manifolds are homeomorphic to the manifolds obtained by Dehn surgeries on the two components of the Whitehead link $\mathcal{L}_{(2,1)}$. Thus the result includes the main result in [9]. Moreover, for $m = 2$ and $d > 1$, our manifolds are homeomorphic to the manifolds obtained by Dehn surgeries on the $(d+1)$ components of the hyperbolic link L_{d+1}. In particular the result also includes the result in Section 5 of [7].

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